m3pm16l23.tex
Lecture 23. 6.3.2012

Proof of the Theorem (continued).

$$I(x,1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x^{it}\phi(1+it)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\lambda}\phi(1+it)dt$$

with $\lambda := \log x$. By $(*)$, $\int_{-\infty}^{\infty} |\phi(1+it)|dt < \infty$ (i.e. $\phi \in L_1$, in the language of Lebesgue integration). So

$$I(x,1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\lambda}\phi(1+it)dt \to 0 \qquad (\lambda, x \to \infty)$$

by the Riemann-Lebesgue Lemma (I.7). So by $(**)$,

$$\int_1^{\infty} \frac{A(y) - \alpha y}{y^2}dy \text{ converges to } \alpha' = \alpha_0 - \alpha. \qquad //$$

6. THE INGHAM-NEWMAN (TAUBERIAN) THEOREM

From the point of view of just proving PNT, one needs only one of this section and the last. But both are worth knowing, so we follow our course text, Jameson [J], in giving both.

The function B below will satisfy the Standing Assumption of II.1 (continuous except at integers; left and right limits there – e.g., a sum function).

Theorem 1 (INGHAM-NEWMAN THEOREM). If $|B(x)| \leq M/x$ $(x \geq 1)$,

$$g(s) := \int_1^{\infty} \frac{B(s)}{x^s}dx \qquad (Re\ s > 0),$$

and g can be continued analytically to a region $E \supset \{s : Re\ s \geq 0\}$ – then

$$\int_1^{\infty} B(x)dx = g(0).$$

Proof. If $g(0) = c$, write $B_0(x) := B(x) - c/x^2$. Then for $Re\ s > 0$,

$$\int_1^{\infty} \frac{B_0(s)}{x^s}dx = g(s) - \frac{c}{s+1} =: g_0(s),$$

where $g_0(0) = g(0) - c = 0$. This reduces to $c = 0$, so take $c = 0$ now.

As $c = g(0) = 0$, and g is holomorphic in a region (= non-empty open connected set in **C**: M2PM3, II.4 L15-16!) containing $\{s : Re\ s \geq 0\}$, g is holomorphic at 0, and $g(0) = 0$. So we may cancel s from its Cauchy-Taylor expansion at 0: $g(s)/s$ is holomorphic at 0. Fix $X > 1$, and let

$$g_X(s) := \int_1^X x^{-s} B(x) dx.$$

Then g_X is entire (for details of the proof, see e.g. J App. D), and $g_X(0) = \int_1^X B(x)dx$. We have to prove

$$g_X(0) \to 0 \qquad (X \to \infty).$$

Choose $\epsilon > 0$, then R so large that $M/R < \epsilon$ (recall $|B(x) \leq M/x$). Let $C := C(0, R)$. Write

$$J(s) := \frac{1}{s}\Big(1 + \frac{s^2}{R^2}\Big)X^s$$

(the 1980 innovation of D. J. NEWMAN (1930-2007), following A. E. INGHAM (1900-1967) in 1935). Write $a := \log X$, so $X^s = e^{as}$. So

$$J(s) = \Big(\frac{1}{s} + \frac{s}{R^2}\Big)(1 + as + \frac{1}{2}a^2 s^2 + \ldots) = \frac{1}{s} + K(s),$$

where K is entire, so J has a simple pole at 0 of residue 1. So Jg_X has a simple pole at 0 of residue $g_X(0)$, and by CRT

$$\frac{1}{2\pi i}\int_C J(s) g_X(s) ds = g_X(0). \qquad (*)$$

Write $C_\pm$ for the semicircles of C in the positive/negative half-planes, L for the line-segment from $-iR$ to iR. As g is holomorphic in E and $g(s)/s$ is holomorphic at 0,

$$J(s)g(s) = g(s)/s + K(s)g(s)$$

is holomorphic in E. So by Cauchy's Theorem applied to the contour Γ formed by C_+ (positive sense) and L (negative sense), $\int_\Gamma Jg = 0$, i.e.

$$\int_L J(s)g(s)ds = \int_{C_+} J(s)g(s)ds.$$

By $(*)$,

$$\begin{aligned} g_X(0) = \frac{1}{2\pi i}\int_C J(s)g_X(s)ds &= \frac{1}{2\pi i}\int_{C_+} Jg_X + \frac{1}{2\pi i}\int_{C_-} Jg_X \\ &= \frac{1}{2\pi i}\int_{C_+} J(g_X - g) + \frac{1}{2\pi i}\int_{C_-} Jg_X + \frac{1}{2\pi i}\int_L Jg_X \\ &= I_1 + I_2 + I_3, \qquad \text{say.} \end{aligned}$$