m3pm16l25.tex

Lecture 25. 13.3.2012

7. AN ELEMENTARY TAUBERIAN THEOREM

Theorem 1. If
$$\int_1^\infty \frac{A(x) - \alpha x}{x^2} dx \qquad \text{converges,}$$
and *either* (i) A is increasing and non-negative,
or (ii) there exists B with B, $B - A$ increasing and non-negative, and
$$\int_1^\infty \frac{B(x) - \beta x}{x^2} dx \qquad \text{converges for some } \beta$$
– then
$$A(x)/x \to \alpha \qquad (x \to \infty).$$

Proof. Part (ii) follows from (i), which applied to B, β and $B - A, \beta - \alpha$ (say) gives $B(x)/x \to \beta$, $(B(x) - A(x))/x \to \beta - \alpha$, so $A(x)/x \to \alpha$.

So assume (i). If $\alpha \neq 0$, w.l.o.g. take $\alpha = 1$ (consider $A(x)/\alpha$). Choose $0 < \delta < \frac{1}{2}$. As the integral converges,
$$\forall \delta_1 > 0 \quad \exists R = R(\delta_1) \ s.t. \ |\int_{x_0}^{x_1} \frac{A(x) - x}{x^2} dx| < \delta_1 \quad \forall x_1 > x_0 \geq R. \qquad (i)$$

If $A(x) \leq (1 + \delta)x$ for all $x \geq R$, $\limsup A(x)/x \leq 1 + \delta$. If not, there exists $x_0 > R$ with $A(x_0) > (1 + \delta)x_0$. Then as A increases,
$$A(x) > (1 + \delta)x_0 \qquad \forall x \geq x_0.$$
Let $x_1 := (1 + \delta)x_0$. Then
$$\begin{aligned} \int_{x_0}^{x_1} \frac{A(x) - x}{x^2} dx &> (1 + \delta)x_0 \int_{x_0}^{x_1} \frac{dx}{x^2} - \int_{x_0}^{x_1} \frac{dx}{x} \\ &= x_1 (\frac{1}{x_0} - \frac{1}{x_1}) - \log(\frac{x_1}{x_0}) \\ &= \delta - \log(1 + \delta) \qquad (x_1 = (1 + \delta)x_0). \qquad (ii) \end{aligned}$$
Take $\delta_1 := \delta - \log(1 + \delta)$. From the power series for $\log(1 + \delta)$, $\delta_1 > 0$ (indeed, $\delta_1 \geq \frac{1}{3}\delta^2$). So (ii) contradicts (i). So no such x_0 exists. So
$$\limsup A(x)/x \leq 1 + \delta.$$

Similarly,

$$\liminf A(x)/x \geq 1 - \delta.$$

As δ can be arbitrarily small, $A(x)/x \to 1$, as required.

The case $\alpha = 0$ is similar but simpler. //

Note. 1. The Theorem is *Tauberian*, in that it goes from information on an integral involving A to information on A itself. The prototype stems from *Abel's Continuity Theorem* (if a power series $f(x) = \sum_0^\infty a_n x^n$ converges at a point on its circle of convergence (w.l.o.g. 1: $\sum_0^\infty a_n = \ell$), then $f(x) \to \ell$ as $x \uparrow 1$; for proof, see e.g. T. M. Apostol, *Mathematical Analysis*, Th. 13.33, p.421-2). Alfred TAUBER (1866-1942) in 1897 proved a partial converse: if $f(x) \to \ell$, and $a_n = o(1/n)$, then $\sum a_n$ converges to ℓ. Think of the power series f as a 'smoothed version' of the sequence of coefficients a_n: an Abelian theorem goes in the smoothing direction; a Tauberian theorem is a partial converse going in the unsmoothing direction, under an additional condition, the *Tauberian condition*. Similarly for Dirichlet series.

2. Littlewood's Tauberian theorem improves Tauber's result by weakening the Tauberian condition from $a_n = o(1/n)$ to $a_n = O(1/n)$ (J. E. LITTLEWOOD (1885-1977) in 1911), and in fact $O(1/n)$ is best-possible here. This is the case relevant to the logarithmic series $\log(1 + x) = \sum_1^\infty (-)^{n-1} x^n/n$, and we have used $\log 2 = \sum_1^\infty (-)^{n-1}/n$ to continue $\zeta(s)$ analytically from $Re\ s > 1$ to $Re\ s > 0$ by the alternating zeta function (Dirichelt eta function $\eta(s)$).

3. In the real case, one-sided bounds suffice: Littlewood's Tauberian theorem holds for $a_n = O_L(1/n)$ (i.e. $a_n \geq M/n$ for some M). The Tauberian condition above, that A is (non-negative and) increasing, is of this kind. The Theorem is false without it.

4. The Ingham-Newman Theorem is Tauberian (and much harder than the elementary result above!) It is a *complex Tauberian theorem*; the *Tauberian condition* is the analytic continuability to the left of the 1-line except possibly for a pole at 1 (visibly motivated by the Riemann zeta function!) It is a variant on the first complex Tauberian theorem, the *Wiener-Ikehara Theorem* (S. IKEHARA (1904-1984) in 1931, Norbert WIENER (1894-1964) in 1932), motivated by PNT.

5. Theorem 1 above and *either* of III.5 Th. 2 and III.6 Th. 2 give us Theorem 2 below, from which we easily deduce PNT in III.8.