WORKED EXAMPLES 3

COVARIANCE CALCULATIONS

EXAMPLE 1 Let X and Y be discrete random variables with joint mass function defined by

$$f_{X,Y}(x,y) = \frac{1}{4}, \qquad (x,y) \in \{(0,0),(1,1),(1,-1),(2,0)\},$$

and zero otherwise. The marginal mass functions, expectations and variances of X and Y are

$$\begin{aligned}
f_X(x) &= \sum_y f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}, & x = 0,2, \\ \frac{1}{2}, & x = 1, \end{cases} \\
\Longrightarrow E_{f_X}[X] &= \sum_{x=0}^{2} x f_X(x) = \left(0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4}\right) = 1, \\
E_{f_X}[X^2] &= \sum_{x=0}^{2} x^2 f_X(x) = \left(0 \times \frac{1}{4} + 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{4}\right) = \frac{3}{2}, \\
\Longrightarrow Var_{f_X}[X] &= E_{f_X}[X^2] - \{E_{f_X}[X]\}^2 = \frac{3}{2} - \{1\}^2 = \frac{1}{2}.
\end{aligned}$$

$$\begin{aligned}
f_Y(y) &= \sum_x f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}, & y = -1,1, \\ \frac{1}{2}, & y = 0, \end{cases} \\
\Longrightarrow E_{f_Y}[Y] &= \sum_{y=0}^{2} y f_Y(y) = \left(-1 \times \frac{1}{4} + 0 \times \frac{1}{2} + 1 \times \frac{1}{4}\right) = 0 \\
E_{f_Y}[Y^2] &= \sum_{y=0}^{2} y^2 f_Y(y) = \left((-1)^2 \times \frac{1}{4} + 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{4}\right) = \frac{1}{2} \\
\Longrightarrow Var_{f_Y}[Y] &= E_{f_Y}[Y^2] - \{E_{f_Y}[Y]\}^2 = \frac{1}{2} - \{0\}^2 = \frac{1}{2},
\end{aligned}$$

and to compute the covariance we also need to compute $\mathrm{E}_{f_{X,Y}}[XY]$

$$\begin{aligned}
E_{f_{X,Y}}[XY] &= \sum_x \sum_y xy f_{X,Y}(x,y) = \left((0 \times 0) \times \frac{1}{4} + (1 \times 1) \times \frac{1}{4} + (1 \times -1) \times \frac{1}{4} + (2 \times 0) \times \frac{1}{4}\right) = 0 \\
\Longrightarrow Cov_{f_{X,Y}}[X,Y] &= E_{f_{X,Y}}[XY] - E_{f_X}[X]\, E_{f_Y}[Y] = 0 - 1 \times 0 = 0, \qquad \text{and} \qquad \mathrm{Corr}_{\mathrm{f_{X,Y}}}[\mathrm{X},\mathrm{Y}] = 0.
\end{aligned}$$

Hence the two variables have covariance and correlation zero. But note that X and Y are **not independent** as it is **not** true that

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

for **all** x and y.

EXAMPLE 2 Let X and Y be continuous random variables with joint pdf

$$f_{X,Y}(x,y) = 3x, \qquad 0 \leq y \leq x \leq 1,$$

and zero otherwise.

The marginal pdfs, expectations and variances of X and Y are

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^x 3xdy = 3x^2, \qquad 0 \leq x \leq 1, \\
\Longrightarrow E_{f_X}[X] &= \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^1 x \times 3x^2 dx = \left[\frac{3}{4}x^4\right]_0^1 = \frac{3}{4}, \\
E_{f_X}[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^1 x^2 \times 3x^2 dx = \left[\frac{3}{5}x^5\right]_0^1 = \frac{3}{5}, \\
\Longrightarrow Var_{f_X}[X] &= E_{f_X}[X^2] - \{E_{f_X}[X]\}^2 = \frac{3}{5} - \left\{\frac{3}{4}\right\}^2 = \frac{3}{80}.
\end{aligned}$$

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_y^1 3xdx = \left[\frac{3}{2}x^2\right]_y^1 = \frac{3}{2}(1-y^2), \qquad 0 \leq y \leq 1, \\
\Longrightarrow E_{f_Y}[Y] &= \int_{-\infty}^{\infty} y f_Y(y)dy = \int_0^1 y \times \frac{3}{2}(1-y^2)dy = \left[\frac{3}{2}\left(\frac{y^2}{2} - \frac{y^4}{4}\right)\right]_0^1 = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8}, \\
E_{f_Y}[Y^2] &= \int_{-\infty}^{\infty} y^2 f_Y(y)dy = \int_0^1 y^2 \times \frac{3}{2}(1-y^2)dy = \left[\frac{3}{2}\left(\frac{y^3}{3} - \frac{y^5}{5}\right)\right]_0^1 = \frac{3}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{5}, \\
\Longrightarrow Var_{f_Y}[Y] &= E_{f_Y}[Y^2] - \{E_{f_Y}[Y]\}^2 = \frac{1}{5} - \left\{\frac{3}{8}\right\}^2 = \frac{19}{320},
\end{aligned}$$

and to compute the covariance we also need to compute $\mathrm{E}_{f_{X,Y}}[XY]$

$$\begin{aligned}
E_{f_{X,Y}}[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)dydx = \int_0^1\int_0^x xy \times 3xdydx \\
&= \int_0^1 \left\{\int_0^x ydy\right\} 3x^2 dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^x 3xdx = \int_0^1 \frac{x^2}{2} \times 3x^2 dx \\
&= \frac{3}{2}\left[\frac{x^5}{5}\right]_0^1 = \frac{3}{10}, \\
\Longrightarrow Cov_{f_{X,Y}}[X,Y] &= E_{f_{X,Y}}[XY] - E_{f_X}[X]\, E_{f_Y}[Y] = \frac{3}{10} - \frac{3}{4} \times \frac{3}{8} = \frac{3}{160} \\
Corr_{f_{X,Y}}[X,Y] &= \frac{Cov_{f_{X,Y}}[X,Y]}{\sqrt{Var_{f_X}[X] \times Var_{f_Y}[Y]}} = \frac{\dfrac{3}{160}}{\sqrt{\dfrac{3}{80} \times \dfrac{19}{320}}} = 0.397.
\end{aligned}$$