m3pm16soln1.tex

M3PM16/M4PM16 SOLUTIONS 1. 24.1.2013

Q1 ([L], 12-13).

$$li(x) := \int_2^x \frac{du}{\log u} = \Big[\frac{u}{\log u}\Big]_2^x - 2\int_2^x d\Big(\frac{1}{\log u}\Big) = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{du}{\log^2 u}.$$

For $x \geq 4$,

$$0 < \int_2^x \frac{du}{\log^2 u} = \int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x < \int_2^{\sqrt{x}} \frac{2}{\log^2 2} + \int_{\sqrt{x}}^x \frac{du}{\log^2 x}$$

$$= \frac{\sqrt{x}-2}{\log^2 2} + \frac{x-\sqrt{x}}{\frac{1}{4}\log^2 x} < \frac{\sqrt{x}}{\log^2 2} + \frac{4x}{\log^2 x} = o\Big(\frac{x}{\log x}\Big).$$

The LH inequality gives

$$\liminf li(x)/\frac{x}{\log x} \geq 1.$$

The RH inequality gives

$$\limsup li(x)/\frac{x}{\log x} \leq 1.$$

Combining,

$$li(x)/\frac{x}{\log x} \to 1: \qquad li(x) \sim \frac{x}{\log x}.$$

Q2 ([L], 13-14). Integrating by parts $m+1$ times,

$$li(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \ldots + \frac{m!x}{\log^{m+1} x} + const + (m+1)! \int_2^x \frac{du}{\log^{m+2} u}.$$

For $x \geq 4$, as before,

$$0 < \int_2^x \frac{du}{\log^{m+2} u} = \int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x < \int_2^{\sqrt{x}} \frac{du}{\log^{m+2} 2} + \int_{\sqrt{x}}^x \frac{du}{\log^{m+2}(\sqrt{x})}$$

$$< \frac{\sqrt{x}-2}{\log^{m+2} 2} + \frac{x-\sqrt{x}}{2^{-m-2}\log^{m+2} x} = o\Big(\frac{x}{\log^{m+1} x}\Big).$$

So

$$li(x) - \Big(\frac{x}{\log x} + \frac{1!x}{\log^2 x} + \ldots + \frac{(m-1)!x}{\log^m x}\Big) = \frac{m!x}{\log^{m+1} x}(1 + o(1)). \qquad //$$

Note. Numerical evidence shows that $li(x)$ gives a much better approximation than $x/\log\ x$, in line with Q1, Q2, and we shall prefer it – particularly in PNT with any error term – see Problems 2.

Q3 ([L], 214-5). Taking $x = p_n$ in $\pi(x) := \sum_{p \le x} 1$ gives

$$\pi(p_n) = \sum_{p \le p_n} 1 = n.$$

By PNT, $\pi(x) \sim x/\log x$, so $n \sim p_n/\log p_n$:

$$\frac{n \log p_n}{p_n} \to 1. \qquad (i)$$

Taking logs of (i), $\log n + \log\log p_n - \log p_n \to 0$. Dividing this by $\log p_n$,

$$\frac{\log n}{\log p_n} + \frac{\log\log p_n}{\log p_n} - 1 \to 0.$$

But $\log x = o(x)$, so $\log\log p_n = o(\log p_n)$, so this says

$$\frac{\log n}{\log p_n} \to 1. \qquad (ii)$$

Multiply (i) and (ii): $n \log n/\log p_n \to 1$, i.e. $p_n \sim n \log n$. //

Q4 ([L], 214-5). By PNT and Q2, $\pi(x) = x/\log x + O(x/\log^2 x)$. So taking $x = p_n$,

$$n = \frac{p_n}{\log p_n} + O(\frac{p_n}{\log^2 p_n}), \quad = \frac{p_n}{\log p_n} + O(\frac{n \log n}{\log^2 n}) = \frac{p_n}{\log p_n} + O(\frac{n}{\log n}),$$

using Q3 and $p_n \ge n$. So

$$p_n = n(1 + O(1/\log n))\ \log p_n. \qquad (iii)$$

By Q3, $p_n = n \log n(1 + o(1))$, so

$$\log p_n = \log n + \log\log n + o(1). \qquad (iv)$$

Substituting (iv) in (iii) gives the result ("big error terms swallow little error terms"). // NHB