M2PM3 HANDOUT: THE ABEL AND DIRICHLET TESTS FOR CONVERGENCE

The Abel and Dirichlet tests for convergence of series belong to Real Analysis rather than, or as much as, to Complex Analysis. We give them here since they are necessary for handling convergence and absolute convergence of *Dirichlet series* – series of the form $\sum_{n=1}^{\infty} a_n/n^s$, with s a complex variable. These, and in particular the most important special case, the *Riemann zeta function* $\zeta(s) := \sum_{n=1}^{\infty} a_n/n^s$, are important in Analytic Number Theory; the zeta functions is crucial for the study of the distribution of *prime numbers*. In this course, the main importance of the zeta functions is as an example of the crucially important concept of *analytic continuation* (III.8). See Coursework 1 (2010).

Abel's Lemma below is the discrete analogue of integration by parts, or *partial integration*. It is accordingly also called *partial summation*. In what follows, a_n, v_n are real.

Theorem *(Abel's Lemma, or Partial Summation).*
Write $s_n := a_1 + \ldots + a_n$. Show that
(i) $a_1 v_1 + \ldots + a_n v_n = s_1(v_1 - v_2) + \ldots + s_{n-1}(v_{n-1} - v_n) + s_n v_n$.
(ii) If $m \leq a_1 + \ldots + a_n \leq M$ for all n, and v_n is positive and decreasing, then $m v_1 \leq a_1 v_1 + \ldots + a_n v_n \leq M v_1$.
(iii) If in (ii) $|s_n| \leq M$ for all n, then $|a_1 v_1 + \ldots a_n v_n| \leq M v_1$ for all n.

Proof. (i)

$$\begin{aligned} a_1 v_1 + \ldots a_n v_n &= s_1 v_1 + (s_2 - s_1) v_1 + \ldots + (s_n - s_{n-1}) v_n \\ &= s_1(v_1 - v_2) + s_2(v_2 - v_3) + \ldots + s_{n-1}(v_{n-1} - v_n) + s_n v_n. \end{aligned}$$

(ii) As $v_n \downarrow$, $v_k - v_{k+1} \geq 0$. This and $m \leq s_k \leq M$ give

$$m(v_k - v_{k+1}) \leq s_k(v_k - v_{k+1}) \leq M(v_k - v_{k+1}) \quad (k = 1, \ldots, n-1), \quad m v_n \leq s_n v_n \leq M v_n.$$

Sum over $k = 1$ to $n - 1$: the left and right telescope. Using (i) for the middle gives

$$m v_1 \leq a_1 v_1 + \ldots + a_n v_n \leq M v_1.$$

(iii) If $|s_n| \leq M$ for all n, taking $m = -M$ in (ii) gives

$$|a_1 v_1 + \ldots + a_n v_n| \leq M v_1. \qquad //$$

Theorem *(Dirichlet's Text for Convergence).*
If (a_n) has bounded partial sums $s_n = \sum_1^n a_k$ and $v_n \downarrow 0$, then $\sum a_n v_n$ is convergent.

Proof. As $v_n \downarrow 0$: $\forall \epsilon > 0$ $\exists N$ such that for $n \geq N$, $0 \leq v_n < \epsilon$. As the partial sums of $\sum a_n$ are bounded, for some M $|\sum_1^n a_k| \leq M$ for all n. So

$$|\sum_m^n a_k| = |\sum_1^n a_k - \sum_1^{m-1} a_k| \leq 2M \qquad \forall m, n \quad (m \leq n).$$

So by (iii) of Abel's Lemma, $|\sum_m^n a_k v_k| \leq 2M\epsilon$ for all $m, n \geq N$. By Cauchy's General Principle of Convergence, $\sum a_n v_n$ converges (as it is Cauchy).//

Theorem *Abel's Test for Convergence.*
If $\sum a_n$ converges and $v_n \downarrow \ell$ for some ℓ, then $\sum a_n v_n$ converges.

Proof. As the series $\sum a_n$ converges, its sequence $s_n := \sum_1^n a_k$ of partial sums converges. So (s_n) is bounded. As $v_n \downarrow \ell$, $w_n := v_n - \ell \downarrow 0$. So by Dirichlet's Test, $\sum a_n w_n$ converges, to c say:

$$a_1 w_1 + \ldots a_n w_n \to c \qquad (n \to \infty).$$

That is

$$a_1 v_1 + \ldots a_n v_n - \ell(a_1 + \ldots + a_n) \to c \qquad (n \to \infty).$$

But $a_1 + \ldots a_n \to b := \sum_1^\infty a_k$. So

$$a_1 v_1 + \ldots a_n v_n \to c + \ell.b \qquad (n \to \infty),$$

i.e. $\sum a_n v_n$ converges. //

We include the following results (used in Coursework 1, 2010, and III.8, 2011) for completeness.

Theorem (*Alternating Series Test*). If $a_n \downarrow 0$, $\sum(-1)^n a_n$ converges.

Proof. Write $s_n := \sum_1^n a_k$.

$$s_{2n} = (a_1 - a_2) + \ldots + (a_{2n-1} - a_{2n}). \qquad (1)$$

Since $a_n \downarrow$, each bracket on RHS is ≥ 0, so $s_{2n} \uparrow$. But bracketing differently,

$$s_{2n} = a_1 - (a_2 - a_3) - \ldots - (a_{2n-2} - a_{2n-1}) - a_{2n}. \qquad (2)$$

Each bracket on RHS is ≥ 0 as $a_n \downarrow$, and $a_{2n} \geq 0$. So $s_{2n} \leq a_1$. So s_{2n} is $\uparrow$ and bounded above, so

$$s_{2n} \uparrow s < \infty.$$

Also $s_{2n+1} = s_{2n} + a_{2n+1}$. But $s_{2n} \to s$, $a_{2n+1} \to 0$, so

$$s_{2n+1} \to s + 0 = s.$$

Combining the odd and even subsequences gives $s_n \to s$, as required. //

Theorem (*Integral Test).* If $f(x)$ is decreasing and non-negative on $[1, \infty)$, $\sum_1^\infty f(n)$ and $\int_1^\infty f(x)dx$ converge or diverge together.

We omit the proof (see a textbook on Analysis); you may quote the result.

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