m3pm16l24.tex
Lecture 24. 8.3.2012

$\underline{I_1}$. If $s = \sigma + it$ and $|s| = R$ (so $s\overline{s} = R^2$),

$$\frac{1}{s} + \frac{s}{R^2} = \frac{R^2}{sR^2} + \frac{s}{R^2} = \frac{1}{R^2}\Big(\frac{s\overline{s}}{s} + s\Big) = \frac{\overline{s} + s}{R^2} = \frac{2\sigma}{R^2}.$$

If $\sigma > 0$, as $|B(x)| \le M/x$, $|x^{-s}B(x)| \le M/x^{\sigma+1}$, so

$$|g(s) - g_X(s)| \le \int_X^\infty \frac{M}{x^{\sigma+1}} dx = \frac{M}{\sigma X^\sigma}.$$

So for $|s| = R$, $\sigma > 0$ (as on C_+),

$$|J(s)(g(s) - g_X(s))| \le \frac{2\sigma X^\sigma}{R^2} . \frac{M}{\sigma X^\sigma} = \frac{2M}{R^2}.$$

This holds also by continuity at $\pm iR$ (on C_+, but with $\sigma = 0$). So by ML (M2PM3)

$$|I_1(X)| \le \frac{\pi R}{2\pi} . \frac{2M}{R^2} = \frac{M}{R} \to 0 \qquad (R \to \infty).$$

$\underline{I_2}$. If $\sigma < 0$, $|g_X(s)| \le M \int_1^X x^{-\sigma-1} dx < MX^{-\sigma}/|\sigma|$. So for $|s| = R$, $\sigma < 0$ (as on C_-),

$$|J(s) g_X(s)| \le \frac{2|\sigma| X^\sigma}{R^2} . \frac{MX^{-\sigma}}{|\sigma|} = \frac{2M}{R^2}$$

as with I_1. As before, ML gives $I_2 \to 0$.
$\underline{I_3}$.

$$\begin{aligned} I_3(X) &= \frac{1}{2\pi} \int_{-R}^R \frac{g(it)}{it} \Big(1 - \frac{t^2}{R^2}\Big) X^{it} dt \\ &= \frac{1}{2\pi} \int_{-R}^R \frac{g(it)}{it} \Big(1 - \frac{t^2}{R^2}\Big) e^{i\lambda t} dt \qquad (\lambda := \log X) \\ &\to 0 \qquad (X, \lambda \to \infty) \end{aligned}$$

by the Riemann-Lebesgue Lemma (I.7).

Combining, $g_X(0) \to 0$ $(X \to \infty)$, as required. //

Cor. 1. If in Theorem 1 $g_1(s) = \int_1^\infty B_1(x) dx / x^{s-1}$ $(Re\ s > 1)$ and g_1 can be continued analytically to a region containing $\{s : Re\ s \ge 1\}$ – then

$\int_1^\infty B_1(x)dx = g_1(1)$.

Proof. Apply the Theorem to $g(s) := g_1(s+1)$. //

Cor. 2. If $f(s) := \int_1^\infty A(x)dx/x^{s+1}$ ($Re\ s > 1$) can be continued analytically to a region containing $\{s : Re\ s \geq 1\}$ except possibly $s = 1$,

$$f(s) = \frac{\sigma}{s-1} + g(s), \qquad g \text{ holomorphic at } 1$$

and $|A(x)| \leq Mx$ $(x \geq 1)$ – then

$$\int_1^\infty \frac{A(x) - \alpha x}{x^2} dx \qquad \text{converges to } g(1).$$

Proof. Put $B(x) := A(x)/x^2 - \alpha/x$. Then $|B(x)| \leq (M + |\alpha|)/x$ $(x \geq 1)$. For $Re\ s > 1$,

$$\int_1^\infty \frac{B(x)}{x^{s-1}} dx = \int_1^\infty \Big(\frac{A(x)}{x^{s+1}} - \frac{\alpha}{x^s}\Big) dx = f(s) - \sigma/(s-1) = g(s)$$

(with g above), and the result follows by Cor. 1. //

Theorem 2. If (i) $f(s) = \sum_1^\infty a_n/n^s$ converges for $Re\ s > 1$, and f can be continued analytically to a region containing $\{s : Re\ s \geq 1\}$ except possibly at $s = 1$,
(ii) $f(s) = \alpha/(s-1) + \alpha_0 + (s-1)h(s)$, h holomorphic at 1,
(iii') $|A(x)| \leq Mx$ $(x \geq 1)$ – then

$$\int_1^\infty \frac{A(x) - \alpha x}{x^2} dx \quad \text{converges to } \alpha_0 - \alpha.$$

Proof. For $Re\ s > 1$, $x^{-s}A(x) \to 0$ $(x \to \infty)$ by (iii'). By Abel summation (I.3, last Cor. – using (iii') again with $f(x) := x^{-s}$ in the notation of I.3),

$$f(s) = sf_1(s), \qquad f_1(s) := \int_1^\infty \frac{A(x)}{x^{s+1}} dx.$$

So

$$\begin{aligned} f_1(s) = \frac{f(s)}{s} &= \frac{\alpha}{s(s-1)} + \frac{\alpha_0}{s} + \frac{(s-1)h(s)}{s} \\ &= \alpha\Big(\frac{1}{s-1} - \frac{1}{s}\Big) + \frac{\alpha_0}{s} + \frac{(s-1)h(s)}{s} = \frac{\alpha}{s-1} + g(s), \end{aligned}$$

with g holomorphic at 1 and $g(1) = \alpha - \alpha_0$. Apply Cor. 2. //