m3pm16l3.tex

Lecture 3. 19.1.2012

Dirichlet Test for Convergence: If a_n have bounded partial sums $A_n = \sum_1^n a_r$, and $v_n \to 0$, then $\sum a_n v_n$ converges.

Proof: If $|A_n| \leq K, |A_n v_n| \leq K v_n \to 0$.
In $\sum_0^{n-1} A_r(v_r - v_{r-1})$, $|A_r(v_r - v_{r-1})| \leq K(v_r - v_{r-1})$. As $v_n \to 0$, $\sum v_r - v_{r-1}$ is a convergent telescoping series, so $\sum A_r(v_r - v_{r-1})$ is convergent by the Comparison Test.
Combining, $\sum a_n v_n$ is convergent by Abel's Lemma. //

Abel's Test for Convergence. If $\sum a_n$ convergent and v_n is real, monotonic and convergent, then $\sum a_n v_n$ converges.

Proof: A_n is convergent, v_n is convergent, so $A_n v_n$ is convergent. A_n is also bounded, $A_n \leq K$. $\sum(v_r - v_{r-1})$ is a convergent telescoping series. The result follows as above by the Comparison Test. //

Abel's Summation Formula. If $y < x$ and f has a continuous derivative on $[y, x]$ (i.e. $f \in C^1[y, x]$), then

$$\sum_{y<r\leq x} a_r f_r = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt.$$

Proof. Let $m = [y], x = [n]$, with $[\cdot]$ denoting the integer part. Then $\sum_{y<r\leq n} a_r f_r = \sum_{m+1}^n a_r f_r$. As $A(x) := \sum_{r\leq x} a_r$, $A(t) = A(r)$ for $r \leq t < r+1$. So

$$\begin{aligned}
\sum_{m+1}^{n-1} A_r(f_r - f_{r+1}) &= -\sum_{m+1}^{n-1} A(r) \int_r^{r+1} f'(t)dt \\
&= -\sum_{m+1}^{n-1} \int_r^{r+1} A(t)f'(t)dt \qquad \text{as } A \text{ is constant on } (r, r+1) \\
&= -\int_{m+1}^{n} A(t)f'(t)dt.
\end{aligned}$$

Similarly, for $n \leq t \leq x$ $A(t) = A(n)$, so

$$A(x)f(x) - A(n)f(n) = A(n)[f(x) - f(n)] = \int_n^x A(t)f'(t)dt,$$

and for $m \leq t \leq y$ $A(t) = A(m)$, so

$$A(m)f(m+1) - A(y)f(y) = A(m)[f(m+1) - f(y)] = \int_y^{m+1} A(t)f'(t)dt.$$

Finally, substituting into $(*)$ in the proof of Abel's Lemma for $A_n f_n - A_m f_{m+1}$ gives the result. //

Corollary 1. (i) $\sum_{r \leq x} a_r f_r = A(x)f(x) - \int\limits_1^x A(t)f'(t)dt.$

(ii) $\sum_{r \leq x} a_r (f(x) - f(r)) = \int\limits_1^\infty A(t)f'(t)dt.$

Corollary 2. If $f \in C^1[2, x]$ and $a(1) = 0$, then $\sum\limits_{2 \leq r \leq x} a_r f_r = A(x)f(x) - \int\limits_2^x A(t)f'(t)dt.$

Proof: Take $y = 2$ and use $A(2) = a_1 + a_2 = a_2$. //

Corollary 3. If $f \in C^1[1, \infty]$, and $A(x)f(x) \to 0$ as $x \to \infty$, then $\sum_1^\infty a_r f_r = -\int_1^\infty A(t)f'(t)dt$, and then
$\sum_{r > q} a_r f_r = -A(x)f(x) - \int_x^\infty A(t)f'(t)dt.$

Proof: Take $y = 1$ and let $x \to \infty$. //

§4. The Integral Test and Euler's Constant

The Integral Test: If $f > 0$ and is monotonic decreasing on $[1, \infty]$, then:
(i) $\int\limits_1^\infty f(x)dx$ and $\sum\limits_1^\infty f(n)$ converge or diverge together;
(ii) $\sum\limits_1^n f(r) - \int_1^n f(x)dx \to l \in [0, f(1)]$ as $n \to \infty$.

Proof: As f is monotonic, it is integrable on each $[1, x]$. If $n - 1 \leq x \leq n$,

$$f(n-1) \geq f(x) \geq f(n).$$

Integrate from $n-1$ to n:

$$f(n-1) \geq \int_{n-1}^{n} f(x)dx \geq f(n).$$

Sum from 1 to $n-1$:

$$\sum_{1}^{n-1} f(r) \geq \int_{1}^{n} f \geq \sum_{2}^{n} f(r) : \quad \sum_{1}^{n} f(r) - f(n) \geq \int_{1}^{n} f \geq \sum_{1}^{n} f(r) - f(1). \ (*)$$

If $\sum_1^\infty f(r) < \infty$, the LH inequality gives $\int\limits_1^\infty f(x)dx < \infty$.
If $\int_1^\infty f(x)dx < \infty$, the RH inequality gives $\sum_1^\infty f(r) < \infty$. Combining, this gives (i). For (ii),

$$f(1) \geq \phi(n) := \sum_{1}^{n} f(r) - \int_{1}^{n} f \geq f(n) \geq 0.$$

Then by $(*)$,

$$\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^{n} f(x)dx \leq 0, \qquad 0 \leq \phi(n) \leq f(1),$$

So $\phi(n)$ is bounded and decreasing, so it is convergent: $\phi(n) \downarrow l \in [0, f(1)]$. //

Corollary (Euler's Constant).

$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log n \to \gamma \qquad (n \to \infty),$$

where γ is Euler's constant.
Proof. Take $f(x) = 1/x$ in the Integral Test. Note that

$$0 < \sum_{1}^{N} \frac{1}{n} - logN < 1$$

and that

$$\sum_{1}^{N} \frac{1}{n} = \log N + \gamma + \frac{1}{2N} + O(\frac{1}{2N}).$$