

A FINITENESS THEOREM FOR THE BRAUER GROUP OF K3 SURFACES IN ODD CHARACTERISTIC

ALEXEI N. SKOROBOGATOV AND YURI G. ZARHIN

ABSTRACT. Let p be an odd prime and let k be a field finitely generated over the finite field with p elements. For any K3 surface X over k we prove that the cokernel of the natural map $\mathrm{Br}(k) \rightarrow \mathrm{Br}(X)$ is finite modulo the p -primary torsion subgroup.

1. INTRODUCTION

In our previous paper [19] we proved that the cokernel of the natural map of Brauer groups $\mathrm{Br}(k) \rightarrow \mathrm{Br}(X)$ is finite when X is a K3 surface over a field k of characteristic zero which is finitely generated over its prime subfield. The crucial ingredients of that result are the Tate conjecture for divisors proved in the case of abelian varieties by Faltings [5, 6] and the Kuga–Satake construction as reworked by Deligne [4]. The main goal of this paper is to prove the analogous statement in finite odd characteristic. In this case the crucial ingredients are the Tate conjecture for K3 surfaces and the existence of the Kuga–Satake variety, as recently established by Madapusi Pera [14, Thm. 4.17], as well as the second author’s results on abelian varieties in finite characteristic [24, 25, 26].

The following semisimplicity property for ℓ -adic Galois representations attached to K3 surfaces is obtained in Theorem 5.1 (i).

Theorem 1.1. *Let k be a field finitely generated over \mathbb{F}_p , where $p \neq 2$, and let X be a K3 surface over k . For any prime $\ell \neq p$ the Galois module $\mathrm{H}_{\text{ét}}^2(\bar{X}, \mathbb{Q}_\ell(1))$ is semisimple.*

The proofs of the following main results of this paper can be found in Section 5.

Theorem 1.2. *Let k be a field finitely generated over \mathbb{F}_p , where $p \neq 2$, and let X be a K3 surface over k . Then for all but finitely many ℓ the Galois module $\mathrm{H}_{\text{ét}}^2(\bar{X}, \mu_\ell)$ is semisimple and $\mathrm{H}_{\text{ét}}^2(\bar{X}, \mu_\ell)^{\mathrm{Gal}(k)} = \mathrm{NS}(\bar{X})^{\mathrm{Gal}(k)}/\ell$.*

Using the method of [19] we deduce from Theorems 1.1 and 1.2 the desired finiteness statement:

Theorem 1.3. *Let k be a field finitely generated over \mathbb{F}_p , where $p \neq 2$, and let X be a K3 surface over k . Then the cokernel of the natural map $\mathrm{Br}(k) \rightarrow \mathrm{Br}(X)$ is finite modulo the p -primary torsion subgroup.*

Although our method is similar to that of [19], the case of finite characteristic requires a more extensive use of the results of the second author on the analogue of the Tate conjecture with finite coefficients and the semisimplicity of mod ℓ Galois

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representations attached to abelian varieties when ℓ is a large prime [25, 26]. Indeed, unlike the classical Kuga–Satake–Deligne construction, Madapusi Pera’s work does not provide us with a Clifford algebra over \mathbb{Z} . The absence of integral cohomology in characteristic p makes the treatment of ℓ -adic Clifford algebras a bit more subtle.

2. PRELIMINARIES

We start with defining our notation. For an abelian group B and a positive integer n we write $B/n = B \otimes \mathbb{Z}/n$ and write $B[n]$ for the kernel of the multiplication by n map $[n] : B \rightarrow B$. If ℓ is a prime we denote by $B(\text{non-}\ell)$ the quotient of B by the ℓ -primary torsion subgroup. The ℓ -adic Tate module $T_\ell(B)$ is defined as the projective limit of abelian groups $B[\ell^i]$, where the transition maps $B[\ell^{i+1}] \rightarrow B[\ell^i]$ are given by the multiplication by ℓ . If $B[\ell]$ is finite, then $T_\ell(B)$ is a free \mathbb{Z}_ℓ -module of finite rank. See [19, p. 485] for more details.

Let k be a field with an algebraic closure \bar{k} , and let $\bar{k}_{\text{sep}} \subset \bar{k}$ be the separable closure of k in \bar{k} . Let $\text{Gal}(k) = \text{Gal}(\bar{k}_{\text{sep}}/k)$ be the absolute Galois group of k , and let $\text{Br}(k)$ be the Brauer group of k .

For a geometrically integral, smooth and projective variety X over k we write \bar{X} for the variety $X \times_k \bar{k}_{\text{sep}}$ over \bar{k}_{sep} . We write $\text{Br}(X) = \text{H}_{\text{ét}}^2(X, \mathbb{G}_m)$ for the Brauer–Grothendieck group of X , and $\text{Br}_0(X)$ for the image of the canonical homomorphism $\text{Br}(k) \rightarrow \text{Br}(X)$ induced by the structure map.

Let X be a K3 surface over k . If $\ell \neq \text{char}(k)$, then $\text{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$ is a free \mathbb{Z}_ℓ -module of rank 22, and the natural map

$$\text{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))/\ell \xrightarrow{\sim} \text{H}_{\text{ét}}^2(\bar{X}, \mu_\ell)$$

is an isomorphism of $\text{Gal}(k)$ -modules. By Grothendieck [7, III.8.2] and Tate [22] there is a short exact sequence of $\text{Gal}(k)$ -modules, which are also free \mathbb{Z}_ℓ -modules of finite rank:

$$0 \rightarrow \text{NS}(\bar{X}) \otimes \mathbb{Z}_\ell \rightarrow \text{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1)) \rightarrow T_\ell(\text{Br}(\bar{X})) \rightarrow 0. \quad (1)$$

We identify $\text{NS}(\bar{X}) \otimes \mathbb{Z}_\ell$ with its image in $\text{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$. Then (1) shows that $\text{NS}(\bar{X}) \otimes \mathbb{Z}_\ell$ is a *saturated* \mathbb{Z}_ℓ -submodule of $\text{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$, that is, the quotient is torsion-free. It is well known that there exists a finite Galois field extension k'/k such that the open finite index subgroup $\text{Gal}(k') \subset \text{Gal}(k)$ acts trivially on $\text{NS}(\bar{X})$ and hence on $\text{NS}(\bar{X}) \otimes \mathbb{Z}_\ell$.

We *define* $\text{NS}(X)$ as the Galois invariant subgroup $\text{NS}(X) := \text{NS}(\bar{X})^{\text{Gal}(k)}$. (Note that the image of $\text{Pic}(X)$ in $\text{NS}(X)$ is a subgroup of finite index.) In particular, $\text{NS}(X)$ is a saturated \mathbb{Z} -submodule of $\text{NS}(\bar{X})$. We have

$$\text{NS}(X) \otimes \mathbb{Z}_\ell \subset \text{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))^{\text{Gal}(k)} \subset \text{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1)).$$

Since $\text{NS}(X) \otimes \mathbb{Z}_\ell$ is a saturated \mathbb{Z}_ℓ -submodule of $\text{NS}(\bar{X}) \otimes \mathbb{Z}_\ell$, it is also a saturated \mathbb{Z}_ℓ -submodule of $\text{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$. The intersection pairing

$$\text{NS}(\bar{X}) \times \text{NS}(\bar{X}) \rightarrow \mathbb{Z}, \quad \alpha, \beta \mapsto \alpha \cdot \beta$$

is non-degenerate, see [9, Prop. 3, p. 64] and also [11, Lemma V.3.27]. In particular, the discriminant d_X of this pairing is a non-zero integer. On the other hand, the cup-product pairing

$$e_{X,\ell} : \text{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1)) \times \text{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1)) \rightarrow \mathbb{Z}_\ell$$

is Galois-invariant and a perfect duality. The restriction of $e_{X,\ell}$ to $\text{NS}(\bar{X})$ coincides with the intersection pairing. We define the group of transcendental cycles $T(\bar{X})_\ell$ as the orthogonal complement to $\text{NS}(\bar{X}) \otimes \mathbb{Z}_\ell$ in $H_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$ with respect to $e_{X,\ell}$. It is clear that $T(\bar{X})_\ell$ is a saturated Galois-stable \mathbb{Z}_ℓ -submodule of $H_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$. It is also clear that if ℓ does not divide d_X , then

$$H_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1)) = (\text{NS}(\bar{X}) \otimes \mathbb{Z}_\ell) \oplus T(\bar{X})_\ell.$$

For any ℓ we have

$$(\text{NS}(\bar{X}) \otimes \mathbb{Z}_\ell) \cap T(\bar{X})_\ell = 0,$$

and the direct sum $(\text{NS}(\bar{X}) \otimes \mathbb{Z}_\ell) \oplus T(\bar{X})_\ell$ has finite index in $H_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$. It is clear that $T(\bar{X})_\ell/\ell$ is a $\text{Gal}(k)$ -submodule of $H_{\text{ét}}^2(\bar{X}, \mu_\ell)$.

Let us consider the \mathbb{Q}_ℓ -vector space $H_{\text{ét}}^2(\bar{X}, \mathbb{Q}_\ell(1)) := H_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ with its natural structure of a $\text{Gal}(k)$ -module. We have a natural embedding of Galois modules $\text{NS}(\bar{X}) \otimes \mathbb{Q}_\ell \subset H_{\text{ét}}^2(\bar{X}, \mathbb{Q}_\ell(1))$, from which we obtain

$$\text{NS}(X) \otimes \mathbb{Q}_\ell \subset H_{\text{ét}}^2(\bar{X}, \mathbb{Q}_\ell(1))^{\text{Gal}(k)}.$$

K. Madapusi Pera [14, Thm. 1] proved that if k is finitely generated over \mathbb{F}_p and $p > 2$, then

$$\text{NS}(X) \otimes \mathbb{Q}_\ell = H_{\text{ét}}^2(\bar{X}, \mathbb{Q}_\ell(1))^{\text{Gal}(k)}.$$

This is an important special case of the Tate conjecture on algebraic cycles [20]. Closely related results were previously obtained by D. Maulik [10] and F. Charles [2, Cor. 2]. Since $\text{NS}(X) \otimes \mathbb{Z}_\ell$ is a saturated \mathbb{Z}_ℓ -submodule in $H_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$, the theorem of Madapusi Pera is equivalent to

$$\text{NS}(X) \otimes \mathbb{Z}_\ell = H_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))^{\text{Gal}(k)}.$$

Another restatement of the same result is

$$T(\bar{X})_\ell^{\text{Gal}(k)} = 0.$$

Note that this still holds if k is replaced by any finite separable field extension.

3. ABELIAN VARIETIES

Let A be an abelian variety over a field k . We write $\text{End}(A)$ for the ring of endomorphisms of A . For a positive integer n not divisible by $\text{char}(k)$, we write $A[n]$ for $A(\bar{k})[n]$. Then $A[n]$ is a $\text{Gal}(k)$ -submodule of $A(\bar{k}_{\text{sep}})$.

Let $\ell \neq \text{char}(k)$ be a prime. It is well known [16] that the ℓ -adic Tate module $T_\ell(A) = T_\ell(A(\bar{k}_{\text{sep}}))$ is a free \mathbb{Z}_ℓ -module of rank $2\dim(A)$ equipped with a natural continuous action of the Galois group

$$\rho_{\ell,A} : \text{Gal}(k) \rightarrow \text{Aut}_{\mathbb{Z}_\ell}(T_\ell(A)).$$

We write $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, and view $T_\ell(A)$ as a \mathbb{Z}_ℓ -lattice in $V_\ell(A)$. Then $\rho_{\ell,A}$ gives rise to the ℓ -adic representation

$$\rho_{\ell,A} : \text{Gal}(k) \rightarrow \text{Aut}_{\mathbb{Z}_\ell}(T_\ell(A)) \subset \text{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)).$$

There are canonical isomorphisms of Galois modules $T_\ell(A)/\ell^i \cong A[\ell^i]$ for all $i \geq 1$.

There is a natural embedding of \mathbb{Z} -algebras $\text{End}(A) \hookrightarrow \text{End}(\bar{A})$, whose image is a saturated subgroup [15, Sect. 4, p. 501]. We will identify $\text{End}(A)$ with its image in $\text{End}(\bar{A})$. The natural action of $\text{Gal}(k)$ on $\text{End}(\bar{A})$ is continuous when $\text{End}(\bar{A})$ is given discrete topology, so the image of $\text{Gal}(k)$ in $\text{Aut}(\text{End}(\bar{A}))$ is finite. The ring $\text{End}(A)$ coincides with the Galois-invariant subring $\text{End}(\bar{A})^{\text{Gal}(k)}$, see [18]. In other

words, there is a finite Galois field extension k'/k such that $\text{Gal}(k) \rightarrow \text{Aut}(\text{End}(\bar{A}))$ factors through $\text{Gal}(k) \rightarrow \text{Gal}(k'/k)$ and

$$\text{End}(A) = \text{End}(\bar{A})^{\text{Gal}(k'/k)}.$$

There are natural embeddings of \mathbb{Z}_ℓ -algebras

$$\text{End}(A) \otimes \mathbb{Z}_\ell \subset \text{End}(\bar{A}) \otimes \mathbb{Z}_\ell \subset \text{End}_{\mathbb{Z}_\ell}(T_\ell(A)),$$

of \mathbb{Q}_ℓ -algebras

$$\text{End}(A) \otimes \mathbb{Q}_\ell \subset \text{End}(\bar{A}) \otimes \mathbb{Q}_\ell \subset \text{End}_{\mathbb{Q}_\ell}(V_\ell(A))$$

and of \mathbb{F}_ℓ -algebras

$$\text{End}(A) \otimes \mathbb{F}_\ell \subset \text{End}(\bar{A}) \otimes \mathbb{F}_\ell \subset \text{End}_{\mathbb{F}_\ell}(A[\ell]).$$

The image of $\text{End}(A) \otimes \mathbb{Z}_\ell$ in $\text{End}_{\mathbb{Z}_\ell}(T_\ell(A))$ is obviously contained in the centraliser $\text{End}_{\text{Gal}(k)}(T_\ell(A))$ of $\text{Gal}(k)$ in $\text{End}_{\mathbb{Z}_\ell}(T_\ell(A))$. Similar statements hold if \mathbb{Z}_ℓ is replaced by \mathbb{Q}_ℓ or \mathbb{F}_ℓ .

For each prime $\ell \neq \text{char}(k)$ let us consider the \mathbb{Q}_ℓ -bilinear *trace form*

$$\psi_\ell : \text{End}_{\mathbb{Q}_\ell}(V_\ell(A)) \times \text{End}_{\mathbb{Q}_\ell}(V_\ell(A)) \rightarrow \mathbb{Q}_\ell, \quad \psi_\ell(u, v) = \text{tr}(uv).$$

This form is Galois-invariant, symmetric and non-degenerate. The same formula defines a perfect (unimodular) \mathbb{Z}_ℓ -bilinear form on $\text{End}_{\mathbb{Z}_\ell}(T_\ell(A))$. The restriction of this form to $\text{End}(\bar{A})$ takes values in \mathbb{Z} and does not depend on the choice of ℓ , see [13, Sect. 19, Thm. 4].

Let us choose a polarisation on A and write $u \mapsto u'$ for the corresponding Rosati involution on $\text{End}(\bar{A}) \otimes \mathbb{Q}$, see [13, Sect. 20]. Then $\text{tr}(u'u)$ is a *positive* rational number if $u \neq 0$, see [8, Ch. V, Sect. 3, Thm. 1] or [13, Sect. 21, Thm. 1]. This implies that the $\text{Gal}(k)$ -invariant form

$$\psi_\ell : \text{End}(\bar{A}) \times \text{End}(\bar{A}) \rightarrow \mathbb{Z}$$

is non-degenerate (but not necessarily perfect) and does not depend on ℓ . Since the Rosati involution commutes with the action of $\text{Gal}(k)$, and $\text{End}(A) = \text{End}(\bar{A})^{\text{Gal}(k)}$, the bilinear form

$$\psi_\ell : \text{End}(A) \times \text{End}(A) \rightarrow \mathbb{Z}$$

is also non-degenerate, and therefore its discriminant d_A is a *non-zero* integer that does not depend on ℓ .

The following assertion was proved by one of the authors [23, 24] in $\text{char}(k) > 2$, by Faltings [5, 6] in $\text{char}(k) = 0$ and by Mori [12] in $\text{char}(k) = 2$. (See also [27, 29].)

Theorem 3.1. *Let k be a field finitely generated over its prime subfield and let A be an abelian variety over k . For a prime $\ell \neq \text{char}(k)$ we have*

- (i) $\text{End}(A) \otimes \mathbb{Z}_\ell = \text{End}_{\text{Gal}(k)}(T_\ell(A))$ and $\text{End}(A) \otimes \mathbb{Q}_\ell = \text{End}_{\text{Gal}(k)}(V_\ell(A))$;
- (ii) *the Galois module $V_\ell(A)$ is semisimple.*

Remark 3.2. When k is finite, the assertion (i) of Theorem 3.1 was proved by Tate [21], who conjectured that it holds for an arbitrary finitely generated field. He also proved that the two formulae in (i) are equivalent for any given k, A, ℓ . The semisimplicity of $V_\ell(A)$ for finite k was established earlier by Weil [13].

The following assertion was proved in [25, 26, 29].

Theorem 3.3. *Let k be a field finitely generated over its prime subfield, and let A be an abelian variety over k . Then for all but finitely many primes $\ell \neq \text{char}(k)$ the Galois module $A[\ell]$ is semisimple and $\text{End}(A)/\ell = \text{End}_{\text{Gal}(k)}(A[\ell])$.*

Proposition 3.4. *Let k be a field finitely generated over its prime subfield, and let A be an abelian variety over k . Let P be an infinite set of primes that does not contain $\text{char}(k)$. Suppose that for each $\ell \in P$ we are given a $\text{Gal}(k)$ -submodule $U_\ell \subset \text{End}_{\mathbb{Z}_\ell}(T_\ell(A))$ such that U_ℓ is a saturated \mathbb{Z}_ℓ -submodule of $\text{End}_{\mathbb{Z}_\ell}(T_\ell(A))$ and $U_\ell^{\text{Gal}(k)} = 0$. Then for all but finitely many $\ell \in P$ we have $(U_\ell/\ell)^{\text{Gal}(k)} = 0$.*

Proof. By Theorem 3.1(ii) the Galois module $V_\ell(A)$ is semisimple, and by a theorem of Chevalley [3, p. 88] this implies that $\text{End}_{\mathbb{Q}_\ell}(V_\ell(A))$ is also semisimple. The Galois submodules $U_\ell \otimes \mathbb{Q}_\ell$ and $\text{End}_{\text{Gal}(k)}(V_\ell(A))$ are orthogonal in $\text{End}_{\mathbb{Q}_\ell}(V_\ell(A))$ with respect to the Galois-invariant bilinear form ψ_ℓ . Indeed, otherwise $(U_\ell \otimes \mathbb{Q}_\ell)^{\text{Gal}(k)} \neq 0$ which contradicts the assumption $U_\ell^{\text{Gal}(k)} = 0$. It follows that $\text{End}(A) \otimes \mathbb{Z}_\ell$ and U_ℓ are orthogonal, too.

Now assume that $\ell \in P$ does not divide d_A . Then the \mathbb{Z}_ℓ -bilinear symmetric Galois-invariant form

$$\psi_\ell : \text{End}(A) \otimes \mathbb{Z}_\ell \times \text{End}(A) \otimes \mathbb{Z}_\ell \rightarrow \mathbb{Z}_\ell$$

is perfect. Thus the Galois module $\text{End}_{\mathbb{Z}_\ell}(T_\ell(A))$ splits into a direct sum

$$\text{End}_{\mathbb{Z}_\ell}(T_\ell(A)) = (\text{End}(A) \otimes \mathbb{Z}_\ell) \oplus S_\ell,$$

where S_ℓ is the orthogonal complement to $\text{End}(A) \otimes \mathbb{Z}_\ell$. It is clear that S_ℓ is a Galois-stable saturated \mathbb{Z}_ℓ -submodule of $\text{End}_{\mathbb{Z}_\ell}(T_\ell(A))$. We have natural isomorphisms of Galois modules

$$\text{End}_{\mathbb{Z}_\ell}(T_\ell(A))/\ell = \text{End}_{\mathbb{Z}_\ell/\ell}(T_\ell(A)/\ell) = \text{End}_{\mathbb{F}_\ell}(A[\ell]).$$

Thus reducing modulo ℓ , we obtain a Galois-invariant decomposition

$$\text{End}_{\mathbb{F}_\ell}(A[\ell]) = \text{End}(A)/\ell \oplus S_\ell/\ell.$$

This gives rise to

$$\text{End}_{\text{Gal}(k)}(A[\ell]) = \text{End}(A)/\ell \oplus (S_\ell/\ell)^{\text{Gal}(k)}.$$

Theorem 3.3 implies that for all but finitely many $\ell \in P$ we have $(S_\ell/\ell)^{\text{Gal}(k)} = 0$. Since U_ℓ is orthogonal to $\text{End}(A) \otimes \mathbb{Z}_\ell$, we have $U_\ell \subset S_\ell$. Moreover, U_ℓ is a saturated \mathbb{Z}_ℓ -submodule of S_ℓ , and this implies that the natural map $U_\ell/\ell \rightarrow S_\ell/\ell$ is injective. We conclude that $(U_\ell/\ell)^{\text{Gal}(k)} = 0$ for all but finitely many $\ell \in P$. \square

Recall that if $\text{char}(k)$ does not divide a positive integer n , then we have a canonical isomorphism of Galois modules

$$H_{\text{ét}}^1(\bar{A}, \mathbb{Z}/n) = \text{Hom}(A[n], \mathbb{Z}/n).$$

For a prime $\ell \neq \text{char}(k)$ we obtain canonical isomorphisms of Galois modules

$$H_{\text{ét}}^1(\bar{A}, \mathbb{Z}_\ell) = \text{Hom}_{\mathbb{Z}_\ell}(T_\ell(A), \mathbb{Z}_\ell), \quad H_{\text{ét}}^1(\bar{A}, \mathbb{Q}_\ell) = \text{Hom}_{\mathbb{Q}_\ell}(V_\ell(A), \mathbb{Q}_\ell),$$

$$\text{End}_{\mathbb{Z}_\ell}(H_{\text{ét}}^1(\bar{A}, \mathbb{Z}_\ell)) \cong \text{End}_{\mathbb{Z}_\ell}(T_\ell(A)),$$

where the last identification is an anti-isomorphism of rings. This allows us to restate Proposition 3.4 in the following form.

Corollary 3.5. *Let k be a field finitely generated over its prime subfield, and let A be an abelian variety over k . Let P be an infinite set of primes that does not contain $\text{char}(k)$. Suppose that for each $\ell \in P$ we are given a $\text{Gal}(k)$ -submodule*

$$U_\ell \subset \text{End}_{\mathbb{Z}_\ell}(\mathbf{H}_{\text{ét}}^1(\bar{A}, \mathbb{Z}_\ell))$$

such that U_ℓ is a saturated \mathbb{Z}_ℓ -submodule of $\text{End}_{\mathbb{Z}_\ell}(\mathbf{H}_{\text{ét}}^1(\bar{A}, \mathbb{Z}_\ell))$ and $U_\ell^{\text{Gal}(k)} = 0$. Then for all but finitely many $\ell \in P$ we have $(U_\ell/\ell)^{\text{Gal}(k)} = 0$.

Theorem 3.6. *Let k be a field finitely generated over its prime subfield, and let A be an abelian variety over k . Then*

- (i) *the $\text{Gal}(k)$ -module $\text{End}_{\mathbb{Q}_\ell}(\mathbf{H}_{\text{ét}}^1(\bar{A}, \mathbb{Q}_\ell))$ is semisimple for any prime $\ell \neq \text{char}(k)$;*
- (ii) *the $\text{Gal}(k)$ -module $\text{End}_{\mathbb{F}_\ell}(\mathbf{H}_{\text{ét}}^1(\bar{A}, \mathbb{Z}/\ell))$ is semisimple for all but finitely many primes $\ell \neq \text{char}(k)$.*

Proof. (i) We have a \mathbb{Q}_ℓ -algebra anti-isomorphism

$$\text{End}_{\mathbb{Q}_\ell}(V_\ell(A)) \cong \text{End}_{\mathbb{Q}_\ell}(\mathbf{H}_{\text{ét}}^1(\bar{A}, \mathbb{Q}_\ell)),$$

which is also an isomorphism of Galois modules. By Theorem 3.1 the Galois module $V_\ell(A)$ is semisimple, and then Chevalley's theorem [3, p. 88] implies the semisimplicity of $\text{End}_{\mathbb{Q}_\ell}(V_\ell(A))$.

(ii) We have an \mathbb{F}_ℓ -algebra anti-isomorphism

$$\text{End}_{\mathbb{F}_\ell}(\mathbf{H}_{\text{ét}}^1(\bar{A}, \mathbb{Z}/\ell)) \cong \text{End}_{\mathbb{F}_\ell}(A[\ell]),$$

which is also an isomorphism of Galois modules. By Theorem 3.3 the Galois module $A[\ell]$ is semisimple for all but finitely many ℓ . By a theorem of Serre [17, Cor. 1], if ℓ is greater than $2 \dim_{\mathbb{F}_\ell}(A[\ell]) - 2 = 4 \dim(A) - 2$, then the semisimplicity of $A[\ell]$ implies the semisimplicity of the Galois module $\text{End}_{\mathbb{F}_\ell}(A[\ell])$. This finishes the proof of (ii). \square

4. CLIFFORD ALGEBRAS AND KUGA-SATAKE CONSTRUCTION

Let Λ be a principal ideal domain and let E be a free Λ -module of finite rank with a non-degenerate quadratic form $q : E \rightarrow \Lambda$. Let $C(E, q)$ be the Clifford algebra of (E, q) . Let $\varrho : E \rightarrow C(E, q)$ be the canonical Λ -linear homomorphism satisfying

$$\varrho(x)^2 = q(x) \in \Lambda \subset C(E, q).$$

The map ϱ is injective by [1, Sect. 19.3, Cor. 2 to Thm. 1], so that $\varrho(E) \cong E$. It follows from [1, Sect. 19.3, Thm. 1] that $C(E, q)$ is a free Λ -module of finite rank, and the Λ -submodule $\varrho(E)$ is a direct summand of the Λ -module $C(E, q)$; in particular, it is a *saturated* submodule. Let

$$\text{mult}_L : C(E, q) \rightarrow \text{End}_\Lambda(C(E, q))$$

be the homomorphism of Λ -algebras that sends $a \in C(E, q)$ to the endomorphism $x \mapsto ax$. Since $1 \in C(E, q)$, the homomorphism mult_L is injective. In particular, the Λ -algebras $C(E, q)$ and $\text{mult}_L(C(E, q))$ are isomorphic.

We claim that $\text{mult}_L(\varrho(E))$ is a saturated Λ -submodule of $\text{End}_\Lambda(C(E, q))$. Taking into account that $\varrho(E)$ is saturated in $C(E, q)$, it is enough to show that $\text{mult}_L(C(E, q))$ is saturated in $\text{End}_\Lambda(C(E, q))$. Indeed, if $a \in C(E, q)$ is such that $\text{mult}_L(a)$ is a divisible element in $\text{End}_\Lambda(C(E, q))$, then $\text{mult}_L(a)(x)$ is divisible in $C(E, q)$ for all $x \in C(E, q)$. In particular, if we put $x = 1$, then we see that $a = \text{mult}_L(a)(1)$ is divisible in $C(E, q)$. This proves our claim.

Now let $\Lambda = \mathbb{Z}_\ell$ and let E_ℓ be a free \mathbb{Z}_ℓ -module of finite rank with a non-degenerate quadratic form $q_\ell : E_\ell \rightarrow \mathbb{Z}_\ell$. Let $\text{Aut}(C(E_\ell, q_\ell))$ be the group of automorphisms of the \mathbb{Z}_ℓ -algebra $C(E_\ell, q_\ell)$. If there is a continuous ℓ -adic representation $\rho_\ell : \text{Gal}(k) \rightarrow \text{Aut}_{\mathbb{Z}_\ell}(E_\ell)$ that preserves q_ℓ , then, by the universal property of Clifford algebras, there is a representation

$$\rho_{\ell, \text{Cliff}} : \text{Gal}(k) \rightarrow \text{Aut}(C(E_\ell, q_\ell))$$

such that for any $\sigma \in \text{Gal}(k)$ and any $x \in E_\ell$ we have $\rho_{\ell, \text{Cliff}}(\sigma)(\rho(x)) = \rho(\rho_\ell(\sigma)(x))$. Using [1, Sect. 19.3, Thm. 1] one easily checks that $\rho_{\ell, \text{Cliff}}$ is continuous.

Proposition 4.1. *Let k be a field finitely generated over its prime subfield. Let P be an infinite set of primes not containing $\text{char}(k)$. Assume that for each $\ell \in P$ we are given a free \mathbb{Z}_ℓ -module E_ℓ with a non-degenerate quadratic form $q_\ell : E_\ell \rightarrow \mathbb{Z}_\ell$ and a continuous ℓ -adic representation $\rho_\ell : \text{Gal}(k) \rightarrow \text{Aut}_{\mathbb{Z}_\ell}(E_\ell)$ preserving q_ℓ .*

Suppose that there is an abelian variety A over a finite separable field extension k' of k , and for each prime $\ell \in P$ there is an isomorphism of \mathbb{Z}_ℓ -modules $\iota_\ell : C(E_\ell, q_\ell) \xrightarrow{\sim} H_{\text{ét}}^1(\bar{A}, \mathbb{Z}_\ell)$ such that the composition of injective maps of \mathbb{Z}_ℓ -modules

$$E_\ell \xrightarrow{\varrho} C(E_\ell, q_\ell) \xrightarrow{\iota_\ell} H_{\text{ét}}^1(\bar{A}, \mathbb{Z}_\ell) \xrightarrow{\text{mult}_L} \text{End}_{\mathbb{Z}_\ell}(H_{\text{ét}}^1(\bar{A}, \mathbb{Z}_\ell))$$

is a homomorphism of $\text{Gal}(k')$ -modules.

Then the $\text{Gal}(k)$ -module $E_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is semisimple for each $\ell \in P$, and the $\text{Gal}(k)$ -module E_ℓ/ℓ is semisimple for all but finitely many $\ell \in P$.

Proof. Replacing k' by its normal closure over k , we can assume that k' is a finite Galois extension of k , so that $\text{Gal}(k') \subset \text{Gal}(k)$ is an (open) normal subgroup of index $[k' : k]$. In the beginning of this section we have seen that $\text{mult}_L(\rho(E_\ell))$ is a saturated \mathbb{Z}_ℓ -submodule of $\text{End}_{\mathbb{Z}_\ell}(C(E_\ell, q_\ell))$. Since ι_ℓ is an isomorphism, $\text{mult}_L(\iota_\ell(\rho(E_\ell)))$ is a saturated \mathbb{Z}_ℓ -submodule of $\text{End}_{\mathbb{Z}_\ell}(H_{\text{ét}}^1(\bar{A}, \mathbb{Z}_\ell))$ isomorphic to E_ℓ as a $\text{Gal}(k')$ -module. This immediately implies that the $\text{Gal}(k')$ -module E_ℓ/ℓ is isomorphic to a submodule of

$$\text{End}_{\mathbb{Z}_\ell}(H_{\text{ét}}^1(\bar{A}, \mathbb{Z}_\ell))/\ell = \text{End}_{\mathbb{F}_\ell}(H_{\text{ét}}^1(\bar{A}, \mathbb{F}_\ell)).$$

The \mathbb{Q}_ℓ -vector space $E_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a $\text{Gal}(k)$ -module. Considered as a $\text{Gal}(k')$ -module, it is isomorphic to a submodule of

$$\text{End}_{\mathbb{Z}_\ell}(H_{\text{ét}}^1(\bar{A}, \mathbb{Z}_\ell)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = \text{End}_{\mathbb{Q}_\ell}(H_{\text{ét}}^1(\bar{A}, \mathbb{Q}_\ell)).$$

By applying Theorem 3.6 to the abelian variety A over k' we obtain that the $\text{Gal}(k')$ -module $E_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is semisimple for each $\ell \in P$, and that the $\text{Gal}(k')$ -module E_ℓ/ℓ is semisimple for all but finitely many $\ell \in P$. The semisimplicity of $E_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ as a $\text{Gal}(k)$ -module follows from its semisimplicity as a $\text{Gal}(k')$ -module by [17, Lemma 5 (b), p. 523]. By the same lemma, if we further exclude the finitely many primes dividing $[k' : k]$, the semisimplicity of the $\text{Gal}(k)$ -module E_ℓ/ℓ follows from the semisimplicity of the $\text{Gal}(k')$ -module E_ℓ/ℓ . \square

We finish this section with the following complement to Proposition 4.1.

Proposition 4.2. *In the assumptions of Proposition 4.1 suppose that for each prime $\ell \in P$ we have a $\text{Gal}(k')$ -submodule U_ℓ of E_ℓ such that U_ℓ is saturated in E_ℓ and $U_\ell^{\text{Gal}(k')} = 0$. Then $(U_\ell/\ell)^{\text{Gal}(k')} = 0$ for all but finitely many $\ell \in P$.*

Proof. Since U_ℓ is saturated in E_ℓ and $\text{mult}_L(\iota_\ell(\varrho(E_\ell))) \cong E_\ell$ is saturated in $\text{End}_{\mathbb{Z}_\ell}(\mathbb{H}_{\text{ét}}^1(A, \mathbb{Z}_\ell))$, we see that $\text{mult}_L(\iota_\ell(\varrho(U_\ell)))$ is a saturated \mathbb{Z}_ℓ -submodule of $\text{End}_{\mathbb{Z}_\ell}(\mathbb{H}_{\text{ét}}^1(A, \mathbb{Z}_\ell))$. It is $\text{Gal}(k')$ -stable and isomorphic to U_ℓ as a Galois module. The proposition now follows from Corollary 3.5. \square

5. K3 SURFACES

Theorem 5.1. *Let k be a field of odd characteristic p that is finitely generated over \mathbb{F}_p . Let X be a K3 surface over k . Then*

- (i) *for any prime $\ell \neq p$ the $\text{Gal}(k)$ -modules $\mathbb{H}_{\text{ét}}^2(\bar{X}, \mathbb{Q}_\ell(1))$ and $T(\bar{X})_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ are semisimple;*
- (ii) *for all but finitely many primes ℓ the $\text{Gal}(k)$ -modules $\mathbb{H}_{\text{ét}}^2(\bar{X}, \mu_\ell)$ and $T(\bar{X})_\ell/\ell$ are semisimple;*
- (iii) *for all but finitely many primes ℓ we have $(T(\bar{X})_\ell/\ell)^{\text{Gal}(k)} = 0$.*

Proof. Choose a polarisation

$$\xi \in \text{Pic}(X) \subset \text{NS}(\bar{X})^{\text{Gal}(k)} \subset \mathbb{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))^{\text{Gal}(k)}.$$

By the adjunction formula the degree of ξ is even, so that $(\xi^2) = e_{X,\ell}(\xi, \xi) = 2d$, where d is a positive integer.

Let $\text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$ be the orthogonal complement to ξ in $\mathbb{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$ with respect to $e_{X,\ell}$. Clearly, $\text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$ is a Galois-stable saturated \mathbb{Z}_ℓ -submodule of $\mathbb{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$. We have

$$\mathbb{Z}_\ell \xi \cap \text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1)) = 0$$

and the direct sum $\mathbb{Z}_\ell \xi \oplus \text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$ is a subgroup of index $2d$ in $\mathbb{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$. Furthermore, if ℓ does not divide $2d$, then

$$\mathbb{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1)) = \mathbb{Z}_\ell \xi \oplus \text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1)) \quad (2)$$

and the restriction

$$e_{X,\ell} : \text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1)) \times \text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1)) \rightarrow \mathbb{Z}_\ell$$

is a perfect Galois-invariant pairing.

By a theorem of K. Madapusi Pera [14, Thm. 4.17] the family of ℓ -adic representations $E_\ell := \text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$ preserving the quadratic form $q_\ell(x) = e_{X,\ell}(x, x)$, where ℓ is a prime not equal to p , satisfies the assumption of Proposition 4.1. It follows that the $\text{Gal}(k)$ -module

$$\text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Q}_\ell(1)) = \text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is semisimple for each $\ell \neq p$. This implies that the $\text{Gal}(k)$ -module

$$\mathbb{H}_{\text{ét}}^2(\bar{X}, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell \xi \oplus \text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Q}_\ell(1))$$

is semisimple, and hence so is its $\text{Gal}(k)$ -submodule $T(\bar{X})_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, so (i) is proved.

Suppose that $(\ell, 2d) = 1$. Let $\bar{\xi}$ be the image of ξ in

$$\mathbb{H}_{\text{ét}}^2(\bar{X}, \mu_\ell(1)) = \mathbb{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))/\ell.$$

From (2) we obtain a direct sum decomposition of $\text{Gal}(k)$ -modules

$$\mathbb{H}_{\text{ét}}^2(\bar{X}, \mu_\ell(1)) = \mathbb{F}_\ell \bar{\xi} \oplus \text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))/\ell,$$

where $\bar{\xi}$ is $\text{Gal}(k)$ -invariant. By Proposition 4.1 the $\text{Gal}(k)$ -module $\text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))/\ell$ is semisimple for all but finitely many primes $\ell \neq p$. Hence $\mathbb{H}_{\text{ét}}^2(\bar{X}, \mu_\ell(1))$ is semisimple too. Since $T(\bar{X})_\ell$ is a Galois-stable saturated \mathbb{Z}_ℓ -submodule of $\text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))$,

the $\text{Gal}(k)$ -module $T(\bar{X})_\ell/\ell$ is isomorphic to a submodule of $\text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1))/\ell$. This proves (ii).

Part (iii) follows from Proposition 4.2 applied to the family of $\text{Gal}(k)$ -submodules

$$U_\ell := T(\bar{X})_\ell \subset E_\ell = \text{PH}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell),$$

for all $\ell \neq p$. Indeed, by the Tate conjecture for K3 surfaces over finitely generated fields of odd characteristic, proved by Madapusi Pera [14, Thm. 1] (see also [10] and [2]), we have $T(\bar{X})_\ell^{\text{Gal}(k')} = 0$ for any finite separable field extension k' of k . \square

Proof of Theorem 1.2. The first statement is already proved in Theorem 5.1 (ii). As recalled in the introduction, for all but finitely many primes ℓ we have a direct sum decomposition of $\text{Gal}(k)$ -modules

$$\text{H}_{\text{ét}}^2(\bar{X}, \mathbb{Z}_\ell(1)) = (\text{NS}(X) \otimes \mathbb{Z}_\ell) \oplus T(\bar{X})_\ell.$$

For these primes ℓ the Galois module $\text{H}_{\text{ét}}^2(\bar{X}, \mu_\ell)$ is isomorphic to the direct sum of $\text{NS}(X)/\ell$ and $T(\bar{X})_\ell/\ell$. By Theorem 5.1 (iii), for all but finitely many ℓ we have $(T(\bar{X})_\ell/\ell)^{\text{Gal}(k)} = 0$. This implies

$$\text{H}_{\text{ét}}^2(\bar{X}, \mu_\ell)^{\text{Gal}(k)} = (\text{NS}(X)/\ell)^{\text{Gal}(k)}.$$

The cokernel of the natural map $\text{NS}(\bar{X})^{\text{Gal}(k)} \rightarrow (\text{NS}(X)/\ell)^{\text{Gal}(k)}$ is a subgroup of the Galois cohomology group $\text{H}^1(k, \text{NS}(\bar{X}))$. This group is finite, since $\text{NS}(\bar{X})$ is a free abelian group of finite rank. Removing the primes that divide the order of $\text{H}^1(k, \text{NS}(\bar{X}))$ we see that the natural map

$$\text{NS}(X) = \text{NS}(\bar{X})^{\text{Gal}(k)} \longrightarrow \text{H}_{\text{ét}}^2(\bar{X}, \mu_\ell)^{\text{Gal}(k)}$$

is surjective for all but finitely many primes ℓ . \square

Proof of Theorem 1.3. The Hochschild–Serre spectral sequence

$$\text{H}^p(k, \text{H}^q(\bar{X}, \mathbb{G}_m)) \Rightarrow \text{H}^{p+q}(X, \mathbb{G}_m)$$

gives rise to the well known filtration $\text{Br}_0(X) \subset \text{Br}_1(X) \subset \text{Br}(X)$, where $\text{Br}_0(X)$ is the image of the natural map $\text{Br}(k) \rightarrow \text{Br}(X)$ and $\text{Br}_1(X)$ is the kernel of the natural map $\text{Br}(X) \rightarrow \text{Br}(\bar{X})$. The spectral sequence shows that $\text{Br}_1(X)/\text{Br}_0(X)$ is contained in the finite group $\text{H}^1(k, \text{NS}(\bar{X}))$, and $\text{Br}(X)/\text{Br}_1(X)$ is contained in $\text{Br}(\bar{X})^{\text{Gal}(k)}$. Hence it is enough to prove that $\text{Br}(\bar{X})^{\text{Gal}(k)}$ is finite.

For a prime $\ell \neq p$ the Kummer exact sequence gives rise to the following exact sequence of $\text{Gal}(k)$ -modules [19, (5), p. 486]:

$$\begin{aligned} 0 \rightarrow (\text{NS}(\bar{X})/\ell)^{\text{Gal}(k)} \rightarrow \text{H}^2(\bar{X}, \mu_\ell)^{\text{Gal}(k)} &\rightarrow \text{Br}(\bar{X})[\ell]^{\text{Gal}(k)} \rightarrow \\ \text{H}^1(k, \text{NS}(\bar{X})/\ell) \rightarrow \text{H}^1(k, \text{H}^2(\bar{X}, \mu_\ell)). \end{aligned}$$

In the proof of Theorem 1.2 we have seen that for all but finitely many primes ℓ the Galois module $\text{NS}(\bar{X})/\ell$ is a direct summand of $\text{H}^2(\bar{X}, \mu_\ell)$, and the second arrow in the above exact sequence is an isomorphism. It follows that $\text{Br}(\bar{X})[\ell]^{\text{Gal}(k)} = 0$ for all but finitely many ℓ .

To finish the proof we note that for any prime $\ell \neq p$ the ℓ -primary subgroup $\text{Br}(\bar{X})\{\ell\}^{\text{Gal}(k)}$ is finite. This follows from [19, Prop. 2.5 (c)] in view of the semisimplicity of $\text{H}^2(\bar{X}, \mathbb{Q}_\ell(1))$, proved in Theorem 5.1 (i), and the Tate conjecture

$$\text{NS}(X) \otimes \mathbb{Q}_\ell = \text{H}_{\text{ét}}^2(\bar{X}, \mathbb{Q}_\ell(1))^{\text{Gal}(k)}$$

proved by Madapusi Pera [14, Thm. 1] for any field k of odd characteristic p finitely generated over \mathbb{F}_p . \square

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INSTITUTE FOR THE INFORMATION TRANSMISSION PROBLEMS, RUSSIAN ACADEMY OF SCIENCES,
19 BOLSHOI KARETNYI, MOSCOW, 127994 RUSSIA
E-mail address: a.skorobogatov@imperial.ac.uk

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA
16802, USA

DEPARTMENT OF MATHEMATICS, THE WEIZMANN INSTITUTE OF SCIENCE, POB 26, REHOVOT
7610001, ISRAEL

INSTITUTE OF MATHEMATICAL PROBLEMS OF BIOLOGY, RUSSIAN ACADEMY OF SCIENCES, PUSHCHINO,
MOSCOW REGION, RUSSIA
E-mail address: zarhin@math.psu.edu