m3pm16prob6.tex

M3PM16/M4PM16 PROBLEMS 6. 21.2.2013

Q1. Show (by using Chebyshev's Upper Estimate and Abel summation, or otherwise) that

$$\sum \frac{1}{p \log p} < \infty.$$

Q2 ($\zeta(2n)$ and the Bernoulli numbers). Define the *Bernoulli numbers* B_n by the generating function

$$\frac{t}{e^t - 1} = \sum_{n-0}^{\infty} \frac{B_n t^n}{n!}$$

(one can check that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, $B_6 = 1/42$, ... ; also all $B_{2n+1} = 0$ and all B_{2n} are rational).
(i) Show that

$$z \cot z = 1 + \sum_{j=2}^{\infty} B_j (2ix)^j / j!$$

(ii) From the Weierstrass product for sin,

$$\sin z = z \prod_{n=1}^{\infty} (1 - z^2/n^2\pi^2),$$

show by taking logs and differentiating that

$$z \cot z = 1 - 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (z/n\pi)^{2k} = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k)(z/\pi)^{2k}.$$

(ii) Deduce Euler's formula

$$\zeta(2n) = (-)^{n+1} (2\pi)^{2n} B_{2n} / 2(2n!);$$

so $\zeta(2) = \pi^2/6$ (again!), $\zeta(4) = \pi^4/90$ (again!), $\zeta(6) = \pi^6/945$, and $\zeta(2n)$ is a rational multiple of π^{2n} (so is irrational, indeed transcendental).
Note. One can show from the functional equation for ζ (III.8 L25) that $\zeta(-2n) = 0$ and $\zeta(1 - 2n) = -B_{2n}/n$.

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