Good reduction of the Brauer–Manin obstruction

A joint work in progress with J-L. Colliot-Thélène

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Notation:

k is a number field,
k_v is the completion of k at a place v,
$\mathcal{O}_v$ is the ring of integers of k_v,
$\mathbb{A}_k$ is the ring of adèles of k,

S is a finite set of places of k containing the archimedean places, $\mathcal{O}_S = \{x \in k | \mathrm{val}_v(x) \geq 0 \text{ for any } v \notin S\}$,

$\overline{k}$ is an algebraic closure of k, $\Gamma = \mathrm{Gal}(\overline{k}/k)$,
X is a variety over k, $\overline{X} = X \times_k \overline{k}$,

$\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$,
$\mathrm{Br}_0(X) = \mathrm{Im}[\mathrm{Br}(k) \to \mathrm{Br}(X)]$,
$\mathrm{Br}_1(X) = \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})]$

The following question was asked by Peter Swinnerton-Dyer.

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Let $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_S)$ be a **smooth** *and projective morphism with geometrically integral fibres. Let $X = \mathcal{X} \times_{\mathcal{O}_S} k$ be the generic fibre. Assume that $\mathrm{Pic}(\overline{X})$ is a finitely generated* **torsion-free** *abelian group. Does there exist a closed subset $Z \subset \prod_{v \in S} X(k_v)$ such that*

$$X(\mathbb{A}_k)^{\mathrm{Br}} = Z \times \prod_{v \notin S} X(k_v) \ ?$$

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In other words: is it true that only the bad reduction primes and the archimedean places show up in the Brauer–Manin obstruction?

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Remark

If for every $v \notin S$ the image of $\mathrm{Br}(X) \to \mathrm{Br}(X_v)$ is contained in the subgroup generated by the images of $\mathrm{Br}(k_v)$ and $\mathrm{Br}(\mathcal{X}_v)$, then the answer is positive.

Proof Since $\mathcal{X}_v/\mathcal{O}_v$ is projective we have $X(k_v) = \mathcal{X}_v(\mathcal{O}_v)$. Thus the value of $A \in \mathrm{Br}(\mathcal{X}_v)$ at any $P \in X(k_v)$ comes from $\mathrm{Br}(\mathcal{O}_v) = 0$. □

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The following proposition generalises an earlier result of Martin Bright.

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Idea of proof: Let k_v^{nr} be the maximal unramified extension of k_v in $\overline{k_v}$,

$$X_v^{\mathrm{nr}} = X_v \times_{k_v} k_v^{\mathrm{nr}}, \quad \overline{X}_v = X_v \times_{k_v} \overline{k}_v.$$

Let $I = \mathrm{Gal}(\overline{k}_v / k_v^{\mathrm{nr}})$ be the inertia group.

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Key claim: Inertia I acts trivially on $\mathrm{Pic}(\overline{X}_v)$.

Let ℓ be a prime different from the residual characteristic of k_v. The Kummer exact sequence

$$1\to\mu_{\ell^n}\to\mathbb{G}_m \xrightarrow{[\ell^n]} \mathbb{G}_m\to 1$$

gives $\mathrm{Pic}(\overline{X}_v)/\ell^n \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}_v,\mu_{\ell^n})$. Passing to the limit we obtain

$$\mathrm{Pic}(\overline{X}_v) \subset \mathrm{Pic}(\overline{X}_v)\otimes\mathbb{Z}_\ell \subset \mathrm{H}^2_{\text{ét}}(\overline{X}_v,\mathbb{Z}_\ell(1)) = \mathrm{lim.proj.}\,\mathrm{H}^2_{\text{ét}}(\overline{X}_v,\mu_{\ell^n}).$$

Smooth base change theorem for the smooth and proper morphism $\pi : \mathcal{X}_v \rightarrow \mathrm{Spec}(\mathcal{O}_v)$ implies that the étale sheaf $R^2\pi_*\mu_{\ell^n}$ is locally constant. It follows that the action of $\mathrm{Gal}(\overline{k_v}/k_v)$ on the generic geometric fibre $\mathrm{H}^2_{\text{ét}}(\overline{X}_v, \mu_{\ell^n})$, i.e. on the fibre at $\mathrm{Spec}(\overline{k_v})$, factors through

$$\pi_1(\mathrm{Spec}(\mathcal{O}_v), \mathrm{Spec}(\overline{k_v})) = \mathrm{Gal}(\overline{k_v}/k_v)/I.$$

This proves the key claim.

One deduces that every element of $\mathrm{Br}_1(X_v)$ belongs to $\mathrm{Ker}[\mathrm{Br}(X_v) \rightarrow \mathrm{Br}(X_v^{\mathrm{nr}})]$. The proposition follows with a little more work (or just use Martin Bright's result). □

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Let ℓ be a prime different from the residual characteristic of k_v.
Write $\mathrm{Br}(X_v)\{\ell\}$ for the ℓ-primary subgroup of $\mathrm{Br}(X_v)$.
Let $\mathbb{F}$ be the residue field of k_v, and let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$.

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Lemma

If the closed geometric fibre $\mathcal{X} \times_{\mathcal{O}_S} \overline{\mathbb{F}}$ has no connected unramified covering of degree ℓ, then $\mathrm{Br}(X_v)\{\ell\}$ is generated by the images of $\mathrm{Br}(k_v)\{\ell\}$ and $\mathrm{Br}(\mathcal{X}_v)\{\ell\}$.

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The proof is an easy consequence of Gabber's purity theorem and results of Kato (Crelle's J., 1986, which use K-theory and the Merkuriev–Suslin theorem).

Sketch of proof:

Let $\mathcal{X}_0 = \mathcal{X} \times_{\mathcal{O}_S} \mathbb{F}$ be the closed fibre of $\pi : \mathcal{X}_v \to \mathrm{Spec}(\mathcal{O}_v)$, $\overline{\mathcal{X}}_0 = \mathcal{X} \times_{\mathcal{O}_S} \overline{\mathbb{F}}$.
$\mathbb{F}(\mathcal{X}_0)$ is the function field of $\mathcal{X}_0$.

Kato proves that the residue map fits into a *complex*

$$\mathrm{Br}(X)[\ell^n] \xrightarrow{\text{res}} \mathrm{H}^1(\mathbb{F}(\mathcal{X}_0), \mathbb{Z}/\ell^n) \longrightarrow \bigoplus_{Y \subset \mathcal{X}_0,} \mathrm{H}^0(\mathbb{F}(Y), \mathbb{Z}/\ell^n(-1)),$$

where the sum is over all irreducible $Y \subset \mathcal{X}_0$ such that $\mathrm{codim}_{\mathcal{X}_0}(Y) = 1$, and $\mathbb{F}(Y)$ is the function field of Y.
A character in

$$\mathrm{H}^1(\mathbb{F}(\mathcal{X}_0), \mathbb{Z}/\ell^n) = \mathrm{Hom}(\mathrm{Gal}(\overline{\mathbb{F}(\mathcal{X}_0)}/\mathbb{F}(\mathcal{X}_0)), \mathbb{Z}/\ell^n)$$

defines a covering of $\mathcal{X}_0$ that corresponds to the invariant field of this character.

If $A \in \mathrm{Br}(X_v)[\ell^n]$, then the covering defined by $\mathrm{res}(A)$ is unramified at every divisor of $\mathcal{X}_0$. Hence it is unramified, i.e. $\mathrm{res}(A) \in \mathrm{H}^1_{\text{ét}}(\mathcal{X}_0, \mathbb{Z}/\ell^n)$.

$$\mathrm{H}^p(\mathbb{F}, \mathrm{H}^q_{\text{ét}}(\overline{\mathcal{X}}_0, \mathbb{Z}/\ell^n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(\mathcal{X}_0, \mathbb{Z}/\ell^n)$$

gives rise to

$$0 \to \mathrm{H}^1(\mathbb{F}, \mathbb{Z}/\ell^n) \to \mathrm{H}^1_{\text{ét}}(\mathcal{X}_0, \mathbb{Z}/\ell^n) \to \mathrm{H}^1_{\text{ét}}(\overline{\mathcal{X}}_0, \mathbb{Z}/\ell^n)$$

Our assumption implies that $\mathrm{H}^1_{\text{ét}}(\overline{\mathcal{X}}_0, \mathbb{Z}/\ell^n) = 0$, hence

$$\mathrm{res}(A) \in \mathrm{H}^1(\mathbb{F}, \mathbb{Z}/\ell^n).$$

By local class field theory $\mathrm{Br}(k_v)\{\ell^n\} \to \mathrm{H}^1(\mathbb{F}, \mathbb{Z}/\ell^n)$ is an isomorphism, so that there exists $\alpha \in \mathrm{Br}(k_v)\{\ell^n\}$ such that $\mathrm{res}(\alpha) = \mathrm{res}(A)$. By Gabber's absolute purity theorem $A - \alpha \in \mathrm{Br}(\mathcal{X}_v)$. □

Question: Is there an analogue when $\ell = p$?

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Remark

Let X be a smooth, projective and geometrically integral variety over k such that $\mathrm{Pic}(\overline{X})$ is a finitely generated torsion-free abelian group, and $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is finite. Then $X(\mathbb{A}_k)^{\mathrm{Br}}$ is **open** *and closed in $X(\mathbb{A}_k)$.*

Proof $\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \subset \mathrm{H}^1(k, \mathrm{Pic}(\overline{X}))$, which is finite since $\mathrm{Pic}(\overline{X})$ is finitely generated and torsion-free.

The sum of local invariants of a given element of $\mathrm{Br}(X)$ is a continuous function on $X(\mathbb{A}_k)$ with finitely many values, and this function is identically zero if the element is in $\mathrm{Br}_0(X)$. $\square$

Main result:

Theorem

Assume
(i) $\mathrm{H}^1(\overline{X}, O_{\overline{X}}) = 0$;
(ii) *the Néron–Severi group* $\mathrm{NS}(\overline{X})$ *has no torsion;*
(iii) $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ *is a finite abelian group of order invertible in* $\mathcal{O}_S$.
Then the answer to our question is positive.

This follows from the previous results by the smooth base change theorem: we can identify

$$\mathrm{H}^1_{\text{ét}}(\overline{\mathcal{X}}_0, \mathbb{Z}/\ell) = \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/\ell) \simeq \mathrm{Pic}(\overline{X})_\ell$$

and so conclude that the closed geometric fibre has no connected étale covering of degree ℓ. □

Condition (iii) is hard to check in general, so we state a particular case where all conditions are only on $\overline{X}$.

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Corollary

Assume
(i) $\mathrm{H}^1(\overline{X}, O_{\overline{X}}) = \mathrm{H}^2(\overline{X}, O_{\overline{X}}) = 0$;
(ii) *the Néron–Severi group* $\mathrm{NS}(\overline{X})$ *has no torsion;*
(iii) *either* $\dim X = 2$, *or* $\mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$ *is torsion-free for every prime* ℓ *outside* S.
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This applies to unirational varieties (some of them are not rational, e.g. Harari's example of a transcendental Brauer–Manin obstruction with $\mathrm{Br}(\overline{X}) = \mathbb{Z}/2$).

What about bad reduction?

If $A \in \mathrm{Br}(X_v)_n$, and assume that the residual characteristic of k_v does not divide n. Let $\mathcal{X}_v \to \mathrm{Spec}(\mathcal{O}_v)$ be a regular model, smooth and projective over $\mathrm{Spec}(\mathcal{O}_v)$. Let $\mathcal{X}_0$ be the closed fibre, $\mathcal{X}_0^{\mathrm{smooth}}$ be its smooth locus, and let V_i be the irreducible components of $\mathcal{X}_0^{\mathrm{smooth}}$ that are geometrically irreducible. Then the reduction of a k_v-point belongs to some V_i.

Kato's complex implies that $\mathrm{res}_{V_i}(A) \in \mathrm{H}^1_{\text{ét}}(V_i, \mathbb{Z}/n)$, but this group is finite. Evaluating at $\mathbb{F}$-points gives finitely many functions $V_i(\mathbb{F}) \to \mathrm{H}^1(\mathbb{F}, \mathbb{Z}/n) = \mathbb{Z}/n$, or one function $f : V_i(\mathbb{F}) \to (\mathbb{Z}/n)^m$. This defines a partition of $V_i(\mathbb{F})$. We get a partition of $X(k_v)$ into a disjoint union of subsets such that $A(P)$ is constant on each subset.

Note that this Corollary does not apply to K3 surfaces. Nevertheless we have the following result.

Theorem

Let D be the diagonal quartic surface over $\mathbb{Q}$ given by

$$x_0^4 + a_1 x_1^4 + a_2 x_2^4 + a_3 x_3^4 = 0,$$

where $a_1, a_2, a_3 \in \mathbb{Q}^*$. Let $\mathcal{S}$ be the set of primes consisting of 2 and the primes dividing the numerators or the denominators of a_1, a_2, a_3. Then

$$D(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = Z \times \prod_{p \notin \mathcal{S}} D(\mathbb{Q}_p)$$

for an open and closed subset $Z \subset D(\mathbb{R}) \times \prod_{p \in \mathcal{S}} D(\mathbb{Q}_p)$.

Proof This follows from the previous theorem by the results of Ieronymou–AS–Zarhin: only the primes from $\{2, 3, 5\} \cap \mathcal{S}$ can divide the order of the finite group $\mathrm{Br}(D)/\mathrm{Br}_1(D)$. □

Note The primes from $\mathcal{S} \setminus \{2, 3, 5\}$ are not too bad, whereas those from $\{2, 3, 5\} \cap \mathcal{S}$ are seriously bad.

Note We do not have an example of a transcendental Azumaya algebra on D of order 2 defined over $\mathbb{Q}$, but Thomas Preu has constructed such an algebra of order 3 which gives a BM obstruction to WA. Order 5?

A full proof of the result of Ieronymou–AS–Zarhin is quite long.

Notation:

X is the Fermat quartic $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$;
E is the elliptic curve $y^2 = x^3 - x$ with CM by $\mathbb{Z}[i]$, $i = \sqrt{-1}$.

The Galois representation on E_n was computed by Gauss:

if p splits in $\mathbb{Z}[i]$, so that $p = a^2 + b^2$, where $a + bi \equiv 1 \bmod 2 + 2i$, then the Frobenius at the prime $(a + b\sqrt{-1})$ of $\mathbb{Z}[\sqrt{-1}]$ acts on E_n as the complex multiplication by $a + b\sqrt{-1}$.

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- Ieronymou–AS–Zarhin paper: an explicit analysis of the Galois representation on E_n, for n odd.

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One can find two elliptic pencils on X intersecting trivially with these 16 curves, which gives a map $X \to \mathbb{P}^1 \times \mathbb{P}^1$ of degree 4 contracting these curves.

C is the curve $v^2 = (u^2 - 1)(u^2 - \frac{1}{2})$,

$A = C \times C/\tau$, where τ changes the signs of all four coordinates,

K is the Kummer surface attached to A.

K is birationally equivalent to the surface

$$z^2 = (x-1)(x-\tfrac{1}{2})(y-1)(y-\tfrac{1}{2}), \quad t^2 = xy.$$

Explicit equations show that $X \to \mathbb{P}^1 \times \mathbb{P}^1$ factors through the degree 4 map $K \to \mathbb{P}^1 \times \mathbb{P}^1$ given by $(x, y, z, t) \mapsto (x, y)$. This gives a birational map $X \to K$, which must be an isomorphism.

Note K is birational to the double covering of $\mathbb{P}^2$:

$$z^2 = (x-1)(x-\tfrac{1}{2})(t^2-x)(t^2-\tfrac{1}{2}x).$$

Easy to write Azumaya algebras on K:

$$(t^2-x, t^2-\tfrac{1}{2}x), \quad (t^2-x, x-1).$$

Open problem For many diagonal quartics D over $\mathbb{Q}$ one expects a transcendental element of order 2 in $\mathrm{Br}(D)$. How to write it explicitly? Can one use Mizukami's isomorphism?

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