Brauer group in arithmetic geometry

with special reference to K3 surfaces and abelian varieties

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Contents

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Introduction

These lectures are a sequel to the book [CTS21]. Our aim was to develop some of the stories that were only touched upon or not even mentioned in this book, with emphasis of the applications of the Brauer group in arithmetic geometry.

Over a field of characteristic zero the geometric Brauer group of a smooth proper variety with $b_2 = \rho$ is finite[1]. Varieties with $b_2 > \rho$, for example abelian varieties and K3 surfaces, exhibit a new phenomenon: they have infinite geometric Brauer group whose Tate module carries a Galois representation. Various finiteness properties of the Brauer group can be related to deep conjectures about these Galois representations.

The philosophy of these lectures is based on the assumption that K3 surfaces is a convenient vantage point to explore the arithmetic of the Brauer group and the Brauer–Manin obstruction beyond the realm of geometrically rational or rationally connected varieties. What makes K3 surfaces more tractable is a classical construction of Kuga–Satake that associates to a complex K3 surface an abelian variety of large dimension. Geometry of K3 surfaces was intensively studied since the proof of the global Torelli theorem by Pyatetskii-Shapiro and Shafarevich in 1971: a complex K3 surface is determined by the periods of the unique holomorphic 2-form, leading to an interpretation of the moduli spaces of polarised K3 surfaces in terms of Shimura varieties. Relations to abelian varieties and Shimura varieties make K3 surfaces a natural 'testing ground' for fundamental conjectures in arithmetic geometry. Indeed, Deligne proved Weil conjectures for K3 surfaces in 1972 vis the Kuga–Satake construction before he proved the Weil conjectures in the general case.

We begin with a tour of Tate and Mumford–Tate conjectures, which are known for K3 surfaces, with proofs crucially based on the Kuga–Satake construction, see Section 1. We explain what these conjectures mean for the finiteness properties of the Brauer group. In Section 1.4 we discuss various uniformity conjectures for abelian varieties and K3 surfaces, and links among them. These conjectures assert the boundedness of certain integer invariants of K3 surfaces and abelian varieties defined over a number field of bounded degree (and bounded dimension in the case of abelian varieties). In Section 2 we discuss the problem of explicit calculation of the Brauer group, which requires understanding differentials in Leray spectral sequences. Section 3 is devoted to explicit determination of the Brauer–Manin set. For this one needs to know the behaviour of the 'evaluation map' attached to a Brauer class and a place of the ground number field. One natural question here is when such a map can be non-constant. We explain known results in this direction. Finally, in Section 4 we discuss the existence of rational points in the Brauer–Manin set in the best understood particular case, namely, that of Kummer varieties, using a method of Swinnerton-Dyer.

[1] This fails in characteristic $p > 0$, e.g., for supersingular K3 surfaces, see Section 1.3.

Notation

For an abelian group G we denote by $G[n]$ the subgroup $\{x \in G | nx = 0\}$. We write G_{odd} for the union of all subgroups $G[n]$ where n is odd. If ℓ is a prime, then the ℓ-primary torsion subgroup $G\{\ell\}$ is the union of $G[\ell^n]$ for $n \geq 1$.

The ℓ-adic Tate module is defined as

$$T_\ell(G) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, G) = \varprojlim_n G[\ell^n],$$

where the transition maps $G[\ell^{n+1}] \to G[\ell^n]$ are multiplications by ℓ. It is easy to check that $T_\ell(G)$ is a torsion-free $\mathbb{Z}_\ell$-module. There are natural injective maps $T_\ell(G)/\ell^n \hookrightarrow G[\ell^n]$. If the group $G[\ell]$ is finite, then the $\mathbb{Z}_\ell$-module $T_\ell(G)$ is finitely generated. By Nakayama's lemma we have $T_\ell(A) \simeq \mathbb{Z}_\ell^r$ where $r \leq \dim_{\mathbb{F}_\ell}(G[\ell])$. If, moreover, G is an ℓ-primary torsion abelian group, then $T_\ell(G) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is the divisible subgroup G_{div} of A. Define

$$V_\ell(G) = T_\ell(G) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

This is a $\mathbb{Q}_\ell$-vector space.

For a field k we denote by $\bar{k}$ an algebraic closure of k, and by k_{s} the separable closure of k in $\bar{k}$. Let $\Gamma_k = \mathrm{Gal}(k_{\mathrm{s}}/k)$. We write $X^{\mathrm{s}} = X_{k_{\mathrm{s}}} = X \times_k k_{\mathrm{s}}$.

A field is called *finitely generated* if it is finitely generated over its prime subfield.

For a scheme X we denote by $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ the (cohomological) Brauer group of X. When X is a variety over a field k, the group $\mathrm{Br}(X^{\mathrm{s}})$ is called the *geometric* Brauer group of X. We use the standard notation

$$\mathrm{Br}_0(X) = \mathrm{Im}[\mathrm{Br}(k) \to \mathrm{Br}(X)], \qquad \mathrm{Br}_1(X) = \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(X^{\mathrm{s}})^\Gamma].$$

$\mathrm{Br}_1(X)$ is called the *algebraic* Brauer group. Following D'Addezio, we introduce the following notation: for a field extension K/k we denote by $\mathrm{Br}(X_K)^k$ the image of the natural map $\mathrm{Br}(X_k) \to \mathrm{Br}(X_K)$. In particular, $\mathrm{Br}(X^{\mathrm{s}})^k$, that is, the image of $\mathrm{Br}(X) \to \mathrm{Br}(X^{\mathrm{s}})^\Gamma$, is called the *transcendental* Brauer group of X.

1 Conjectures of Tate and Mumford–Tate

References: Moonen's survey paper [Moo17].

1.1 Tate conjecture for divisors

The Tate conjecture is stated for varieties over finitely generated fields of arbitrary characteristic.

Tate conjecture for divisors. *The equivalent properties in the following theorem hold when the ground field k is finitely generated.*

Theorem 1.1 *Let X be a smooth, projective, geometrically integral variety over a field k. Let $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$. Let $\ell \neq \mathrm{char}(k)$ be a prime. The following conditions are equivalent.*

(i) *The injective map* $c_1\colon (\mathrm{NS}\,(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell)^\Gamma \to \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))^\Gamma$ *is an isomorphism.*

(ii) *The injective map* $c_1\colon (\mathrm{NS}\,(X^{\mathrm{s}}) \otimes \mathbb{Q}_\ell)^\Gamma \to \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell(1))^\Gamma$ *is an isomorphism.*

(iii) $(T_\ell(\mathrm{Br}(X^{\mathrm{s}})))^\Gamma = 0$.

(iv) $(V_\ell(\mathrm{Br}(X^{\mathrm{s}})))^\Gamma = 0$.

(v) $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma$ *is finite.*

Proof. The Kummer exact sequence gives rise to an exact sequence of finitely generated $\mathbb{Z}_\ell$-modules

$$0 \longrightarrow \mathrm{NS}\,(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) \longrightarrow T_\ell(\mathrm{Br}(X^{\mathrm{s}})) \longrightarrow 0,$$

with a continuous action of Γ. When tensored with $\mathbb{Q}_\ell$, it gives the exact sequence

$$0 \longrightarrow \mathrm{NS}\,(X^{\mathrm{s}}) \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell(1)) \longrightarrow V_\ell(\mathrm{Br}(X^{\mathrm{s}})) \longrightarrow 0. \tag{1}$$

By [CTS21, Thm. 5.3.1 (ii)] this sequence is split as a sequence of Γ-modules. Thus (ii) is equivalent to (iv). The group $T_\ell(\mathrm{Br}(X^{\mathrm{s}}))$ is torsion-free. Thus (iii) is equivalent to (iv). It is clear that (iii) implies (i). That (i) implies (ii) follows from the simple observation that for any finitely generated $\mathbb{Z}_\ell$-module M with an action of Γ, the map $M^\Gamma \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to (M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)^\Gamma$ is surjective.

For any abelian group A with an action of a group Γ, one has a natural isomorphism $T_\ell(A)^\Gamma \cong T_\ell(A^\Gamma)$. The group $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}$ is of cofinite type [CTS21, Prop. 5.2.9], hence so is $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma$, that is, $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^m \oplus B$, where B is a finite abelian group. Then $T_\ell(\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma) \simeq \mathbb{Z}_\ell^m$. It follows that (iii) is equivalent to (v), because both statements are equivalent to $m = 0$. □

Remark 1.2 Let k be a finitely generated field of characteristic p. For a smooth, projective, and geometrically integral variety X over k, finiteness of $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^{\Gamma}$ for one prime $\ell \neq p$ implies finiteness of the prime-to-p subgroup of $\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$. This is due to Cadoret–Hui–Tamagawa[CHT, Cor. 1.4], see also Yanshuai Qin's paper [Qin, Thm. 1.2].

Theorem 1.3 (i) *Tate conjecture for divisors holds for abelian varieties.*
(ii) *Tate conjecture for divisors holds for K3 surfaces.*

Proof. (i) Over a finite field, this was proved by Tate; over a field finitely generated over the prime field, it was proved by Zarhin in characteristic $p > 2$ [Zar75, Zar76], by Faltings in characteristic zero [Fal83, Fal86], and by Mori in characteristic 2, see [Mor85].

(ii) In general, the case of K3 surfaces is reduced to the case of abelian varieties via the Kuga–Satake construction which associates to a K3 surface X an abelian variety A of large dimension. Deligne observed that A can be defined over a finite extension of the ground field. Over a field of characteristic zero, the Tate conjecture for K3 surfaces is due to Tankeev and, independently, Y. André, see [CTS21, Thm. 16.7.1]. More recently, Tate conjecture was proved for K3 surfaces in finite characteristic in growing generality by Nygaard, Ogus, F. Charles, Maulik, and finally in full generality by Madapusi Pera for $p > 2$ and by Madapusi Pera and Wansu Kim for $p = 2$. □

Corollary 1.4 *Let X be an abelian variety or a K3 surface over a field k that is finitely generated over its prime subfield. Then* $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^{\Gamma}$ *is finite for all primes ℓ not equal to* $\mathrm{char}(k)$.

Proof. This follows from Theorems 1.1 and 1.3. □

1.2 Mumford–Tate conjecture

Unlike Tate conjecture, the conjecture of Mumford–Tate is stated for varieties over finitely generated fields of *characteristic zero*.

The original Mumford–Tate conjecture was stated for an abelian variety A in terms of the natural Hodge structure on the first homology group $H_1 = H_1(A_{\mathbb{C}}, \mathbb{Z})$. The free $\mathbb{Z}_\ell$-module $H_1 \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is identified with the ℓ-adic Tate module $T_\ell(A_{\bar{k}})$ and so carries a natural Galois representation. The idea is that the image of the Galois group is roughly as large as is allowed by the symmetries of the cohomology of A, as reflected in the Hodge structure.

Let X be a smooth, projective and geometrically integral variety over a field k that is finitely generated over $\mathbb{Q}$. Let $i \geq 1$ and j be integers. We choose an embedding $k \hookrightarrow \mathbb{C}$ and define H as the quotient of $\mathrm{H}^i(X_{\mathbb{C}}, \mathbb{Z}(j))$ by the torsion subgroup. We write $H_{\mathbb{Q}} = H \otimes_{\mathbb{Z}} \mathbb{Q}$, $H_{\mathbb{R}} = H \otimes_{\mathbb{Z}} \mathbb{R}$, $H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C}$, and for a prime ℓ write $H_\ell = H \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. Let $\mathbf{GL}(H)$ be the group $\mathbb{Z}$-scheme such that for any commutative ring R we have $\mathbf{GL}(H)(R) = \mathrm{GL}(H \otimes_{\mathbb{Z}} R)$. The generic fibre $\mathbf{GL}(H)_{\mathbb{Q}}$ is the algebraic group $\mathbf{GL}(H_{\mathbb{Q}})$ over $\mathbb{Q}$.

Let us first discuss the Galois side. The comparison theorems between Betti and étale cohomology give an isomorphism between H_ℓ and the quotient of $\mathrm{H}^i_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell(j))$ by the torsion subgroup. Let $\rho_\ell : \Gamma_k \to \mathbf{GL}(H)(\mathbb{Z}_\ell)$ be the resulting continuous representation. Here the twist $\mathbb{Z}_\ell(j)$ is understood as the Tate twist; it changes the Galois representation by tensoring it with the j-th power of the cyclotomic character. Since Γ_k is compact and ρ_ℓ is continuous, $\rho_\ell(\Gamma_k)$ is a compact, hence closed, subgroup of the ℓ-adic Lie group $\mathrm{GL}(H_{\mathbb{Q}_\ell})$. It is therefore a Lie subgroup. Let $\mathfrak{g}_\ell$ be its Lie algebra. (This is a $\mathbb{Q}_\ell$-vector space.)

Let $G_{k,\ell}$ be the algebraic group over $\mathbb{Z}_\ell$ defined as the Zariski closure of $\rho_\ell(\Gamma_k)$ in $\mathbf{GL}(H)_{\mathbb{Z}_\ell}$. The generic fibre $G_{K,\ell,\mathbb{Q}_\ell} = G_{k,\ell} \times_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is the Zariski closure of the image of the $\mathbb{Q}_\ell$-representation $\Gamma_k \to \mathbf{GL}(H)_{\mathbb{Q}_\ell}$. The algebraic $\mathbb{Q}_\ell$-group $G_{K,\ell,\mathbb{Q}_\ell}$ is called the ℓ-adic *monodromy group*. By theorems of Bogomolov [Bog80], Serre [Ser81] and Henniart [Hen82], the group $\rho_\ell(\Gamma_k)$ is an open subgroup of $G_{K,\ell,\mathbb{Q}_\ell}(\mathbb{Q}_\ell)$ with respect to the ℓ-adic topology. Hence $\rho_\ell(\Gamma_k)$ is open in $G_{k,\ell}(\mathbb{Z}_\ell) = G_{K,\ell,\mathbb{Q}_\ell}(\mathbb{Q}_\ell) \cap \mathrm{GL}(H_\ell)$, but since the latter group is compact, it has finite index in it. (For an abelian variety, this index is know to be bounded, see Remark 1.3 of Zywina's paper.)

The ℓ-adic monodromy group $G_{K,\ell,\mathbb{Q}_\ell}$ is not necessarily connected (for example, for $i = 2$ and $j = 1$, because the action of Γ_k on the Néron–Severi group $\mathrm{NS}\,(X_{\bar{k}})$ is via a finite quotient). By a result of Serre there exists a finite field extension k^{conn} of k such that for every field $K \subset \bar{k}$ containing k^{conn} and every prime ℓ the group $G_{K,\ell,\mathbb{Q}_\ell}$ is connected, see [LP97]. We have $G_{k^{\mathrm{conn}},\ell,\mathbb{Q}_\ell} = G^\circ_{k,\ell,\mathbb{Q}_\ell}$, where $G^\circ_{k,\ell,\mathbb{Q}_\ell}$ is the connected component of identity in $G_{k,\ell,\mathbb{Q}_\ell}$.

Now we discuss the Hodge side. The free $\mathbb{Z}$-module H of finite rank carries a natural Hodge structure. Let us recall what this means.

The $\mathbb{R}$-torus $\mathbb{S} = R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ is called the *Deligne torus*. A $\mathbb{Q}$-Hodge structure on $H_{\mathbb{Q}}$ of pure weight n can be described as a representation $h\colon \mathbb{S} \to \mathbf{GL}(H)_{\mathbb{R}}$ whose restriction to $\mathbb{G}_{m,\mathbb{R}} \subset \mathbb{S}$ is $x \mapsto x^{-n}$. Then $H_{\mathbb{C}}$ is a direct sum of the subspaces $(H_{\mathbb{C}})^{p,q}$, $p+q = n$, which are the eigenspaces of $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ acting by $(z_1, z_2) \mapsto z_1^{-p} z_2^{-q}$. We have $\overline{(H_{\mathbb{C}})^{p,q}} = (H_{\mathbb{C}})^{q,p}$. In the context of Hodge structures, $\mathbb{Z}(j)$ is understood as the Hodge structure on $(2\pi i)^j\mathbb{Z}$ of pure weight $-j$; twisting by j means tensoring with $\mathbb{Z}(j)$. For example, $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ carries a natural Hodge structure of weight 0.

The *Mumford–Tate group* $\mathbf{G}_{\mathbb{Q}} \subset \mathbf{GL}(H_{\mathbb{Q}})$ of the Hodge structure on $H_{\mathbb{Q}}$ is the smallest algebraic group over $\mathbb{Q}$ such that $\mathbf{G}_{\mathbb{R}}$ contains the image of the homomorphism $h\colon \mathbb{S} \to \mathbf{GL}(H)_{\mathbb{R}}$. It follows that $\mathbf{G}_{\mathbb{Q}}$ is connected. The Mumford–Tate group $\mathbf{G}_{\mathbb{Q}}$ is known to be reductive if the Hodge structure is polarisable, which we assume from now on.

The key property of the Mumford–Tate group is that an element of the full tensor algebra of $H_{\mathbb{Q}}$ and $H_{\mathbb{Q}}^\vee$ is fixed by $\mathbf{G}_{\mathbb{Q}}$ if and only if it has Hodge type $(0,0)$. Since in characteristic zero a reductive group is determined by its tensor invariants, $\mathbf{G}_{\mathbb{Q}}$

can be characterised by this property.

Let $\mathbf{G}$ be the group $\mathbb{Z}$-scheme which is the Zariski closure of the Mumford–Tate group $\mathbf{G}_{\mathbb{Q}}$ in $\mathbf{GL}(H)$.

Mumford–Tate conjecture at a prime ℓ. *We have* $\mathbf{G}_{\mathbb{Q}_\ell} = G_{k^{\rm conn},\ell,\mathbb{Q}_\ell} = G^\circ_{k,\ell,\mathbb{Q}_\ell}$.

An equivalent form of the conjecture is: $\mathbf{G}_{\mathbb{Z}_\ell} = G_{k^{\rm conn},\ell}$. Another equivalent form is in terms of Lie algebras, namely, the equality of Lie subalgebras $\mathfrak{g}_\ell = \mathfrak{h}\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$ of $\mathfrak{gl}(H_{\mathbb{Q}_\ell})$, where $\mathfrak{h}$ is the Lie algebra of the Mumford–Tate group $\mathbf{G}_{\mathbb{Q}}$.

By Tankeev and [LP95, Thm. 4.3], Mumford–Tate conjecture at ℓ implies Mumford–Tate conjecture at any other prime.

The Mumford–Tate conjecture $\mathbf{G}_{\mathbb{Z}_\ell} = G_{k^{\rm conn},\ell}$ implies that $\rho_\ell(\Gamma_{k^{\rm conn}})$ is an open subgroup of $\mathbf{G}(\mathbb{Z}_\ell)$ of finite index. This prompts the following stronger version of the conjecture.

Serre's integral Mumford–Tate conjecture. *There is a constant C such that for all primes ℓ the image $\rho_\ell(\Gamma_{k^{\rm conn}})$ is a subgroup of $\mathbf{G}(\mathbb{Z}_\ell)$ of index at most C.*

This was conjectured by Serre to hold for all varieties, see [Ser77, Conjecture C.3.7] and [Ser94, 10.3].

Definition 1.5 *A (polarisable) $\mathbb{Q}$-Hodge structure H is said to be of CM type if its Mumford–Tate group is a torus. Equivalently, the endomorphism algebra $\mathrm{End}_{\mathbb{Q}-\mathrm{HS}}(H)$ contains a commutative semisimple $\mathbb{Q}$-algebra F such that H is free of rank 1 as an F-module.*

A complex abelian variety A (respectively, a K3 surface X) has CM type if the Hodge structure on $\mathrm{H}^1(A,\mathbb{Q})$ (respectively, on $\mathrm{H}^2(X,\mathbb{Q})$) is of CM type.

Mumford–Tate conjecture is known for abelian varieties of CM type. More generally, for arbitrary abelian varieties, it is 'true for centres' (Vasiu [Vas08, Thm. 1.3.1], Ullmo–Yafaev). Commelin showed that if Mumford–Tate conjecture holds for two abelian varieties, then it holds for their product. Results of Serre imply that this conjecture holds for elliptic curves; Tankeev proved that it holds for simple abelian varieties of prime dimension. For general abelian varieties (already in dimension 4) Mumford–Tate conjecture conjecture is open.

For K3 surfaces, in contrast, the situation is much better understood.

Theorem 1.6 (Tankeev, Y. André, Cadoret–Moonen) *The integral Mumford–Tate conjecture holds for K3 surfaces over fields finitely generated over $\mathbb{Q}$.*

Proof. The Mumford–Tate conjecture for K3 surfaces was proved by Tankeev, and independently by Y. André. Cadoret and Moonen [CM20] showed, using that the moduli spaces of K3 surfaces are closely related to Shimura varieties, that the usual Mumford–Tate conjecture for K3 surfaces implies its integral version. □

Actually, more is true. Let $\rho : \Gamma_k \to \mathbf{GL}(H)(\hat{\mathbb{Z}})$ be the continuous representation of Γ_k whose ℓ-adic component is ρ_ℓ. Cadoret and Moonen proved in [CM20] that $\rho(\Gamma_{k^{\mathrm{conn}}})$ is an open subgroup of $\mathbf{G}(\hat{\mathbb{Z}})$ and therefore (since $\mathbf{G}(\hat{\mathbb{Z}})$ is compact) has finite index. This adelic version of the Mumford–Tate conjecture can only be expected to hold if the Hodge structure on H is Hodge-maximal [CM20, 2.6], which is the case when X is a K3 surface [CM20, Prop. 6.2].

Example 1.7 Mumford–Tate conjecture holds for diagonal hypersurfaces. Indeed, in the language of André's motives, varieties dominated by products of curves give rise to 'abelian motives'. In particular, this is the case for diagonal hypersurfaces. As explained in [Moo17], Vasiu's result that Mumford–Tate conjecture for abelian varieties is 'true for centres' implies that the same holds for abelian motives. The middle cohomology group of the Fermat hypersurface with coefficients $\mathbb{C}$ is the direct sum of 1-dimensional eigenspaces of $(\mu_d)^n$ acting by automorphisms, so the Mumford–Tate group is a torus. This proves the claim.

Question 1.8 The integral Mumford–Tate conjecture holds for diagonal surfaces. Does the adelic version of Mumford–Tate conjecture hold for diagonal hypersurfaces?

The integral Mumford–Tate conjecture has the following strong consequence for the finiteness of the Galois invariant subgroup of the Brauer group. In particular, the integral Mumford–Tate conjecture for X implies the finiteness of $\mathrm{Br}(X_{\bar{k}})^{\Gamma}$.

Proposition 1.9 (M. Orr, A.S.) *Let X be a smooth, projective and geometrically integral variety defined over a field k which is finitely generated over $\mathbb{Q}$. Assume that the integral Mumford–Tate conjecture is true for X. Then for every positive integer m there exists a constant $C = C_{m,X}$ such that for every subgroup $\Gamma' \subset \Gamma_k$ of index at most m we have $|\mathrm{Br}(X_{\bar{k}})^{\Gamma'}| < C$.*

Proof. See [OS18, §5]. The Kummer sequence for a prime ℓ gives that $\mathrm{Br}(X_{\bar{k}})\{\ell\}$ is an extension of the torsion subgroup of $\mathrm{H}^3_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$, which is finite and is zero for almost all ℓ, by the divisible subgroup $\mathrm{Br}(X_{\bar{k}})\{\ell\}_{\mathrm{div}} \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2-\rho}$, see [CTS21, Thm. 5.2.9]. Thus we need to bound the size of $\mathrm{Br}(X_{\bar{k}})^{\Gamma'}_{\mathrm{div}}\{\ell\}$ for each ℓ and to prove that $\mathrm{Br}(X_{\bar{k}})^{\Gamma'}_{\mathrm{div}}[\ell] = 0$ for all $\ell > \ell_0$. We have an isomorphism of Galois modules

$$\mathrm{Br}(X_{\bar{k}})_{\mathrm{div}}\{\ell\} \cong T_\ell(\mathrm{Br}(X_{\bar{k}})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

so $\mathrm{Br}(X_{\bar{k}})_{\mathrm{div}}[\ell] \cong T_\ell(\mathrm{Br}(X_{\bar{k}}))/\ell$. The integral Mumford–Tate conjecture allows us to replace the Galois group with the Mumford–Tate group. Let us indicate why $(T_\ell(\mathrm{Br}(X_{\bar{k}}))/\ell)^S = 0$ for almost all ℓ if $S \subset \mathbf{G}(\mathbb{Z}_\ell)$ is a subgroup of bounded index.

Denote by N the image of $\mathrm{NS}\,(X_{\mathbb{C}})$ in H. From the Kummer sequence and the comparisons theorem between singular and ℓ-adic étale cohomology we obtain an isomorphism $T_\ell(\mathrm{Br}(X_{\bar{k}}))/\ell \cong (H/N)/\ell$. We arrange that $\mathbf{G} \times_{\mathbb{Z}} \mathbb{F}_\ell$ is a connected algebraic group over $\mathbb{F}_\ell$ for all $\ell > \ell_0$. We want to show that no subgroup of $\mathbf{G}(\mathbb{F}_\ell)$ of bounded index fixes a non-zero vector in $(H/N)/\ell$ if ℓ_0 is large enough.

By the key property of the Mumford–Tate group, we have

$$(H_{\mathbb{Q}})^{\mathbf{G}_{\mathbb{Q}}} = H_{\mathbb{Q}} \cap H^{(0,0)}_{\mathbb{C}} = N_{\mathbb{Q}},$$

where the second equality is by the Lefschetz $(1,1)$-theorem. Since $\mathbf{G}_{\mathbb{Q}}$ is reductive, $H_{\mathbb{Q}}$ is a semisimple $\mathbf{G}_{\mathbb{Q}}$-module, thus $(H_{\mathbb{Q}}/N_{\mathbb{Q}})^{\mathbf{G}_{\mathbb{Q}}} = 0$ and hence $(H_{\mathbb{C}}/N_{\mathbb{C}})^{\mathbf{G}_{\mathbb{C}}} = 0$. It follows that the dimensions of the stabilisers of non-zero points of $H_{\mathbb{C}}/N_{\mathbb{C}}$ are less than $d = \dim(\mathbf{G}_{\mathbb{Q}})$. Using semicontinuity, one shows the same for the points of $(H/N)/\ell$, if ℓ_0 is large enough. These stabilisers are algebraic groups over $\mathbb{F}_\ell$. The number of their connected components is bounded by an absolute constant. For an arbitrary connected algebraic group G over $\mathbb{F}_\ell$ we have

$$(\ell-1)^{\dim(G)} \leq |G(\mathbb{F}_\ell)| \leq (\ell+1)^{\dim(G)}.$$

This shows that the index of a stabiliser of a non-zero point of $(H/N)/\ell$ cannot be bounded as ℓ grows. Thus no subgroup of $\mathbf{G}(\mathbb{F}_\ell)$ of bounded index fixes a non-zero vector in $(H/N)/\ell$ if $\ell > \ell_0$, where ℓ_0 is large enough.

To prove the finiteness of $\mathrm{Br}(X_{\bar{k}})^{\Gamma'}_{\mathrm{div}}\{\ell\}$ for a fixed ℓ, we use the property that $\mathbf{G}(\mathbb{Z}_\ell)$ has only finitely many open subgroups of bounded index. Hence the same is true for $\rho_\ell(\Gamma_k)$. For each such subgroup S we need to show that $\mathrm{Br}(X_{\bar{k}})\{\ell\}^S$ is finite. This is equivalent to $V_\ell(\mathrm{Br}(X_{\bar{k}}))^S = 0$. We have

$$V_\ell(\mathrm{Br}(X_{\bar{k}}))^S = (H_{\mathbb{Q}_\ell}/N_{\mathbb{Q}_\ell})^S = (H_{\mathbb{Q}_\ell}/N_{\mathbb{Q}_\ell})^{\mathbf{G}_{\mathbb{Q}_\ell}} = 0,$$

where the second equality is due to the fact that S is Zariski dense in $\mathbf{G}_{\mathbb{Q}_\ell}$, and the last one is due to semisimplicity of the $\mathbf{G}_{\mathbb{Q}_\ell}$-module $H_{\mathbb{Q}_\ell}$ and the key property that $(H_{\mathbb{Q}})^{\mathbf{G}_{\mathbb{Q}}} = N_{\mathbb{Q}}$. □

Corollary 1.10 *Let X be a smooth, projective and geometrically integral variety defined over a field k which is finitely generated over $\mathbb{Q}$. If the integral Mumford–Tate conjecture is true for X, then for each positive integer n there exists a constant $C = C_{n,X}$ such that for every $(\bar{k}/L)$-form Y of X defined over a field extension L/k of degree $[L:k] \leq n$ we have $|\mathrm{Br}(\overline{Y})^{\Gamma_L}| < C$.*

See [Amb21] for an analogue of this statement in finite characteristic.

1.3 Finiteness properties of the Brauer group

Let X be a smooth, projective and geometrically integral variety over a field k. Then $\mathrm{Br}(X)$ is a torsion group [CTS21, Thm. 3.5.5], thus a direct sum of its ℓ-primary torsion subgroups $\mathrm{Br}(X)\{\ell\}$, for all primes ℓ.

Separably closed fields

For $\ell \neq \mathrm{char}(k)$, the structure of $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}$ is computed by the Kummer exact sequence: the divisible subgroup of $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}$ is isomorphic to $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2-\rho}$ and the quotient by the divisible subgroup is isomorphic to the torsion subgroup of $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell)$, see [CTS21, Thm. 5.2.9]. We also note that the natural map

$$\mathrm{Br}(X^{\mathrm{s}})\{\ell\} \to \mathrm{Br}(X_{\bar{k}})\{\ell\}$$

is an isomorphism [CTS21, Prop. 5.2.3].

The structure of $\mathrm{Br}(X^{\mathrm{s}})\{p\}$, where $p = \mathrm{char}(k)$, is more involved. First of all, one needs to distinguish between separably closed fields that are not algebraically closed and algebraically closed fields. One problem here is that $\bar{k}$ is not necessarily finitely generated over k_{s}, so $f\colon \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k_{\mathrm{s}})$ is not an fppf covering.

Proposition 1.11 *The natural map* $\mathrm{Br}(X^{\mathrm{s}})\{p\} \to \mathrm{Br}(X_{\bar{k}})\{p\}$ *is injective if the Picard scheme* $\mathbf{Pic}_{X^{\mathrm{s}}/k_{\mathrm{s}}}$ *is smooth over* k_{s}. *This holds if one of the following conditions is satisfied:*

- X *is an abelian variety;*
- X *is a K3 surface;*
- $\mathrm{H}^1(X, \mathcal{O}) = 0$*;*
- $\mathrm{H}^2(X, \mathcal{O}) = 0$.

If $\mathrm{Br}(X^{\mathrm{s}})\{p\} \to \mathrm{Br}(X_{\bar{k}})\{p\}$ *is injective, then* $\mathrm{Br}(X^{\mathrm{s}})^k \xrightarrow{\sim} \mathrm{Br}(X_{\bar{k}})^k$.

Proof. See [CTS21, Thm. 5.2.5]. We write $k = k_{\mathrm{s}}$. Let $f\colon X \to \mathrm{Spec}(k)$ be the structure morphism. Consider the spectral sequence

$$\mathrm{H}^p_{\mathrm{fppf}}(k, R^q f_*\mathbb{G}_{m,X}) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{fppf}}(X, \mathbb{G}_{m,X}) \cong \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_{m,X}).$$

Here the isomorphism is due to the fact that fppf cohomology with coefficients in a smooth group scheme coincides with étale cohomology. For any k-scheme T the map $\mathcal{O}_T \to f_*\mathcal{O}_{T\times_k X}$ is an isomorphism, hence we have an isomorphism of fppf sheaves $\mathbb{G}_{m,k} \xrightarrow{\sim} f_*\mathbb{G}_{m,X}$. We have $\mathrm{H}^2_{\mathrm{fppf}}(k, \mathbb{G}_{m,k}) \cong \mathrm{H}^2_{\text{ét}}(k, \mathbb{G}_{m,k}) = \mathrm{Br}(k) = 0$ because k is separably closed. Moreover, $R^1 f_*\mathbb{G}_m$ is representable by the Picard group scheme $\mathbf{Pic}_{X/k}$. Thus the spectral sequence gives an exact sequence

$$0 \to \mathrm{H}^1_{\mathrm{fppf}}(k, \mathbf{Pic}_{X/k}) \to \mathrm{Br}(X) \to \mathrm{H}^0(k, R^2 f_*\mathbb{G}_{m,X}).$$

If $\mathbf{Pic}_{X/k}$ is smooth, then $\mathrm{H}^1_{\mathrm{fppf}}(k, \mathbf{Pic}_{X/k}) = 0$. With some work one shows that $\mathrm{H}^0(k, R^2 f_*\mathbb{G}_{m,X})$ embeds into $\mathrm{H}^0(\bar{k}, R^2 f_*\mathbb{G}_{m,X})$, see [D'Ad, §3]. But the injective map $\mathrm{Br}(X) \to \mathrm{H}^0(\bar{k}, R^2 f_*\mathbb{G}_{m,X})$ factors through $\mathrm{Br}(X_{\bar{k}})$, hence the result.

It is well known that the Picard scheme of an abelian variety is smooth. □

Example 1.12 The map $\mathrm{Br}(X^{\mathrm{s}})\{p\} \to \mathrm{Br}(X_{\bar{k}})\{p\}$ is not always injective. Indeed, in characteristic $p = 2$ there exist Enriques surfaces such that the connected component of the origin in $\mathbf{Pic}_{X^{\mathrm{s}}/k_{\mathrm{s}}}$ is the group k-scheme α_2. The exact sequence of fppf sheaves

$$0 \to \alpha_p \to \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \to 0,$$

where $F(x) = x^p$, gives $\mathrm{H}^1_{\mathrm{fppf}}(k_{\mathrm{s}}, \alpha_p) \cong k_{\mathrm{s}}/k_{\mathrm{s}}^p$, which can be infinite.

Example 1.13 Artin showed that if X is a *supersingular* K3 surface over an algebraically closed field $\bar{k}$ of characteristic $p > 0$ (by definition, this means that $\rho = b_2 = 22$), then $\mathrm{Br}(X_{\bar{k}}) \cong \bar{k}$, so this group is not divisible (we cannot divide by p), of exponent p, and can be uncountable[2]. In contrast, if X is a non-supersingular K3 surface over an algebraically closed field $\bar{k}$ of characteristic $p > 0$, then $\mathrm{Br}(X_{\bar{k}})$ is countable and divisible. Note that in the last case, assuming k is finitely generated, the group $\mathrm{Br}(X_{\bar{k}})^\Gamma$ does not always have finite exponent, see below.

Transcendental Brauer group

We have the following general result over an arbitrary ground field.

Theorem 1.14 *Let X be a smooth, projective, geometrically integral variety over a field k of characteristic exponent p. Then $\mathrm{Br}(X^{\mathrm{s}})^\Gamma/\mathrm{Br}(X^{\mathrm{s}})^k$, that is, the cokernel of the natural map $\mathrm{Br}(X) \to \mathrm{Br}(X^{\mathrm{s}})^\Gamma$, is the direct sum of a finite group of order coprime to p and a p-torsion group of finite exponent. In particular, if $\mathrm{char}(k) = 0$, then $\mathrm{Br}(X^{\mathrm{s}})^k$ is finite if and only if $\mathrm{Br}(X^{\mathrm{s}})^\Gamma$ is finite.*

This was proved in [CTS13b] when $\mathrm{char}(k) = 0$. As pointed out by Xinyi Yuan [Yua20], the same method works over any k.

For abelian varieties and K3 surfaces over finitenely generated field, building on Corollary 1.4, one strengthens that corollary as follows.

Theorem 1.15 *Let k be a finitely generated field. If X is an abelian variety or a K3 surface, then the subgroup of $\mathrm{Br}(X^{\mathrm{s}})^\Gamma$ consisting of the elements of order coprime to $\mathrm{char}(k)$ is finite.*

Proof. This is a result of Skorobogatov and Zarhin [SZ08, SZ15], except for the case of K3 surfaces over a field of characteristic 2 where it is due to K. Ito [Ito18]. For the case of abelian varieties see also [CTS21, Thm. 16.2.3]. The proof uses results of Faltings and Zarhin who showed that for abelian varieties A, B over a field finitely generated over a prime subfield the natural injective map

$$\mathrm{Hom}(A, B)/\ell \hookrightarrow \mathrm{Hom}_\Gamma(A[\ell], B[\ell])$$

is an isomorphism for almost all ℓ. Let us explain how this implies that for almost all primes ℓ we have $\mathrm{Br}(A^{\mathrm{s}})^\Gamma[\ell] = 0$. In view of Corollary 1.4 this is all that remains to do.

[2]Remarkably, in this case, the fppf sheaf $R^2 p_* \mu_p$ is a smooth group k-scheme of finite type with connected component $\mathbb{G}_a$. This is due to Artin, Artin–Mazur, and Milne, using the Tate conjecture. Bragg and Olsson prove that $R^i p_* G$, where X is projective and of finite type, and G is a finite flat group X-scheme of finite type, is always representable by a group k-scheme of finite type, which is an iterated extension of finite commutative group schemes and k-forms of vector bundles.

The theory of abelian varieties over separably closed fields gives isomorphisms of Γ-modules

$$\mathrm{H}^i_{\text{ét}}(A^{\mathrm{s}},\mathbb{Z}/\ell)\cong\wedge^i\mathrm{H}^1_{\text{ét}}(A^{\mathrm{s}},\mathbb{Z}/\ell),\quad i\geq 1,$$

and

$$\mathrm{H}^1_{\text{ét}}(A^{\mathrm{s}},\mu_\ell)\cong\mathbf{Pic}_{A^{\mathrm{s}}/k_{\mathrm{s}}}[\ell]\cong A^\vee[\ell]\cong\mathrm{Hom}(A[\ell],\mu_\ell).$$

Here the first isomorphism comes from the Kummer sequence, the second one follows from the definition of the dual abelian variety $A^\vee$ as the connected component of $\mathbf{Pic}_{A/k}$, and the third one is due to the non-degeneracy of the Weil pairing

$$e_A\colon A[\ell]\times A^\vee[\ell]\to\mu_\ell.$$

For $i=2$ we use the Weil pairing to define an embedding

$$\mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}},\mu_\ell)\cong\mathrm{Hom}(\wedge^2A[\ell],\mu_\ell)\hookrightarrow\mathrm{Hom}(A[\ell]^{\otimes 2},\mu_\ell)\cong\mathrm{Hom}(A[\ell],A^\vee[\ell]).$$

Now assume that $\ell\neq 2$. Let us call a homomorphism $\phi\colon A[\ell]\to A^\vee[\ell]$ *self-dual* if $e_A(x,\phi y)=e_{A^\vee}(\phi x,y)$ for all $x,y\in A[\ell]$. Denote by $\mathrm{Hom}(A[\ell],A^\vee[\ell])_{\mathrm{sym}}$ the group of self-dual homomorphisms $A[\ell]\to A^\vee[\ell]$. The subtle but crucial fact that the Weil pairings for A and $A^\vee$ *differ by sign*:

$$e_A(x,y)=-e_{A^\vee}(y,x),$$

implies that for $\ell\neq 2$ the image of $\mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}},\mu_\ell)$ in $\mathrm{Hom}(A[\ell],A^\vee[\ell])$ is $\mathrm{Hom}(A[\ell],A^\vee[\ell])_{\mathrm{sym}}$.

On the other hand, there is a canonical isomorphism $\mathrm{NS}\,(A^{\mathrm{s}})=\mathrm{Hom}(A^{\mathrm{s}},(A^\vee)^{\mathrm{s}})_{\mathrm{sym}}$, where the subscript sym stands for self-dual maps of abelian varieties $A^{\mathrm{s}}\to(A^\vee)^{\mathrm{s}}$. Using these isomorphisms, the cycle class map $\mathrm{NS}\,(A^{\mathrm{s}})/\ell\hookrightarrow\mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}},\mu_\ell)$ becomes the natural map

$$\mathrm{Hom}(A^{\mathrm{s}},(A^\vee)^{\mathrm{s}})_{\mathrm{sym}}/\ell\hookrightarrow\mathrm{Hom}(A[\ell],A^\vee[\ell])_{\mathrm{sym}}.$$

By Zarhin and Faltings, for almost all ℓ a Γ-invariant element $\phi\in\mathrm{Hom}_\Gamma(A[\ell],A^\vee[\ell])_{\mathrm{sym}}$ comes from a morphism $\widetilde{\phi}\colon A\to A^\vee$. Since $\ell\neq 2$ we can consider $(\widetilde{\phi}+\widetilde{\phi}^\vee)/2 \bmod \ell$, which is an element of $\mathrm{Hom}(A^{\mathrm{s}},(A^\vee)^{\mathrm{s}})_{\mathrm{sym}}/\ell$ that maps to ϕ. The exact sequence of Γ-modules (coming from the Kummer sequence)

$$0\to\mathrm{Hom}(A^{\mathrm{s}},(A^\vee)^{\mathrm{s}})_{\mathrm{sym}}/\ell\to\mathrm{Hom}(A[\ell],A^\vee[\ell])_{\mathrm{sym}}\to\mathrm{Br}(A^{\mathrm{s}})[\ell]\to 0$$

is split for almost all ℓ (we already used this in the proof of Theorem 1.1). It follows that $\mathrm{Br}(A^{\mathrm{s}})[\ell]^\Gamma=0$ for almost all ℓ.

For the case of K3 surfaces see [CTS21, Thm. 16.7.2, Remark 16.7.3]. The proof uses the Kuga–Satake construction to reduce to the case of abelian varieties. In the finite characteristic case, the proof uses Tate conjecture for K3 surfaces established by Madapusi Pera, and by W. Kim and Madapusi Pera when $\mathrm{char}(k)=2$. □

Corollary 1.16 *Let X be a K3 surface over a finitely generated field. The subgroup of elements of order not divisible by* $\mathrm{char}(k)$ *in* $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ *is finite.*

Proof. Use that $\mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}}))$ is finite. □

A recent result of M. D'Adezzio [D'Ad, Thm. 1.1] concerns p-primary torsion of the Brauer group of abelian varieties over finitely generated fields of characteristic p.

Theorem 1.17 (D'Adezzio) *Let A be an abelian variety over a finitely generated field k of characteristic $p > 0$. Then the transcendental Brauer group $\mathrm{Br}(A^{\mathrm{s}})^k \cong \mathrm{Br}(A_{\bar{k}})^k$ is a direct sum of a finite group and a finite exponent p-group.*

The following general statement appears to be a correct generalisation of Theorem 1.15. (Note that this is about k_{s}, not $\bar{k}$.)

Corollary 1.18 *Let A be an abelian variety over a finitely generated field k. Then $\mathrm{Br}(A^{\mathrm{s}})^{\Gamma}$ has finite exponent.*

Proof. Combine Theorems 1.14 and 1.17. □

Question 1.19 Does $\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$ have finite exponent if X is a K3 surface? It is easy to show that this holds for Kummer surfaces over a field k of characteristic $p \neq 2$. Indeed, let $X = \mathrm{Kum}(A)$, where A is an abelian surface. By Theorem 1.15 it remains to deal with p-primary torsion. The rational map of degree 2 from A to X induces a map $\mathrm{Br}(X) \to \mathrm{Br}(A)$ and a compatible map $\mathrm{Br}(X^{\mathrm{s}}) \to \mathrm{Br}(A^{\mathrm{s}})$. The standard restriction-corestriction argument shows that both maps are injective on the subgroups of elements of odd order, see, e.g., [CTS21, Prop. 3.8.4]. Hence $\mathrm{Br}(X^{\mathrm{s}})^k\{p\} \subset \mathrm{Br}(A^{\mathrm{s}})^k\{p\}$, so $\mathrm{Br}(X^{\mathrm{s}})^k\{p\}$ has finite exponent by Theorem 1.17. Thus $\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}\{p\}$ has finite exponent by Theorem 1.14.

Example 1.20 Note that the transcendental Brauer group of A may well be infinite, cf. [D'Ad, Cor. 5.4]. Let E_1, E_2 be supersingular elliptic curves over an infinite finitely generated field k of characteristic p, and let $A = E_1 \times_k E_2$. Infinitely many elements of $\mathrm{Br}(A)$ that survive in $\mathrm{Br}(A_{\bar{k}})$ are easy to construct. Indeed, multiplication by p map $[p]\colon E_i \to E_i$ is an fppf-torsor with structure group $E_i[p]$, for $i = 1, 2$. The cup-product of these torsors for the two factors is a class in $\mathrm{H}^2_{\mathrm{fppf}}(A, E_1[p] \otimes E_2[p])$. The Weil pairing induces an isomorphism of fppf sheaves over $\mathrm{Spec}(k)$:

$$\mathrm{Hom}(E_1[p] \otimes E_2[p], \mu_p) \cong \mathrm{Hom}(E_1[p], E_2[p]).$$

This gives a map

$$\mathrm{Hom}_{\bar{k}-\text{group schemes}}(E_1[p], E_2[p]) \to \mathrm{H}^2_{\mathrm{fppf}}(A_{\bar{k}}, \mu_p),$$

which one shows to be injective, and a similar map over k. The Kummer exact sequence in fppf topology gives a map

$$\mathrm{H}^2_{\mathrm{fppf}}(A_{\bar{k}}, \mu_p) \to \mathrm{Br}(A_{\bar{k}}),$$

since the Brauer group can be computed in the fppf topology because $\mathbb{G}_m$ is smooth. The group scheme $E_i[p]$, $i = 1, 2$, is an extension of α_p by α_p, hence there is an embedding $\mathrm{End}_k(\alpha_p) \subset \mathrm{Hom}_k(E_1[p], E_2[p])$. We have $\mathrm{End}_k(\alpha_p) \cong k$. This gives a map $k \to \mathrm{Br}(A)$ and a compatible map $\bar{k} \to \mathrm{Br}(A_{\bar{k}})$. Since $\mathrm{NS}\,(A_{\bar{k}})/p$ is finite, we have infinitely many elements of $\mathrm{Br}(A)$ surviving in $\mathrm{Br}(A_{\bar{k}})$.

However, [D'Ad, Thm. 1.1] says that the transcendental Brauer group of A is finite when the p-rank of A is g or $g-1$, where $g = \dim(A)$. For $g = 2$ this means exactly that A is not supersingular.

Example 1.21 When $p \neq 2$ the above classes descend to $X = \mathrm{Kum}(A)$. This gives an example of a K3 surface with an infinite transcendental Brauer group, answering [SZ08, Questions 1, 2] in the negative. Indeed, let $\widetilde{A}$ be the surface obtained by blowing up the subscheme $A[2] \subset A$. By the birational invariance of the Brauer group we have $\mathrm{Br}(A) \cong \mathrm{Br}(\widetilde{A})$, see [CTS21, Cor. 6.2.11]. There is a finite surjective morphism $f\colon \widetilde{A} \to X$ such that $k(\widetilde{A})$ is a Galois extension of $k(X)$ with Galois group $G \cong \mathbb{Z}/2$. By [CTS21, Thm. 3.8.5] we have $\mathrm{Br}(X)\{p\} \cong \mathrm{Br}(A)\{p\}^G$. It is easy to check that the above classes are G-invariant. Indeed, the generator of G multiplies the class of each torsor $[p]\colon E_i \to E_i$ by -1, hence the cup-product class in $\mathrm{H}^2_{\mathrm{fppf}}(A, E_1[p] \otimes E_2[p])$ is fixed by G, and thus so is the resulting Brauer class.

Example 1.22 D'Addezio gives an example to show that in the case of finite characteristic, $\mathrm{Br}(A_{\bar{k}})^\Gamma$ does not always have finite exponent, see [D'Ad, Cor. 6.7]. Take $A = E \times_k E$, where E is an ordinary elliptic curve over k with $\mathrm{End}(E) \cong \mathbb{Z}$. Then $T_p(\mathrm{Br}(A_{\bar{k}}))$ contains the quotient of $\mathrm{End}(E_{\bar{k}}[p^\infty])$ by $\mathrm{End}(E_{\bar{k}}) \otimes \mathbb{Z}_p$. Taking Galois invariants we obtain that $T_p(\mathrm{Br}(A_{\bar{k}}))^\Gamma$ contains the quotient of $\mathrm{End}(E_{\bar{k}}[p^\infty])^\Gamma$ by $\mathrm{End}(E) \otimes \mathbb{Z}_p \cong \mathbb{Z}_p$, so it is enough to show that the rank of the $\mathbb{Z}_p$-module $\mathrm{End}(E_{\bar{k}}[p^\infty])^\Gamma$ is at least 2. Since $\mathrm{End}(E) \cong \mathbb{Z}$, the elliptic curve E is not supersingular, so the p-divisible group $E[p^\infty]$ has at least two slopes. By the Dieudonné–Manin classification, this implies that $E[p^\infty]$ is isogenous to the direct sum of two non-zero p-divisible groups, hence the rank of $\mathrm{End}(E_{\bar{k}}[p^\infty])^\Gamma$ is at least 2.

As in Remark 1.21, if $p \neq 2$, then these classes also descend to $X = \mathrm{Kum}(A)$, showing that $\mathrm{Br}(X_{\bar{k}})^\Gamma$ does not always have finite exponent when X is a K3 surface.

1.4 Uniformity

The aim of this section is to discuss links among several conjectures about K3 surfaces and abelian varieties defined over number fields. These conjectures state that certain invariants take only finitely many values provided the degree of the field of definition and the dimension (in the case of abelian varieties) are bounded.

Coleman's conjecture about $\mathrm{End}(A_{\bar{k}})$. *Let d and g be positive integers. Consider all abelian varieties A of dimension g defined over number fields of degree d. Then there are only finitely many isomorphism classes among the rings $\mathrm{End}(A_{\bar{k}})$.*

This or a closely related conjecture is attributed to Robert Coleman in [Sha96a, Remark 4]. There is a version of this conjecture in which $\mathrm{End}(A_{\bar{k}})$ is replaced by the ring $\mathrm{End}(A)$ of endomorphisms of A defined over k. It is not too hard to show that Coleman's conjecture about $\mathrm{End}(A_{\bar{k}})$ is equivalent to Coleman's conjecture about $\mathrm{End}(A)$, see [OSZ21, Thm. 3.4].

Rémond proved that Coleman's conjecture implies the uniform boundedness of torsion $A(k)_{\mathrm{tors}}$ and of the minimal degree of an isogeny between isogenous abelian varieties, see [Rem18, Thm. 1.1].

Shafarevich's conjecture about $\mathrm{NS}\,(X_{\bar{k}})$**.** *Let d be a positive integer. There are only finitely many lattices L, up to isomorphism, for which there exists a K3 surface X defined over a number field of degree d such that $\mathrm{NS}\,(X_{\bar{k}}) \cong L$.*

It is in this form that Shafarevich has stated his conjecture in [Sha96a]. Since there are only finitely many lattices of bounded rank and discriminant, Shafarevich's conjecture is equivalent to the boundedness of the discriminant of $\mathrm{NS}\,(X_{\bar{k}})$. One can also state a variant of Shafarevich's conjecture in which $\mathrm{NS}\,(X_{\bar{k}})$ is replaced by its Galois-invariant subgroup $\mathrm{NS}\,(X_{\bar{k}})^{\Gamma}$, or, alternatively, by $\mathrm{Pic}(X)$. By [OSZ21, Thm. 3.5] all these versions of Shafarevich's conjecture are equivalent.

Similarly to Shafarevich's conjecture, Coleman's conjecture can be restated in terms of lattices. Recall that $\mathrm{End}(A)$ is an order in the semisimple $\mathbb{Q}$-algebra $\mathrm{End}(A)_{\mathbb{Q}} = \mathrm{End}(A) \otimes \mathbb{Q}$. Let us define $\mathrm{discr}(A)$ as the discriminant of the integral symmetric bilinear form $\mathrm{tr}(xy)$ on $\mathrm{End}(A)$, where $\mathrm{tr} : \mathrm{End}(A)_{\mathbb{Q}} \to \mathbb{Q}$ is the reduced trace. An equivalent form of Coleman's conjecture says that $\mathrm{discr}(A)$ is uniformly bounded for abelian varieties A of bounded dimension defined over number fields of bounded degree.

Várilly-Alvarado's conjecture. [VA17, Conj. 4.6] *Let d be a positive integer and let L be a primitive sublattice of the K3 lattice $E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. If X is a K3 surface defined over a number field of degree d such that $\mathrm{NS}\,(X_{\bar{k}}) \cong L$, then the cardinality of $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is bounded.*

A stronger form of this conjecture omits the reference to the Néron–Severi lattice. It concerns the uniform boundedness of the Galois invariant subgroup of the geometric Brauer group.

It is well known that the cardinality of the finite group $\mathrm{H}^1(k, \mathrm{Pic}(X_{\bar{k}}))$, where X is a K3 surface over an arbitrary field k of characteristic zero, is bounded, see e.g. [VAV17, Lemma 6.4]. This is based on the well known lemma of Minkowski that says that the order of finite subgroups of $\mathrm{GL}(\mathbb{Z}, n)$ is bounded by a function of n.

Conjecture Br(K3). *Let d be a positive integer. There is a constant $C = C(d)$ such that, if X is a K3 surface defined over a number field of degree d, then $|\mathrm{Br}(X_{\bar{k}})^{\Gamma}| < C$.*

A similar conjecture can be stated for abelian varieties of given dimension.

Conjecture Br(AV). *Let d and g be positive integers. There is a constant $C =$*

$C(d,g)$ such that, if A is an abelian variety of dimension g defined over a number field of degree d, then $|\mathrm{Br}(\overline{A})^{\Gamma}| < C$.

As proved in [OSZ21], these conjectures are logically related as follows:

$$\left.\begin{array}{ccc} \text{Coleman's conjecture} & \Longrightarrow & \text{Shafarevich's conjecture} \\ \Downarrow & & \\ \mathrm{Br(AV)} & \Longrightarrow & \text{Várilly-Alvarado's conjecture} \end{array}\right\} \Longrightarrow \mathrm{Br(K3)}$$

Theorem 1.23 *The conjectures featuring in this diagram hold for abelian varieties and K3 surfaces with complex multiplication.*

Proof. See [OS18]. The proof uses a lower bound for the size of Galois orbits of CM points from work of Tsimerman building on the proof of the averaged Colmez conjecture by Andreatta, Goren, Howard and Madapusi Pera, and by X. Yuan and S. Zhang. □

Coleman's conjecture for elliptic curves follows from the Brauer–Siegel theorem, but in the general case not much is known about these conjectures.

All these conjectures may be stated in the form "in a certain class of moduli spaces, only finitely many spaces in the class have rational points over number fields of degree d, excluding points which lie in subvarieties of positive codimension parameterising objects with extra structures."

Some of the above conjectures are known for the fibres of one-parameter families. In particular, a result of Cadoret and Tamagawa implies Coleman's conjecture within a one-parameter family of abelian varieties. Cadoret and Charles have proved uniform boundedness of the ℓ-primary subgroup of the Brauer group for one-parameter families of abelian varieties and K3 surfaces. Várilly-Alvarado and Viray obtained bounds for the Brauer group for one-parameter families of Kummer surfaces attached to products of isogenous elliptic curves [VAV17, Thm. 1.8].

2 Computing the Brauer group

2.1 Leray spectral sequence with coefficients $\mathbb{G}_m$

Let X be a variety over a field k. The Leray spectral sequence for the structure morphism $p\colon X \to \mathrm{Spec}(k)$ and the étale sheaf $\mathbb{G}_m$ is the spectral sequence

$$E_2^{pq} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(X^{\mathrm{s}}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_m). \tag{2}$$

Let us denote the differential $E_i^{p,q} \to E_i^{p+i,q-i+1}$ by $d_i^{p,q}$.

When $k_{\mathrm{s}}[X]^{\times} = k_{\mathrm{s}}^{\times}$, this Leray spectral sequence gives exact sequences

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}})^{\Gamma} \xrightarrow{d_2^{0,1}} \mathrm{Br}(k) \longrightarrow \mathrm{Br}_1(X)$$
$$\longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}})) \xrightarrow{d_2^{1,1}} \mathrm{Ker}[\mathrm{H}^3(k, k_{\mathrm{s}}^{\times}) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)]$$

and

$$0 \longrightarrow \mathrm{Br}_1(X) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Ker}[\mathrm{Br}(X^{\mathrm{s}}) \xrightarrow{d_2^{0,2}} \mathrm{H}^2(k, \mathrm{Pic}(X^{\mathrm{s}}))]$$
$$\xrightarrow{d_3^{0,2}} \mathrm{Ker}[\mathrm{H}^3(k, k_{\mathrm{s}}^\times) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)].$$

Thus to compute $\mathrm{Br}_1(X)$ we need to know $\mathrm{Im}(d_2^{0,1})$ and $\mathrm{Ker}(d_2^{1,1})$, and then to compute $\mathrm{Br}(X)$ we need to know $\mathrm{Ker}(d_2^{0,2})$ and finally $\mathrm{Ker}(d_3^{0,2})$. Let us explain what is known about these four problems.

- If $X(k) \neq \emptyset$, then $\mathrm{H}^i(k, \mathbb{G}_m) \to \mathrm{H}^i(X, \mathbb{G}_m)$ has a retraction, hence is injective, for all $i \geq 0$. Then $d_2^{0,1} = 0$, $d_2^{1,1} = 0$, $d_3^{0,2} = 0$.
- More generally, if there is a morphism $Y \to X$, where Y is a variety over k such that $k_{\mathrm{s}}[Y]^\times = k_{\mathrm{s}}^\times$, then $d_2^{p,1} = 0$ for Y implies the same for X. (Example: for a conic bundle $X \to \mathbb{P}^1_k$ containing a smooth k-fibre Y we have $d_2^{1,1} = 0$.)
- Likewise, in the same situation, if $d_3^{0,2} = 0$ for Y, then the same holds for X.
- If $\mathrm{H}^3(k, k_{\mathrm{s}}^\times) = 0$, then $d_2^{1,1} = 0$ and $d_3^{0,2} = 0$. This holds when k is a number field or a p-adic field.

Proposition 2.1 *Let X be a smooth and geometrically integral variety over a field k such that $k_{\mathrm{s}}[X]^\times = k_{\mathrm{s}}^\times$. For each $p \geq 0$ the differential $d_2^{p,1}$ from the spectral sequence* (2) *coincides, up to sign, with the connecting map defined by the* 2*-extension of* Γ*-modules*

$$0 \longrightarrow k_{\mathrm{s}}^\times \longrightarrow k_{\mathrm{s}}(X)^\times \longrightarrow \mathrm{Div}(X^{\mathrm{s}}) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}}) \longrightarrow 0. \quad (3)$$

The differential $d_2^{p,1}$ comes from the map attached to the exact triangle

$$p_*(\mathbb{G}_{m,X}) \longrightarrow \tau_{[0,1]}(\mathbf{R}p_*(\mathbb{G}_{m,X})) \longrightarrow (R^1p_*)(\mathbb{G}_{m,X})[-1]$$

in the bounded below derived category $\mathcal{D}^+(k)$ of Γ-modules. Here $\mathbf{R}p_*\colon \mathcal{D}^+(X) \to \mathcal{D}^+(k)$ is the derived functor from the bounded below derived category $\mathcal{D}^+(X)$ of étale sheaves on X to $\mathcal{D}^+(k)$, and $\tau_{[0,1]}$ is the truncation functor. Proposition 2.1 then follows from the fact that $\tau_{[0,1]}(\mathbf{R}p_*(\mathbb{G}_{m,X}))$ is represented by the 2-term complex $k_{\mathrm{s}}(X)^\times \to \mathrm{Div}(X^{\mathrm{s}})$, see [BvH09, Lemma 2.3].

There are many examples where the differential $d_2^{0,1}$ is non-zero, the easiest one is a conic without rational points (the generator of $\mathrm{Pic}(X_{\bar{k}}) \cong \mathbb{Z}$ goes to the class of the conic X in $\mathrm{Br}(k)$). See also Section 2.2 below.

There is a partial description of the differential $d_2^{0,2}$.

Proposition 2.2 *Let X be a smooth, projective, geometrically integral variety over a field k of characteristic zero. Let $N(X^{\mathrm{s}}) = \mathrm{NS}\,(X^{\mathrm{s}})/\mathrm{tors}$. The composition*

$$(\mathrm{Br}(X^{\mathrm{s}})_{\mathrm{div}})^\Gamma \hookrightarrow \mathrm{Br}(X^{\mathrm{s}})^\Gamma \xrightarrow{d_2^{0,2}} \mathrm{H}^2(k, \mathrm{Pic}(X^{\mathrm{s}})) \longrightarrow \mathrm{H}^2(k, N(X^{\mathrm{s}})) \quad (4)$$

coincides (up to sign) with the connecting map $\partial\colon (\mathrm{Br}(X^{\mathrm{s}})_{\mathrm{div}})^{\Gamma} \to \mathrm{H}^2(k, N(X^{\mathrm{s}}))$ *defined by the exact sequence*

$$0 \longrightarrow N(X^{\mathrm{s}}) \longrightarrow N(X^{\mathrm{s}}) \otimes \mathbb{Q} \longrightarrow \bigoplus_{\ell} \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{Br}(X^{\mathrm{s}})_{\mathrm{div}} \longrightarrow 0.$$

Proof. This is [CTS13b, Cor. 3.5]. □

For K3 surfaces we have $\mathrm{Pic}(X^{\mathrm{s}}) = N(X^{\mathrm{s}})$ and $\mathrm{Br}(X^{\mathrm{s}}) = \mathrm{Br}(X^{\mathrm{s}})_{\mathrm{div}}$, so this result completely describes $d_2^{0,2}$. Unfortunately, it is not quite enough for abelian varieties (where $\mathrm{Pic}(X^{\mathrm{s}})_{\mathrm{div}}$ is non-zero, see however Corollary 2.3 below) and gives no information at all for Enriques surfaces (where $\mathrm{Br}(X^{\mathrm{s}})_{\mathrm{div}}$ is zero).

I am not aware of any results describing $d_3^{0,2}$. This is fortunately not a problem for abelian varieties (which have a k-point) or over number (or p-adic) fields.

In the case of abelian varieties we have the following consequence of the functoriality of the spectral sequence (2) in X.

Corollary 2.3 *Let A be an abelian variety over a field k of characteristic zero. If $x \in \mathrm{Br}(A_{\bar{k}})^{\Gamma}$ is such that $\partial(x) = 0$, then $2x \in \mathrm{Br}(A_{\bar{k}})^{k}$. In particular, an element of $\mathrm{Br}(A_{\bar{k}})^{\Gamma}_{\mathrm{odd}}$ lifts to $\mathrm{Br}(A)$ if and only if $\partial(x) = 0$.*

Proof. This formally follows from the fact that the antipodal involution $[-1]\colon A \to A$ induces multiplication by -1 on the dual abelian variety $A^{\vee}$, but acts trivially on $\mathrm{H}^2(A_{\bar{k}}, \mathbb{Z}_\ell(1))$ for all ℓ, hence it also acts trivially on $\mathrm{NS}\,(A_{\bar{k}})$ and $\mathrm{Br}(A_{\bar{k}})$. Indeed, since $\partial(x) = 0$, we have $d_2^{0,2}(x) = i_*(y)$ for some $y \in \mathrm{H}^2(k, A^{\vee})$, where

$$i\colon A^{\vee}(\bar{k}) \cong \mathrm{Pic}^0(A_{\bar{k}}) \hookrightarrow \mathrm{Pic}(A_{\bar{k}})$$

is the natural injective map. We have $[-1]^*(x) = x$ and $[-1]^*(y) = -y$, therefore $d_2^{0,2}(2x) = 2d_2^{0,2}(x) = i_*(y) + i_*(-y) = 0$. □

2.2 Example: generic diagonal surfaces

Let k be a field of characteristic zero, and let $K = k(a_1, a_2, a_3)$ be a field extension. For a positive integer d consider the surface $X \subset \mathbb{P}^3_K$ given by

$$x_0^d + a_1 x_1^d + a_2 x_2^d + a_3 x_3^d = 0. \tag{5}$$

Theorem 2.4 *If a_0, a_1, a_2 are algebraically independent over k (that is, K is a purely transcendental extension of k of transcendence degree 3), then the natural map $\mathrm{Br}(K) \to \mathrm{Br}(X)$ is an isomorphism.*

Proof. This is [GS, Thm. 1.5]. For a different proof see [CTS]. □

In particular, $d_2^{0,1} = 0$. Next, we have the following

Proposition 2.5 *If K contains the cyclotomic field $\mathbb{Q}(\mu_d)$, and the subgroup of $K^\times/K^{\times d}$ generated by a_1, a_2, a_3 is isomorphic to $(\mathbb{Z}/d)^3$, then $\mathrm{H}^1(K, \mathrm{Pic}(X_{\overline{K}}))$ is the cyclic group of order d if d is odd, and $d/2$ if d is even.*

Proof. This is [GS, Thm. 1.2 (1)]. □

In this case the differential $d_2^{1,1}$ is injective 'generically', that is, when a_1, a_2, a_3 are independent variables over a number field k, but is zero after specialisation, that is, after making a_1, a_2, a_3 elements of k. Since $\mathrm{Br}_1(X) = \mathrm{Br}_0(X)$, 'universal Brauer classes do not exist' in this situation.

To show that $d_2^{1,1}$ is injective, one computes that $d_2^{1,1}$ sends the generator of $\mathrm{H}^1(K, \mathrm{Pic}(X_{\bar{K}}))$ to the image of twice the triple cup-product

$$2[a_1] \cup [a_2] \cup [a_3] \in \mathrm{H}^3(K, \mu_d^{\otimes 3})$$

under the map

$$\mathrm{H}^3(K, \mu_d^{\otimes 3}) \cong \mathrm{H}^3(K, \mu_d) \to \mathrm{H}^3(K, K^\times)$$

given by the choice of a primitive d-th root of unity. Here $[a] \in \mathrm{H}^1(K, \mu_d)$ is the class of the μ_d-torsor $x^d = a$, where $a \in K^\times$. One shows that this image has order d or $d/2$ (depending on the parity of d) by taking successive residues as in the following diagram, where we assume without loss of generality that k is algebraically closed:

$$\begin{tikzcd}
\mathrm{H}^3(K, \overline{K}^\times) \arrow[r] & \mathrm{H}^2(k(a_1,a_2), \mathbb{Q}/\mathbb{Z}) \arrow[r, "\cong"] & \mathrm{H}^2(k(a_1,a_2), \overline{k(a_1,a_2)}^\times) \arrow[r] & \mathrm{H}^1(k(a_1), \mathbb{Q}/\mathbb{Z}) \\
\mathrm{H}^3(K, \mu_d) \arrow[r] \arrow[u] & \mathrm{H}^2(k(a_1,a_2), \mathbb{Z}/d) \arrow[r, "\cong"] \arrow[u] & \mathrm{H}^2(k(a_1,a_2), \mu_d) \arrow[r] \arrow[u] & \mathrm{H}^1(k(a_1), \mathbb{Z}/d) \arrow[u, hook]
\end{tikzcd}$$

Note that the square in the middle is commutative, whereas the two outer squares are anticommutative. See [GS, §3] for details.

Remark 2.6 Using a theorem of Bright–Browning–Loughran and a follow-up paper of Bright, we obtain, under the above assumptions, that for 100% of k-points[3] $P = (c_0 : c_1 : c_2 : c_3) \in \mathbb{P}_k^3(k)$ such that the surface $X_P \subset \mathbb{P}_k^3$ given by

$$c_0 x_0^d + c_1 x_1^d + c_2 x_2^d + c_3 x_3^d = 0$$

is everywhere locally soluble, X_P is a counterexample to weak approximation and, moreover, $\mathrm{Br}_1(X)$ does not obstruct the Hasse principle on X_P.

In the assumptions of Theorem 2.4 we have $\mathrm{Br}(X_{\overline{K}})^K = 0$. This vanishing is a particular case of a general situation for certain *isotrivial* varieties. Indeed, we have the following result for cyclic twists.

[3]with respect to the usual height on $\mathbb{P}_k^3$

Theorem 2.7 *Let k be a field of characteristic zero. Let $K = k(t)$ and $L = k(\sqrt[d]{t})$. Let $c\colon \Gamma_K \to \mu_d$ be the 1-cocycle given by the action of Γ_K on $\sqrt[d]{t}$. Let Y be a smooth, projective, geometrically integral variety over k with an action of the group k-scheme μ_d. Suppose that $Y_{\bar{k}}$ satisfies the following conditions: for all primes ℓ the group $\mathrm{H}^i_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell)$ is torsion-free for $i = 2, 3$, and*

(a) $\mathrm{H}^1(\mu_d, \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell)) = 0$;

(b) *the cycle class map* $c_1\colon (\mathrm{NS}\,(Y_{\bar{k}}) \otimes \mathbb{Q}_\ell)^{\mu_d} \to \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mathbb{Q}_\ell)^{\mu_d}$ *is an isomorphism.*

Let X be the (L/K)-twist of Y_K by the 1-cocycle $c\colon \Gamma_K \to \mu_d$. Then $\mathrm{Br}(X_{\overline{K}})^K = 0$.

As a corollary, we get that $\mathrm{Br}(X_{\overline{K}})^K = 0$ in the situation of Theorem 2.4. Indeed, let $Y \subset \mathbb{P}^3_F$ be the surface with equation

$$x_0^d + s_1 x_1^d + x_2^d + s_2 x_3^d = 0, \tag{6}$$

where $s_1, s_2 \in F^\times$. The integral ℓ-adic étale cohomology groups of smooth surfaces in $\mathbb{P}^3_{\overline{F}}$ are torsion-free. The group scheme μ_d acts on Y so that x_2 and x_3 are multiplied by the same d-th root of 1, and x_0 and x_1 are not altered. By [GS, Proof of Thm. 2.2] this action satisfies the conditions of Theorem 2.7. (This is non-trivial and is based on topological arguments of F. Pham and A. Degtyarev.) The surface X over $K = F(t)$ obtained by twisting Y as described in Theorem 2.7, is given by

$$x_0^d + s_1 x_1^d + t(x_2^d + s_2 x_3^d) = 0.$$

Suppose that $F = k(s_1, s_2)$ is a purely transcendental extension of a field k of transcendence degree 2. Writing $a_1 = s_1$, $a_2 = t$, $a_3 = s_2 t$, we see that X is the surface over $K = k(a_1, a_2, a_3)$ given by (5). Theorem 2.4 gives $\mathrm{Br}(X_{\overline{K}})^K = 0$.

Proof of Theorem 2.7. We have $K \subset K\bar{k} \subset \overline{K}$, hence the map $\mathrm{Br}(X) \to \mathrm{Br}(X_{\overline{K}})$ factors through $\mathrm{Br}(X_{K\bar{k}}) \to \mathrm{Br}(X_{\overline{K}})$. Thus we can assume without loss of generality that k is algebraically closed.

Take any prime ℓ. There is a natural commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{NS}\,(X_{\overline{K}})/\ell^n)^{\Gamma_K} \arrow[r] & \mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mu_{\ell^n})^{\Gamma_K} \arrow[r] & \mathrm{Br}(X_{\overline{K}})[\ell^n]^{\Gamma_K} & \\
& & \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \arrow[r] \arrow[u] & \mathrm{Br}(X)[\ell^n] \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

We need to prove that the right-hand vertical map is zero for all n. For this it is enough to prove that the middle map in the top row is an isomorphism.

Since $X_L \cong Y_L$ and $k = \bar{k}$, the action of Γ_K on $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/\ell^n)$ is identified with the action of μ_d on $\mathrm{H}^i_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}/\ell^n)$, for all primes ℓ and all integers $n \geq 1$, $i \geq 0$. A similar statement holds for $\mathrm{NS}\,(X_{\overline{K}})$. Thus it is enough to prove that the cycle class map

$$(\mathrm{NS}\,(Y_{\bar{k}})/\ell^n)^{\mu_d} \to \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mu_{\ell^n})^{\mu_d}$$

is an isomorphism. This map fits into the following commutative diagram

$$\begin{tikzcd}
(\mathrm{NS}\,(Y_{\bar{k}}) \otimes \mathbb{Z}_\ell)^{\mu_d} \arrow[r, hook] \arrow[d] & \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell(1))^{\mu_d} \arrow[d] \\
(\mathrm{NS}\,(Y_{\bar{k}})/\ell^n)^{\mu_d} \arrow[r, hook] & \mathrm{H}^2_{\text{ét}}(Y_{\bar{k}}, \mu_{\ell^n})^{\mu_d}
\end{tikzcd}$$

Since $\mathrm{H}^i_{\text{ét}}(Y_{\bar{k}}, \mathbb{Z}_\ell(1))$ is torsion-free for $i = 2$ and $i = 3$, by condition (a) the right-hand vertical map is surjective. By condition (b) the top horizontal arrow is an isomorphism. The commutativity of the diagram implies that the bottom horizontal arrow is an isomorphism. $\square$

2.3 Semiabelian varieties

Another approach to calculating $\mathrm{Br}(X)$ is to use the Kummer sequence over k (with n coprime to $\mathrm{char}(k)$)

$$0 \longrightarrow \mathrm{Pic}(X)/n \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mu_n) \longrightarrow \mathrm{Br}(X)[n] \longrightarrow 0$$

and determine $\mathrm{H}^2_{\text{ét}}(X, \mu_n)$ from the Leray spectral sequence

$$E_2^{pq} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(X^{\mathrm{s}}, \mu_n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mu_n). \tag{7}$$

Here we take the view that $\mathrm{Pic}(X)$, being a subgroup of $\mathrm{Pic}(X_{\bar{k}})^\Gamma$ of finite index, is easier to determine in practice than the more mysterious $\mathrm{Br}(X)$. When X has a k-point, e.g., when $X = A$ is an abelian variety, we have $\mathrm{Pic}(X) = \mathrm{Pic}(X_{\bar{k}})^\Gamma$. When $X = T$ is a torus, then $\mathrm{Pic}(T_{\bar{k}}) = 0$ and $\mathrm{Pic}(T) \cong \mathrm{H}^1(k, \widehat{T})$.

Let us denote the differentials in this spectral sequence by $\delta_i^{p,q}\colon E_i^{p,q} \to E_i^{p+i,q-i+1}$.

The following statement links $d_2^{0,2}$, $\delta_2^{0,2}$, and ∂, under a mild assumption on the Néron–Severi group.

Proposition 2.8 *Let X be a proper, smooth and geometrically integral variety over a field k of characteristic zero such that $\mathrm{NS}\,(X_{\bar{k}})$ is torsion-free. Assume that Γ acts trivially on $\mathrm{NS}\,(X_{\bar{k}})$. Let A be the Albanese variety of X. For any $n \geq 1$ and any $x \in \mathrm{Br}(X_{\bar{k}})^\Gamma[n]$ the following statements hold.*

(a) $\partial(x) = 0$ *if and only if x lifts to an element of $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mu_n)^\Gamma$.*

(b) $d_2^{0,2}(x) = 0$ *if and only if x lifts to an element of $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mu_n)^\Gamma$ contained in the kernel of the composition*

$$\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mu_n)^\Gamma \xrightarrow{\delta_2^{0,2}} \mathrm{H}^2(k, A^\vee[n]) \to \mathrm{H}^2(k, A^\vee). \tag{8}$$

Proof. From the construction of ∂ we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Br}(X_{\bar{k}})^\Gamma[n] \arrow[r, "d_2^{0,2}"] \arrow[d] \arrow[dr, "\partial"] & \mathrm{H}^2(k, \mathrm{Pic}(X_{\bar{k}})) \arrow[d] \\
\mathrm{H}^1(k, \mathrm{NS}\,(X_{\bar{k}})/n) \arrow[r, hook] & \mathrm{H}^2(k, \mathrm{NS}\,(X_{\bar{k}}))
\end{tikzcd}$$

The left-hand vertical map comes from the Kummer exact sequence. The bottom arrow is the Bockstein map attached to multiplication by 2^m on the torsion-free abelian group $\mathrm{NS}\,(X_{\bar{k}})$; it is injective since $\mathrm{H}^1(k, \mathrm{NS}\,(X_{\bar{k}})) = 0$. Part (a) immediately follows from the diagram.

Let us prove (b). Suppose that x lifts to an element of $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mu_n)^\Gamma$ contained in the kernel of (8). By the functoriality of the Leray spectral sequence with respect to the map $\mu_n \to \mathbb{G}_m$ we then have $d_2^{0,2}(x) = 0$. Conversely, suppose that $d_2^{0,2}(x) = 0$. By part (a), x lifts to some $y \in \mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mu_n)^\Gamma$. Let $z \in \mathrm{H}^2(k, A^\vee)$ be the image of y under the map (8). The functoriality of the Leray spectral sequence with respect to the map $\mu_{2^m} \to \mathbb{G}_m$ gives that z goes to $d_2^{0,2}(x) = 0$ in $\mathrm{H}^2(k, \mathrm{Pic}(X_{\bar{k}}))$. Now our condition $\mathrm{H}^1(k, \mathrm{NS}\,(X_{\bar{k}})) = 0$ implies that $z = 0$. $\square$

We now investigate the computation of $\mathrm{H}^i_{\text{ét}}(A, \mathbb{Z}/n)$, where A is a semiabelian variety and n is not divisible by $\mathrm{char}(k)$, via the spectral sequence (7). The fact that $[-1]$ acts as $(-1)^i$ on $\mathrm{H}^i_{\text{ét}}(A^{\mathrm{s}}, \mathbb{Z}/n) = \wedge^i \mathrm{H}^1_{\text{ét}}(A^{\mathrm{s}}, \mathbb{Z}/n)$ implies that all differentials $\delta_2^{p,q}$ are zero when n is odd. Thus the interesting problem is to describe what is going on when $n = 2^m$. The answer is simpler for the coefficients $\mathbb{Z}_2$.

For a free finitely generated $\mathbb{Z}_2$-module M we write $Q^2(M)$ for the $\mathbb{Z}_2$-module of quadratic functions on $M^\vee = \mathrm{Hom}_{\mathbb{Z}_2}(M, \mathbb{Z}_2)$, that is, functions $f \colon M^\vee \to \mathbb{Z}_2$ such that $f(x+y) - f(x) - f(y)$ is a bilinear function of x and y. (An instructive example of a quadratic function on $\mathbb{Z}_2$ is $(x^2 - x)/2$.) There is a natural injection $M \to Q^2(M)$ sending an element of M to the linear function on $M^\vee$ that it defines. The cokernel of this map is $\mathrm{Hom}_{\mathbb{Z}_2}(S^2(M^\vee), \mathbb{Z}_2)$. One immediately checks (for example, by choosing a basis of M and the dual basis of $M^\vee$) that under the natural pairing

$$M^{\otimes 2} \times (M^\vee)^{\otimes 2} \to \mathbb{Z}_2$$

the $\mathbb{Z}_2$-submodules $(M \otimes M)^{S_2} = \langle m \otimes m | m \in M \rangle \subset M^{\otimes 2}$ and $\langle a \otimes b - b \otimes a | a, b \rangle \subset (M^\vee)^{\otimes 2}$ are exact annihilators of each other. Thus there is a canonical isomorphism $\mathrm{Hom}_{\mathbb{Z}_2}(S^2(M^\vee), \mathbb{Z}_2) \cong (M \otimes M)^{S_2}$. We obtain a canonical exact sequence

$$0 \to M \to Q^2(M) \to M^{\otimes 2} \to \wedge^2 M \to 0. \tag{9}$$

The $\mathbb{Z}_2$-module $Q^2(M)$ contains the submodule of quadratic forms $M^\vee \to \mathbb{Z}_2$. The map $Q^2(M) \to M^{\otimes 2}$ sends a quadratic form to the associated bilinear form. This is an isomorphism onto the $\mathbb{Z}_2$-submodule $\langle a \otimes b + b \otimes a | a, b \in M \rangle$ of $\langle m \otimes m | m \in M \rangle$. If we pass to the quotients, we obtain from (9) an equivalent extension

$$0 \to M \xrightarrow{[2]} M \to (M^{\otimes 2})_{S_2, \mathrm{sgn}} \to \wedge^2 M \to 0, \tag{10}$$

where the middle arrow is given by $m \mapsto m \otimes m$. The chain maps are $Q^2(M) \to M$, which sends a quadratic function f to the linear function $4f(m) - f(2m)$, and the natural map $M^{\otimes 2} \to (M^{\otimes 2})_{S_2, \mathrm{sgn}}$ to the maximal quotient on which S_2 acts by the sign character.

Theorem 2.9 (A. Petrov) *Let k be a field of characteristic different from 2. Let A be a semiabelian variety over k. Consider the spectral sequence*

$$E_2^{pq} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(A^{\mathrm{s}}, \mathbb{Z}_2)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(A, \mathbb{Z}_2).$$

Write $M = \mathrm{H}^1_{\text{ét}}(A^{\mathrm{s}}, \mathbb{Z}_2)$ so that $\mathrm{H}^i_{\text{ét}}(A^{\mathrm{s}}, \mathbb{Z}_2) = \wedge^i M$ for $i \geq 0$. The differential

$$\delta_2^{p,2} \colon \mathrm{H}^p_{\text{cont}}(\Gamma, \wedge^2 M) \to \mathrm{H}^{p+2}_{\text{cont}}(\Gamma, M)$$

equals (up to sign) the connecting map of the 2-extension of Γ-modules (9).

Proof. The 2-part of $\pi_1(A^{\mathrm{s}}, 0)$ is the Tate module $T_2(A)$. Hence we have a canonical morphism in the derived category of bounded below complexes of continuous Γ-modules

$$R\Gamma_{\text{cont}}(M^\vee, \mathbb{Z}_2) \to Rp_*(\mathbb{Z}_2),$$

where $p \colon A \to \mathrm{Spec}(k)$ is the structure morphism. It induces identity maps on the cohomology groups, which are canonically isomorphic to $\wedge^i M$, so this is a quasi-isomorphism.

Thus we can calculate $Rp_*(\mathbb{Z}_2)$ using the standard bar complex

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{d_0=0} \mathrm{Func}(M^\vee, \mathbb{Z}_2) \xrightarrow{d_1} \mathrm{Func}(M^\vee \times M^\vee, \mathbb{Z}_2) \longrightarrow \ldots, \qquad (11)$$

where Func is the $\mathbb{Z}_2$-module of continuous functions. The differential d_1 sends a function $f \colon M^\vee \to \mathbb{Z}_2$ to the function of two arguments $f(x+y) - f(x) - f(y)$. The inclusion $Q^2(M) \subset \mathrm{Func}(M^\vee, \mathbb{Z}_2)$ gives rise to a commutative diagram

$$\begin{tikzcd}
Q^2(M) \arrow[r] \arrow[d] & M^{\otimes 2} \arrow[d] \\
\mathrm{Func}(M^\vee, \mathbb{Z}_2) \arrow[r, "d_1"] & \mathrm{Func}(M^\vee \times M^\vee, \mathbb{Z}_2)
\end{tikzcd}$$

The right-hand vertical map is the inclusion of bilinear functions on $M^\vee \times M^\vee$ into all continuous functions. The commutativity is immediate from the definitions of maps. The induced maps on the cohomology groups are isomorphisms $M \xrightarrow{\sim} \mathrm{H}^1(M^\vee, \mathbb{Z}_2)$ and $\wedge^2 M \xrightarrow{\sim} \mathrm{H}^2(M^\vee, \mathbb{Z}_2)$ (since the last group is the image of the cup-product pairing on $\mathrm{H}^1(M^\vee, \mathbb{Z}_2)$, which in this case is alternating). This gives a quasi-isomorphism between the two-term complex $Q^2(M) \to M^{\otimes 2}$ and the truncation $\tau_{[1,2]} Rp_*(\mathbb{Z}_2)$. $\square$

The description of the same differential with coefficients $\mathbb{Z}/\ell^m$ is a little more complicated. Let $\mathcal{A}$ be the abelian category of $\mathbb{Z}_2$-modules with continuous action of Γ, and let $\mathcal{D}(\mathcal{A})$ be the bounded derived category of $\mathcal{A}$. When N is a free finitely generated $\mathbb{Z}_2$-module with continuous action of Γ, we shall denote by Bock the morphism $N/2^m \to N[1]$ defined by the exact sequence

$$0 \to N \xrightarrow{[2^m]} N \to N/2^m \to 0.$$

For $i \geq 1$, abusing notation, we shall also denote by Bock for the composition $N/2^m \to N[1] \to (N/2^i)[1]$. This map is defined by the exact sequence

$$0 \to N/2^i \to N/2^{m+i} \to N/2^m \to 0,$$

which is the push-out of the previous exact sequence by the map $N \to N/2^i$.

Recall that $M = \mathrm{H}^1_{\text{ét}}(A^{\mathrm{s}}, \mathbb{Z}_2)$ so that $\mathrm{H}^i_{\text{ét}}(A^{\mathrm{s}}, \mathbb{Z}/2^m) \cong \wedge^i(M/2^m)$ for $i \geq 0$. We have a crucial exact sequence of Γ-modules

$$0 \to M/2 \to S^2(M/2) \to \wedge^2(M/2) \to 0. \tag{12}$$

Note that $S^2(M/2) \cong ((M/2)^{\otimes 2})_{S_2,\mathrm{sgn}}$. We write $\alpha\colon \wedge^2 (M/2) \to (M/2)[1]$ for the morphism in $\mathcal{D}(\mathcal{A})$ defined by (12). Consider the following morphisms in $\mathcal{D}(\mathcal{A})$:

$$\wedge^2(M/2^m) \longrightarrow \wedge^2(M/2) \xrightarrow{\alpha} (M/2)[1] \xrightarrow{\mathrm{Bock}} (M/2^m)[2], \tag{13}$$

$$\wedge^2(M/2^m) \xrightarrow{\mathrm{Bock}} \wedge^2(M/2)[1] \xrightarrow{\alpha[1]} (M/2)[2] \longrightarrow (M/2^m)[2]. \tag{14}$$

The unmarked arrows in (13) and (14) are the natural surjection and the natural injection, respectively.

Corollary 2.10 (A. Petrov) *Let k be a field of characteristic different from 2. Let A be a semiabelian variety over k. Consider the spectral sequence*

$$E_2^{pq} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(A^{\mathrm{s}}, \mathbb{Z}/2^m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(A, \mathbb{Z}/2^m).$$

The differential $\delta_2^{p,2}\colon \mathrm{H}^p_{\mathrm{cont}}(\Gamma, \wedge^2(M/2^m)) \to \mathrm{H}^{p+2}_{\mathrm{cont}}(\Gamma, M/2^m)$ is (up to sign) the difference of the maps obtained by applying $\mathrm{H}^p_{\mathrm{cont}}(\Gamma, -)$ to (13) and (14).

Proof. This can be proved by an explicit calculation with complexes, see [PSk]. □

Tori

For any torus T over a field k we have $\mathrm{Pic}(T_{\bar{k}}) = 0$. Then the Kummer sequence gives rise to an isomorphism

$$\kappa\colon \mathrm{H}^2_{\text{ét}}(T_{\bar{k}}, \mu_n) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(T_{\bar{k}}, \mathbb{G}_m)[n].$$

Using the isomorphism of Γ-modules $\bar{k}[T]^\times \cong \bar{k}^\times \times \widehat{T}$, we also deduce from the Kummer sequence a natural isomorphism

$$\rho\colon \widehat{T}/n \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(T_{\bar{k}}, \mu_n).$$

The following proposition, combined with Corollary 2.10, answers the question raised on top of [CTS21, p. 220].

Proposition 2.11 *Let T be a torus over a field k of characteristic zero. Let $\bar{d}_3^{0,2}$ be the composition*

$$\mathrm{H}^2_{\text{ét}}(T_{\bar{k}}, \mathbb{G}_m)^\Gamma \xrightarrow{d_3^{0,2}} \mathrm{H}^3(k, \bar{k}[T]^\times) \to \mathrm{H}^3(k, \widehat{T}),$$

where the second map is induced by the projection $\bar{k}[T]^\times \cong \bar{k}^\times \times \widehat{T} \to \widehat{T}$. The restriction of $\bar{d}_3^{0,2}$ to the n-torsion subgroup is (up to sign) the composition

$$\mathrm{H}^2_{\text{ét}}(T_{\bar{k}}, \mathbb{G}_m)[n]^\Gamma \xrightarrow{\kappa^{-1}} \mathrm{H}^2_{\text{ét}}(T_{\bar{k}}, \mu_n)^\Gamma \xrightarrow{\delta_2^{0,2}} \mathrm{H}^2(k, \mathrm{H}^1_{\text{ét}}(T_{\bar{k}}, \mu_n)) \xrightarrow{\rho^{-1}} \mathrm{H}^2(k, \widehat{T}/n) \xrightarrow{\text{Bock}} \mathrm{H}^3(k, \widehat{T}).$$

Looking at the action of $[-1]\colon T \to T$ we see that $2\bar{d}_3^{0,2} = 0$. Thus the differential $d_3^{0,2}$ is zero on $\mathrm{Br}(T_{\bar{k}})^\Gamma_{\text{odd}}$, so that the natural map $\mathrm{Br}(T)_{\text{odd}} \to \mathrm{Br}(T_{\bar{k}})^\Gamma_{\text{odd}}$ is surjective.

3 Computing the Brauer–Manin set

For a variety X over a field k, an element $A \in \mathrm{Br}(X)$ and a k-algebra R we define the *evaluation map* $\mathrm{ev}_A\colon X(R) \to \mathrm{Br}(R)$ by $\mathrm{ev}_A(P) = A(P)$.

Let k be a number field, let Ω_k be the set of all places of k, and let $\mathbf{A}_k$ be the ring of adèles of k. For a subset $S \subset \Omega_k$ we denote by $\mathbf{A}_k^S$ the adèles without components for the places in S.

The set of adelic points $X(\mathbf{A}_k)$ is a locally compact Hausdorff topological space. If X is proper, then $X(\mathbf{A}_k) = \prod_v X(k_v)$ is a product of compact spaces, so $X(\mathbf{A}_k)$ is compact by Tychonoff's theorem.

By local class field theory we have the local invariant $\mathrm{inv}_v\colon \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$, which is an isomorphism for non-archimedean v, injective onto $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ if $k_v \simeq \mathbb{R}$, and zero if $k_v \simeq \mathbb{C}$. The local invariant allows us to think of ev_A as a map $X(k_v) \to \mathbb{Q}/\mathbb{Z}$. The standard spreading-out argument [CTS21, Prop. 13.3.1] based on the fact that $\mathrm{Br}(\mathcal{O}_K) = 0$, where $\mathcal{O}_K$ is the ring of integers of a p-adic field K, shows that the map

$$\mathrm{ev}_A\colon X(\mathbf{A}_k) \to \prod_{v \in \Omega_k} \mathbb{Q}/\mathbb{Z}$$

factors through the direct sum $\bigoplus_{v \in \Omega_k} \mathbb{Q}/\mathbb{Z}$. Thus we have a well defined, continuous pairing, called the *Brauer–Manin pairing*,

$$X(\mathbf{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z} \tag{15}$$

sending $(M_v)_{v \in \Omega}$ and $A \in \mathrm{Br}(X)$ to the sum $\sum_{v \in \Omega_k} \mathrm{ev}_A(M_v) \in \mathbb{Q}/\mathbb{Z}$ (which is actually a finite sum). The *Brauer–Manin set* $X(\mathbf{A}_k)^{\mathrm{Br}}$ is the left kernel of (15). We can also consider larger sets $X(\mathbf{A}_k)^B$, where $B \subset \mathrm{Br}(X)$.

It is natural to ask: which places of k and which elements of $\mathrm{Br}(X)$ are actually needed to define the Brauer–Manin set?

Definition 3.1 *A place v of k is* **irrelevant** *if* $\mathrm{ev}_A\colon X(k_v) \to \mathbb{Q}/\mathbb{Z}$ *is constant for all $A \in \mathrm{Br}(X)$. In the opposite case, v will be called* **relevant**. *If v can be extended to a relevant place of a finite extension of k, then v will be called* **potentially relevant**.

For example, complex places are irrelevant. A real place v such that $X(k_v)$ is connected, is irrelevant.

Lemma 3.2 *Assume $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$. For a set S of places of k the following conditions are equivalent.*

(i) *All primes not in S are irrelevant.*

(ii) *The set $X(\mathbf{A}_k^S)$ is a direct factor of $X(\mathbf{A}_k)^{\mathrm{Br}}$.*

In this case $X(\mathbf{A}_k)^{\mathrm{Br}} = Z \times X(\mathbf{A}_k^S)$, where Z is a closed subset of $X(\mathbf{A}_k^{\Omega_k \setminus S})$.

Proof. Assume (i). Then the evaluation map away from S defines a homomorphism $\mathrm{Br}(X) \to \bigoplus_{v \notin S} \mathbb{Q}/\mathbb{Z}$. Let B be its kernel.

Take any $(P_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$. For any $A \in \mathrm{Br}(X)$ we have $\sum_{v \in \Omega_k} \mathrm{inv}_v(A(P_v)) = 0$, so there exists an element $A_0 \in \mathrm{Br}(k)$ with $\mathrm{inv}_v(A_0) = \mathrm{inv}_v(A(P_v))$ for all v. Then $A - A_0 \in B$, so $\mathrm{Br}(X)$ is generated by B and the image of $\mathrm{Br}(k) \to \mathrm{Br}(X)$. Hence the Brauer–Manin pairing (15) factors through a pairing $X(\mathbf{A}_k^{\Omega_k \setminus S}) \times B \to \mathbb{Q}/\mathbb{Z}$. Thus $X(\mathbf{A}_k)^{\mathrm{Br}}$ is the product of the left kernel of this pairing and $X(\mathbf{A}_k^S)$.

Assume (ii). Take any $(P_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$. If (i) is false, then there are an element $A \in \mathrm{Br}(X)$ and a place $w \notin S$ such that there exists a $(P'_w) \in X(k_w)$ with $\mathrm{ev}_A(P'_w) \neq \mathrm{ev}_A(P_w)$. Replacing P_w in the adelic point (P_v) by P'_w we get an element not in $X(\mathbf{A}_k)^{\mathrm{Br}}$, which contradicts (ii). $\square$

Definition 3.3 *Let $B \subset \mathrm{Br}(X)$ be a subgroup. We call B* **irrelevant** *if it does not obstruct weak approximation, that is, if we have $X(\mathbf{A}_k)^B = X(\mathbf{A}_k)$. If $X(\mathbf{A}_k)^B = X(\mathbf{A}_k)^{\mathrm{Br}}$, then we say that B* **captures** *the Brauer–Manin obstruction.*

For example, $\mathrm{Br}_0(X)$ is an irrelevant subgroup. There is a maximal irrelevant subgroup of $\mathrm{Br}(X)$.

We would like to address the following questions:

Question 3.4 Which places $v \in \Omega_k$ are irrelevant?

In other words, which primes show up in the Brauer–Manin obstruction?

Question 3.5 Which subgroups of $\mathrm{Br}(X)$ are irrelevant and which capture the Brauer–Manin obstruction?

If $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$, then the topological space $X(\mathbf{A}_k)$ has an open covering by the open subsets $X(\mathbf{A}_k) \setminus X(\mathbf{A}_k)^b$, for $b \in \mathrm{Br}(X)$. If X is proper, then $X(\mathbf{A}_k)$ is compact, so there is a finite subcovering. If X is smooth, then $\mathrm{Br}(X)$ is a torsion group, so there is a finite subgroup $B \subset \mathrm{Br}(X)$ such that $X(\mathbf{A}_k)^B = \emptyset$, so B captures the Brauer–Manin obstruction. In a recent paper by J. Berg, C. Pagano, B. Poonen, M. Stoll, N. Triantafillou, B. Viray, and I. Vogt, it is shown that the number of generators of B cannot be bounded already for conic bundles over $\mathbb{P}^1_k$, for any global field k of characteristic different from 2.

3.1 Relevant and irrelevant places

If X is proper, then for any $A \in \mathrm{Br}(X)$ the map ev_A is identically zero on $X(k_v)$ outside of finitely many places. When $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite, this immediately gives the rough general shape of the Brauer–Manin set:

Lemma 3.6 *Let X be a proper variety over a number field k. If $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite, then there exists a finite set S of places of k such that all places not in S are irrelevant, and we have*

$$X(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{v \notin S} X(k_v) \tag{16}$$

for an open and closed set $Z \subset \prod_{v \in S} X(k_v)$.

Proof. This is [CTS21, Lemma 13.3.13]. □

A natural question is: how small can S be?

Question 3.7 (Swinnerton-Dyer) *Let X be a smooth, projective and geometrically integral variety over a number field k such that $\mathrm{Pic}(X_{\bar{k}})$ is a finitely generated torsion-free abelian group. Are the primes of good reduction irrelevant?*

It is likely that Swinnerton-Dyer was actually asking about $k = \mathbb{Q}$. The following result gives sufficient conditions under which the answer is positive.

Theorem 3.8 *Let X be a smooth, projective and geometrically integral variety over a number field k such that $\mathrm{Pic}(X_{\bar{k}})$ is a finitely generated torsion-free abelian group. Let S be a finite set of primes of k containing the primes of bad reduction. Assume that the transcendental Brauer group $\mathrm{Br}(X_{\bar{k}})^k = \mathrm{Br}(X)/\mathrm{Br}_1(X)$ is a finite abelian group of order invertible outside S. Then the primes outside of S are irrelevant.*

Proof. This is [CTS21, Thm. 13.3.15]. Let us sketch a proof. Let v be a place of good reduction. Let $k_{v,\mathrm{nr}} \subset \overline{k}_v$ be the maximal unramified extension, and let $X_{\mathrm{nr}} = X \times_k k_{v,\mathrm{nr}}$.

A crucial observation is that any element of $\mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathrm{nr}})]$ is a sum of a constant class from $\mathrm{Br}(k_v)$ and an element of $\mathrm{Br}(\mathcal{X})$, the Brauer group of a smooth proper model $\mathcal{X}$ of X, see [CTS21, Lemma 10.4.1]. By properness, a k_v-point of X extends to an $\mathcal{O}_v$-point of $\mathcal{X}$. But $\mathrm{Br}(\mathcal{O}_v)=0$, so such classes have constant evaluation map.

This kernel contains all the algebraic classes in $\mathrm{Br}(X)$. Indeed, take any prime $\ell\neq p$, where p is the residue characteristic of K. By the smooth base change theorem the inertia subgroup $I=\mathrm{Gal}(\overline{k}_v/k_{v,\mathrm{nr}})\subset\mathrm{Gal}(\overline{k}_v/k_v)$ acts trivially on $\mathrm{H}^2_{\text{ét}}(X_{\overline{k}_v},\mu_{\ell^n})$ for any $n\geq 1$, hence also on $\mathrm{H}^2_{\text{ét}}(X_{\overline{k}_v},\mathbb{Z}_\ell(1))$, and thus also on $\mathrm{Pic}(X_{\overline{k}_v})$ which in our assumptions has no torsion and so embeds into the last group by the Kummer sequence. The spectral sequence

$$\mathrm{H}^p(k_{v,\mathrm{nr}},\mathrm{H}^q_{\text{ét}}(X_{\overline{k}_v},\mathbb{G}_m))\Rightarrow\mathrm{H}^{p+q}_{\text{ét}}(X,\mathbb{G}_m)$$

gives an exact sequence

$$\mathrm{Br}(k_{v,\mathrm{nr}})\to\mathrm{Ker}[\mathrm{Br}(X_{\mathrm{nr}})\to\mathrm{Br}(X_{\overline{k}_v})]\to\mathrm{H}^1(I,\mathrm{Pic}(X_{\overline{K}}))=0.$$

We have $\mathrm{Br}(k_{v,\mathrm{nr}})=0$ since the residue field $\mathbb{F}$ is perfect. It follows that $\mathrm{Br}_1(X)$ is killed by base change to $k_{v,\mathrm{nr}}$.

Let $A\in\mathrm{Br}(X)[\ell^n]$, where $\ell\neq p$ and $n\geq 1$. Such classes are also killed by base change to $k_{v,\mathrm{nr}}$. From the Gysin exact sequence in étale topology we see that $\mathrm{Br}(\mathcal{X})[\ell^n]$ is the kernel of the residue map $r\colon\mathrm{Br}(X)[\ell^n]\to\mathrm{H}^1(\mathcal{X}_\mathbb{F},\mathbb{Z}/\ell^n)$, where $\mathcal{X}_\mathbb{F}$ is the special fibre of $\mathcal{X}\to\mathrm{Spec}(O_v)$. Since $\mathrm{Pic}(X_{\overline{k}_v})$ is torsion-free, using proper and smooth base change, we obtain that $\mathrm{H}^1(\mathcal{X}_{\overline{\mathbb{F}}},\mathbb{Z}/\ell^n)=0$. The standard spectral sequence then implies that

$$\mathbb{Z}/\ell^n\cong\mathrm{H}^1(\mathbb{F},\mathbb{Z}/\ell^n)\to\mathrm{H}^1(\mathcal{X}_\mathbb{F},\mathbb{Z}/\ell^n)$$

is an isomorphism. By functoriality of residues, the residue of $\alpha(P)\in\mathrm{Br}(k_v)[\ell^n]$, which gives the local invariant $\mathrm{inv}_v(\alpha(P))\in\frac{1}{\ell^n}\mathbb{Z}/\mathbb{Z}$, is the specialisation of the residue $r(A)\in\mathrm{H}^1(\mathcal{X}_\mathbb{F},\mathbb{Z}/\ell^n)$. Thus $\alpha(P)$ does not depend on P.

We have dealt with all cases, because our condition implies $\mathrm{Br}(X_{\bar{k}})^k\{p\}=0$. □

The assumption on the order of the transcendental Brauer group is satisfied if we assume, moreover, that $\mathrm{H}^2(X,\mathcal{O}_X)=0$ and that $\mathrm{H}^3_{\text{ét}}(X_{\bar{k}},\mathbb{Z}_\ell)$ is torsion-free for all ℓ coprime to the primes in S. This condition cannot be dropped, as the following result shows. The issue is the behaviour of p-primary elements of the Brauer group at a place of good reduction v of residue characteristic p. A new phenomenon in comparison to the prime-to-p torsion is the role of the absolute ramification index of k_v.

A variety Y over a perfect field of characteristic p is *ordinary* if $\mathrm{H}^j(Y,B^i_Y)=0$ for all i and j, where $B^i_Y=\mathrm{Im}[\Omega^{i-1}_Y\xrightarrow{d}\Omega^i_Y]$ is the sheaf of exact i-forms. A K3 surface Y over a finite field $\mathbb{F}$ of characteristic p is *ordinary* if and only if the trace of Frobenius acting on $\mathrm{H}^2_{\text{ét}}(Y_{\overline{\mathbb{F}}},\mathbb{Q}_\ell)$, $\ell\neq p$, is not divisible by p, see [BZ09, Lemma 1.1]. Equivalently, the absolute Frobenius acting on $\mathrm{H}^2(Y_{\overline{F}},\mathcal{O})$ is non-zero. For example, the Fermat quartic is ordinary if $p\equiv 1 \bmod 4$ and supersingular if $p\equiv 3 \bmod 4$.

Theorem 3.9 (Bright–Newton) *Let X be a smooth, projective and geometrically integral variety over a number field k such that $\mathrm{H}^2(X, \mathcal{O}_X) \neq 0$. Then every prime v of k of good, ordinary reduction, with residue characteristic p, is potentially relevant: there exist a finite extension k'/k, a place w of k' over v, and an element $A \in \mathrm{Br}(X_{k'})\{p\}$ such that the evaluation map $\mathrm{ev}_A \colon X(k'_w) \to \mathrm{Br}(k'_w)$ is non-constant.*

Proof. This is [BN23, Thm. C]. Here is a very brief sketch of the proof.

Let $i \colon \mathrm{Spec}(\mathbb{F}_v) \to \mathrm{Spec}(\mathcal{O}_v)$ and $j \colon \mathrm{Spec}(k_v) \to \mathrm{Spec}(\mathcal{O}_v)$ be the natural closed and open immersions, respectively. Let $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_v)$ be a smooth proper morphism with generic fibre X_v and special fibre $X_{\mathbb{F}_v}$. By an abuse of notation, we denote by i and j the embeddings of $X_{\mathbb{F}_v}$ and of X_v, respectively, into $\mathcal{X}$.

Let $\overline{k}_v$ be an algebraic closure of k_v. The ring of integers of $\overline{k}_v$ is the normalisation of $\mathcal{O}_v \subset k_v$ in $\overline{k}_v$. Let $\overline{\mathbb{F}}_v$ be its residue field. We denote by $\bar{i}$ and $\bar{j}$ the embeddings of $X_{\overline{\mathbb{F}}_v}$ and $X_{\overline{k}_v}$, respectively, into the pullback of $\mathcal{X}$ to the ring of integers of $\overline{k}_v$.

Consider the spectral sequence of p-adic vanishing cycles

$$\mathrm{H}^p_{\text{ét}}(X_{\overline{\mathbb{F}}_v}, \bar{i}^* R^q \bar{j}_* (\mathbb{Z}/p^r)(1)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X_{\overline{k}_v}, (\mathbb{Z}/p^r)(1)),$$

and similar sequences with coefficients in $\mathbb{Z}_p(1)$ and $\mathbb{Q}_p(1)$. Let $\mathrm{gr}^0 \mathrm{H}^2_{\text{ét}}(X_{\overline{k}_v}, \mathbb{Q}_p(1))$ be the image of

$$\mathrm{H}^2_{\text{ét}}(X_{\overline{k}_v}, \mathbb{Q}_p(1)) \to \mathrm{H}^0_{\text{ét}}(X_{\overline{\mathbb{F}}_v}, \bar{i}^* R^2 \bar{j}_* \mathbb{Q}_p(1))$$

Using the assumption that $X_{\mathbb{F}_v}$ is ordinary, the Hodge–Tate decomposition [BK86, Thm. (0.7)(iii)] gives an isomorphism of $\mathrm{Gal}(\overline{k}_v/k_v)$-modules

$$\mathrm{gr}^0 \mathrm{H}^2_{\text{ét}}(X_{\overline{k}_v}, \mathbb{Q}_p(1)) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \mathrm{H}^0(X_v, \Omega^2) \otimes_{k_v} \mathbb{C}_p(-2),$$

where $\mathrm{Gal}(\overline{k}_v/k_v)$ naturally acts on $\mathbb{C}_p$, the completion of $\overline{k}_v$. Thus the assumption $\mathrm{H}^2(X, \mathcal{O}_X) \neq 0$ implies that $\mathrm{gr}^0 \mathrm{H}^2_{\text{ét}}(X_{\overline{k}_v}, \mathbb{Q}_p(1)) \neq 0$. It follows that the image of

$$\mathrm{H}^2_{\text{ét}}(X_{\overline{k}_v}, (\mathbb{Z}/p^r)(1)) \to \mathrm{H}^0_{\text{ét}}(X_{\overline{\mathbb{F}}_v}, \bar{i}^* R^2 \bar{j}_* (\mathbb{Z}/p^r)(1))$$

is non-zero for some $r \geq 1$. Take an element of $\mathrm{H}^2_{\text{ét}}(X_{\overline{k}_v}, (\mathbb{Z}/p^r)(1))$ with non-zero image. It comes from $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, (\mathbb{Z}/p^r)(1))$, because the natural map between these groups is an isomorphism by proper base change. After a finite extension of k we may assume that it comes from $\mathrm{H}^2_{\text{ét}}(X, (\mathbb{Z}/p^r)(1))$, thus giving a desired Brauer class.

Let $K = k_v(X)$ and let K^{h} be the henselisation of K for the discrete valuation inherited from k_v. We have a spectral sequence of vanishing cycles

$$\mathrm{H}^p_{\text{ét}}(X_{\mathbb{F}_v}, i^* R^q j_* (\mathbb{Z}/p^r)(1)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X_v, (\mathbb{Z}/p^r)(1)).$$

A similar sequence in Galois cohomology is

$$\mathrm{H}^p(\mathbb{F}_v(X_{\mathbb{F}_v}), \mathrm{H}^q(K^{\mathrm{h}}_{\mathrm{nr}}, (\mathbb{Z}/p^r)(1))) \Rightarrow \mathrm{H}^{p+q}(K^{\mathrm{h}}, (\mathbb{Z}/p^r)(1)).$$

These sequences are compatible under restriction to the generic point, so we get a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^2_{\text{ét}}(X_v, (\mathbb{Z}/p^r)(1)) \arrow[r] \arrow[d] & \mathrm{H}^0_{\text{ét}}(X_{\mathbb{F}_v}, i^*R^2j_*(\mathbb{Z}/p^r)(1)) \arrow[d] \\
\mathrm{H}^2(K^{\mathrm{h}}, (\mathbb{Z}/p^r)(1)) \arrow[r] & \mathrm{H}^2(K^{\mathrm{h}}_{\mathrm{nr}}, (\mathbb{Z}/p^r)(1))
\end{tikzcd}$$

One proves that the right-hand vertical map is injective [BN23, Lemma 3.4]. (This is non-trivial. The case $r = 1$ is due to Bloch–Kato [BK86, Prop. 6.1 (i)] which was originally proved using Gabber's injectivity results for étale cohomology generalising work of Bloch–Ogus on the Gersten conjecture.) So our Brauer class gives an element of $\mathrm{H}^2(K^{\mathrm{h}}, (\mathbb{Z}/p^r)(1))$ with non-zero image in $\mathrm{H}^2(K^{\mathrm{h}}_{\mathrm{nr}}, (\mathbb{Z}/p^r)(1))$. Using the crucial relation between the refined Swan conductor and evaluation map [BN23, Thm. B], Bright and Newton show in [BN23, Thm. A] that Brauer elements that have constant evaluation maps over all extensions of k_v, give rise to elements that go to zero in $\mathrm{H}^2(K^{\mathrm{h}}_{\mathrm{nr}}, (\mathbb{Z}/p^r)(1))$. □

Thus, without assumption (iii) of Theorem 3.8, Question 3.7 has a negative answer. By being slightly more precise, the same argument can be used to show that in this result one can take $A \in \mathrm{Br}(X_K)[p]$, see [P, Thm. 4.5].

For a prime v of k we denote by p_v the residue characteristic of k_v and by e_v the absolute ramification index of k_v. Recall that if X has good reduction at v, we denote by $X_{\mathbb{F}_v}$ the special fibre of a smooth projective scheme over $\mathrm{Spec}(\mathcal{O}_v)$ with generic fibre $X \times_k k_v$. (It may depend on the choice of a model.)

Bright and Newton give an explicit lower bound on the set of irrelevant primes. For $A \in \mathrm{Br}(X)\{\ell\}$, $\ell \neq p_v$, the statement follows from the fact that $X_{\overline{\mathbb{F}}_v}$ has no connected unramied cyclic covering of degree ℓ.

Theorem 3.10 (Bright–Newton) *Let X be a smooth, projective and geometrically integral variety over a number field k such that $\mathrm{Pic}(X_{\bar{k}})$ is torsion-free. If v is a prime of good reduction for X such that $e_v < p_v - 1$ and $\mathrm{H}^0(X_{\mathbb{F}_v}, \Omega^1) = 0$, then v is irrelevant.*

Proof. This is [BN23, Thm. D]. One shows that in these assumptions all of $\mathrm{Br}(X)$ goes to zero in $\mathrm{Br}(K^{\mathrm{h}}_{\mathrm{nr}})$. For this one needs to show that the refined Swan conductor is zero on the higher terms of Kato's filtration $\mathrm{fil}_n\mathrm{Br}(X)$, $n \geq 1$. This is deduced from explicit formulae describing the action of multiplication by p on Kato's filtration fil_n and on the refined Swan conductor in terms of the Cartier operator on differential forms, see [BN23, Section 2], [P, Section 3]. □

In particular, for a K3 surface over $\mathbb{Q}$, odd primes of good reduction are irrelevant, see [BN23, Remark 7.5]. M. Pagano showed that for K3 surfaces the prime 2 can be relevant.

Exercise 3.11 (M. Pagano) The K3 surface $X \subset \mathbb{P}^3_{\mathbb{Q}}$ given by

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0$$

has good reduction at the prime 2. The class of the quaternion algebra

$$A := \left(\frac{z^3 + xw^2 + xyz}{x^3}, -\frac{z}{x} \right) \in \mathrm{Br}(\mathbb{Q}(X))$$

is contained in $\mathrm{Br}(X)$. Moreover, $\mathrm{ev}_A \colon X(\mathbb{Q}_2) \to \mathrm{Br}(\mathbb{Q}_2)$ is a non-constant function, so A gives an obstruction to weak approximation on X. See [P22, Thm. 1] for details and the relation to the ideas and constructions of [BN23].

In the ordinary case there is a somewhat stronger version.

Theorem 3.12 (M. Pagano) *Let X be a smooth, projective and geometrically integral variety over a number field k. Let v be a prime of k at which X has good ordinary reduction. Assume that $\mathrm{H}^0(X_{\mathbb{F}_v}, \Omega^1) = 0$ and $\mathrm{H}^1(X_{\overline{\mathbb{F}}_v}, \mathbb{Z}/p_v) = 0$. If p_v–1 does not divide e_v, then v is irrelevant.*

A K3 surface satisfies these conditions by a theorem of Rudakov and Shafarevich.

This can be compared to [Ier22, Prop. 8] which is a similar statement but allowing Brauer elements to be defined over k_v. The same statement is true if the special fibre is a surface in $\mathbb{P}^3_k$, e.g., a diagonal surface or a surface given by $f(x_0, x_1) = g(x_2, x_3)$, see [Ier22, Prop. 13].

Here is a result applicable to non-ordinary reduction of K3 surfaces.

Theorem 3.13 (M. Pagano, Thm. 1.4) *Let X be a K3 surface with good non-ordinary reduction at v. If $e_v \leq p$–1, then v is irrelevant.*

In contrast to the previous results, in the next statement, which applies to the elements of order coprime to the residue characteristic, bad reduction is allowed.

Theorem 3.14 [Ier23, Thm. A] *Let K be a p-adic field. Let X be a diagonal quartic surface in $\mathbb{P}^3_K$. Then for any $A \in \mathrm{Br}(X)$ of odd order not divisible by p the evaluation function is constant.*

The case $K = \mathbb{Q}_p$ was treated earlier in [IS15, §5.2]. Compare with an example from this paper when the order of $\mathcal{A}$ is $p = 5$.

3.2 Relevant and irrelevant Brauer elements

This section is to be written. It should cover results of Creutz–Viray, Creutz on the Brauer groups of abelian varieties, Skorobogatov–Zarhin on the Brauer group of Kummer varities attached to 2-coverings, examples for the odd torsion subgroup on the Brauer group.

3.3 Good reduction of K3 surfaces

It is conjectured that CM K3 surfaces have potential good reduction everywhere.

Proposition 3.15 *Diagonal quartic surfaces over number fields have potential good reduction everywhere (with an algebraic space model at the prime 2).*

Proof. It is enough to treat the Fermat quartic surface F over $\mathbb{Q}$ at the prime 2, because F obviously has good reduction modulo all odd primes.

Let C be the elliptic curve $u^2 = (v^2-1)(v^2-1/2)$ and let C' be the elliptic curve $y^2 = x^3 - 4x$, both over $\mathbb{Q}$. There is a unique (obvious) isogeny $C \to C' = C/K$ of degree $|K| = 2$. (K consists of the origin and a point or order two, which are the two points of C at infinity.) Let A be the quotient of $C' \times C'$ by the diagonal subgroup $K \subset K \times K$. By a result of Mizukami (see the appendix to [ISZ11]), there is an isomorphism $F_{\mathbb{Q}(\mu_8)} \cong \mathrm{Kum}(A)_{\mathbb{Q}(\mu_8)}$. Since C and C' are CM elliptic curves, they have potential good reduction at 2. Thus A has potential good reduction at 2 (for example, by Néron–Ogg–Shafarevich).

By Matsumoto [Mat] all Kummer surfaces attached to abelian surfaces with good supersingular reduction at 2 have potential good reduction (with an algebraic space model). In particular, diagonal quartic surfaces have potential good reduction everywhere. □

Explicit smooth proper models of the Fermat quartic surface in residue characteristic 2 (after an appropriate base change) are not known; it is not known if schemes suffice for this purpose. Also, we do not know how the reduced K3 surface looks like.

For completeness let us mention that in the case of good, non-supersingular reduction, Lazda and Skorobogatov [LS23] give a criterion of good reduction of Kummer surfaces modulo 2. They show that in this case good reduction with a scheme model is available.

4 Hasse principle for Kummer varieties

Research on rational points on surfaces fibred into curves of genus 1 (including many types of K3 surfaces) was initiated by Swinnerton-Dyer. His method sometimes allows to obtain results about the Hasse principle for rational points conditionally on the finiteness of the Tate–Shafarevich group of elliptic curves. This method was developed by Swinnerton-Dyer jointly with Colliot-Thélène and Skorobogatov, and was later simplified and generalised by Harpaz–Skorobogatov and Harpaz in the case of Kummer surfaces. Based on such evidence, Skorobogatov conjectured that the Brauer–Manin obstruction is the only obstruction for the local-to-global principle for rational points on K3 surfaces.

The aim of this section is to sketch a version of Swinnerton-Dyer's method that was recently found by Adam Morgan. It is more flexible, and thus is stronger than [HS16], on which it builds.

4.1 Cohomological preliminaries

References: Poonen–Rains [PR11], Harpaz–Skorobogatov [HS16]

Let k be a field of characteristic not equal to 2. Let A be an abelian variety of dimension $g \geq 2$ over k. There is a Weil pairing

$$e_2 \colon A[2] \times A^\vee[2] \to \{\pm 1\}.$$

It differs by sign from the same pairing for $A^\vee$.

Let $L = k(A[2])$ with $G = \mathrm{Gal}(L/k) \subset \mathrm{GL}(2g)(\mathbb{F}_2)$. Suppose that A has a principal polarisation λ defined over k. It allows us to define a bilinear pairing

$$e_2^\lambda \colon A[2] \times A[2] \to \{\pm 1\}$$

and a class $c_\lambda \in \mathrm{H}^1(G, A^\vee[2])$. Let us recall the definitions of these objects.

There is an exact sequence of Γ_k-modules

$$0 \longrightarrow A^\vee(\bar{k}) \longrightarrow \mathrm{Pic}(A_{\bar{k}}) \longrightarrow \mathrm{NS}\,(A_{\bar{k}}) \longrightarrow 0. \tag{17}$$

The antipodal involution $[-1] \colon A \to A$ induces an action of $\mathbb{Z}/2$ on $\mathrm{Pic}(A_{\bar{k}})$ which turns (17) into an exact sequence of $\mathbb{Z}/2$-modules. The induced action on $\mathrm{NS}\,(A_{\bar{k}})$ is trivial. The involution $[-1]_A$ induces the involution $[-1]_{A^\vee}$ on $A^\vee$. Since $A^\vee(\bar{k})$ is divisible, we obtain $\mathrm{H}^1(\mathbb{Z}/2, A^\vee(\bar{k})) = 0$. Thus the long exact sequence of cohomology gives an exact sequence

$$0 \longrightarrow A^\vee[2] \longrightarrow \mathrm{Pic}(A_{\bar{k}})^{[-1]^*} \longrightarrow \mathrm{NS}\,(A_{\bar{k}}) \longrightarrow 0. \tag{18}$$

The Galois module $\mathrm{NS}\,(A_{\bar{k}})$ is canonically isomorphic to $\mathrm{Hom}(A_{\bar{k}}, A^\vee_{\bar{k}})^{\mathrm{sym}}$, the group of self-dual homomorphisms of abelian varieties $A_{\bar{k}} \to A^\vee_{\bar{k}}$. Recall that under this isomorphism the class λ of a line bundle $\mathcal{L}$ is sent to the morphism $\varphi_\lambda \colon A \to A^\vee$ defined as follows: an element $x \in A(\bar{k})$ goes to the class of the line bundle

$$T_x^*\mathcal{L} \otimes \mathcal{L}^{-1} \in \mathrm{Pic}^0(A_{\bar{k}}) \cong A^\vee(\bar{k}),$$

where T_x is the translation by x. It follows that $\mathrm{NS}\,(A_{\bar{k}})^{\Gamma_k}$ is canonically isomorphic to the group $\mathrm{Hom}(A, A^\vee)^{\mathrm{sym}}$ of self-dual k-homomorphisms of abelian varieties $A \to A^\vee$. A *polarisation* on A is an element $\lambda \in \mathrm{NS}\,(A_{\bar{k}})^{\Gamma_k}$ that comes from an ample line bundle $\mathcal{L}$ on $A_{\bar{k}}$. The polarisation is called *principal* if $\varphi_\lambda : A \to A^\vee$ is an isomorphism.

Let us define $e_2^\lambda(x, y) = e_2(x, \varphi_\Lambda(y))$. This pairing is alternating.

Following Poonen and Rains, we shall write c_λ for the image of λ under the differential $\mathrm{NS}\,(A_{\bar{k}})^{\Gamma_k} \to \mathrm{H}^1(k, A^\vee[2])$ attached to (18). In particular, c_λ vanishes if and only if λ lifts to an element of $(\mathrm{Pic}(A_{\bar{k}})^{[-1]^*})^{\Gamma_k} \cong \mathrm{Pic}(A)^{[-1]^*}$, i.e. a symmetric line bundle. Thus c_λ is the torsor of such liftings. For example, if A is the Jacobian of a smooth projective curve C and λ is the canonical principal polarisation of A, then c_λ is the image of the class of the torsor of theta characteristics (the closed subset of $\mathbf{Pic}^{g-1}_{C/k}$ given by $2x = K_C$), see [PR11, Thm. 3.9]. In particular, $c_\lambda = 0$ if C is a hyperelliptic curve of odd genus or with a rational Weierstrass point [PR11, Prop. 3.11].

An important fact is that (18) is compatible with the short exact sequence (12) discussed in Section 2.3, so that there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A^\vee[2] \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{Pic}(A_{\bar{k}})^{[-1]^*} \arrow[r] \arrow[d] & \mathrm{NS}\,(A_{\bar{k}}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A^\vee[2] \arrow[r] & S^2(A^\vee[2]) \arrow[r] & \wedge^2(A^\vee[2]) \arrow[r] & 0 \end{tikzcd}$$

where the right-hand vertical map sends λ to e_2^λ. Here the right-hand vertical map is the composition

$$\mathrm{NS}\,(A_{\bar{k}}) \to \mathrm{Hom}(T_2(A), T_2(A^\vee))^{\mathrm{sym}} \cong \wedge^2 T_2(A^\vee)(-1) \to \mathrm{H}^2_{\text{ét}}(A_{\bar{k}}, \mathbb{Z}_2(1)) \to \mathrm{H}^2_{\text{ét}}(A_{\bar{k}}, \mathbb{Z}/2),$$

where $T_2(A)$ is the 2-adic Tate module. The bottom exact sequence has trivial action of $\mathrm{Gal}(\bar{k}/L)$, hence c_λ is in the image of the inflation map

$$\mathrm{H}^1(G, A^\vee[2]) \hookrightarrow \mathrm{H}^1(k, A^\vee[2]).$$

We can also think of c_λ as an element of $\mathrm{H}^1(G, A[2])$, after identifying $A[2]$ and $A^\vee[2]$ by the isomorphism $\varphi_\lambda\colon A \xrightarrow{\sim} A^\vee$.

The class $c_\lambda \in \mathrm{H}^1(k, A^\vee[2])$ plays the following important role. By Poonen and Stoll [PS99, Cor. 2] we have $c_\lambda \in \mathrm{Sel}_2(A^\vee)$. Let c'_λ be the image of c_λ in $Ш(A^\vee)[2]$. By [PS99, Thm. 5] we have

$$\langle x, \varphi_{\lambda *}(x) + c'_\lambda \rangle = 0$$

for any $x \in Ш(A)$, where

$$\langle x, y \rangle \colon Ш(A) \times Ш(A^\vee) \to \mathbb{Q}/\mathbb{Z}$$

is the Cassels–Tate pairing. Thus $c'_\lambda = 0$ if and only if the bilinear pairing

$$\langle x, y \rangle_\lambda := \langle x_1, \varphi_{\lambda *} x_2 \rangle$$

on $Ш(A)$ is alternating.

4.2 Main result

Let A be an abelian variety over a field k of characteristic different from 2. Let Z be a k-torsor for the group k-scheme $A[2]$. Recall that the 2-covering $f : Y \to A$ associated to Z is a k-torsor for A defined as the quotient of $A \times_k Z$ by the diagonal action of $A[2]$. In other words, Y is the twisted form of A by Z with respect to the action of $A[2]$ by translations. The morphism f is induced by the first projection, and we have $Z = f^{-1}(0)$. Let L be the étale k-algebra $k[Z]$, so that $Z \cong \mathrm{Spec}(L)$.

Let $\tilde{Y}$ be the blowing-up of Z in Y. The antipodal involution $\iota_A : A \to A$ induces the map $(\iota_A, \mathrm{Id}) : A \times_k Z \to A \times_k Z$ which commutes with the action of $A[2]$ and hence induces an involution $\iota_Y : Y \to Y$. As ι_Y fixes $Z = f^{-1}(0) \subseteq Y$ it extends to an involution $\iota_{\tilde{Y}} : \tilde{Y} \to \tilde{Y}$ whose fixed point set is precisely the exceptional divisor. It is easy to see that the quotient $X = \mathrm{Kum}(Y) = \tilde{Y}/\iota_{\tilde{Y}}$ is smooth. We call X the *Kummer variety* attached to A and Z.

Let F be an extension of k of degree at most 2. Recall that A^F denotes the quadratic twist of A by F, that is, the abelian variety over k obtained by twisting A by the quadratic character of F with respect to the action of μ_2 via the antipodal involution ι_A. Similarly, Y^F denotes the quadratic twist of Y with respect to the involution ι_Y. Since ι_A commutes with translations by the elements of $A[2]$, the quadratic twist Y^F of Y is a k-torsor for A^F. We have a natural embedding $i_F : Z \to Y^F$. Then $\tilde{Y}^F$, defined as the blowing-up of $i_F(Z)$ in Y^F, is the quadratic twist of $\tilde{Y}$ by the quadratic character of F with respect to the action of μ_2 on $\tilde{Y}$ via $\iota_{\tilde{Y}}$. We can also consider $\tilde{Y}^F$ as a quadratic twist of the 2-covering $\tilde{Y} \to X$, and consequently consider every $\tilde{Y}^F$ as a (ramified) 2-covering of X.

Theorem 4.1 (A. Morgan) *Let A be an abelian variety of dimension $g \geq 2$ with principal polarisation λ defined over a number field k such that the following conditions hold:*

(a) *c_λ generates $\mathrm{H}^1(G, A[2])$,*

(b) *$A[2]$ is a simple $\mathbb{F}_2[G]$-module with $\mathrm{End}_G(A[2]) = \mathbb{F}_2$,*

(c) *$A[2]^g = 0$ for some $g \in G$.*

Let $X = \mathrm{Kum}(Y)$, where Y is a 2-covering of A associated to an element $a \in \mathrm{H}^1(k, A[2])$, $a \neq c_\lambda$. Assume that there is an odd prime $\mathfrak{p}$ of k such that a is unramified at $\mathfrak{p}$, A has semistable reduction at $\mathfrak{p}$, the conductor of A has odd exponent at $\mathfrak{p}$, and the group of geometric components of the Néron model at $\mathfrak{p}$ has odd order.

If the 2-primary torsion subgroup of the Tate–Shafarevish group of each quadratic twist of A is finite, then X satisfies the Hasse principle.

Here is an instructive particular case.

Corollary 4.2 *Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 6 with Galois group S_6. Let C be the genus 2 curve with equation $y^2 = f(x)$, and let J be the Jacobian of C. Consider the set of classes $a \in \mathrm{H}^1(\mathbb{Q}, J[2])$ such that there is an odd prime p at which a is unramified, the leading coefficient of $f(x)$ is coprime to p, and $\mathrm{discr}(f)$ is simply divisible by p. Assume the finiteness of the 2-primary part of the Tate–Shafarevich group of every quadratic twist of J. Then the Kummer surfaces associated to 2-coverings of J defined by such classes satisfy the Hasse principle.*

Sketch of proof of Theorem 4.1. The proof is based on the fact that we can twist A and Y by quadratic characters $\chi \in \mathrm{Hom}(\Gamma_k, \mu_2)$ without changing X, as explained above. Indeed, twisting produces A^χ and Y^χ such that $X = \mathrm{Kum}(Y^\chi)$. We have canonical isomorphisms of Γ_k-modules $A[2] \cong A^\chi[2]$ for all χ (and similarly for $A^\vee$). Note that A^χ inherits the principal polarisation of A. The Weil pairing associated to A^χ is the same as the one associated to A.

Let us first explain the strategy of [HS16]. In that paper it is assumed that $\mathrm{H}^1(G, A[2]) = 0$, so $c_\lambda = 0$.

Step (1) Assuming X is everywhere locally soluble, one can use the fibration method to prove the existence of a χ_0 such that

$$a \in \mathrm{Sel}_2(A^{\chi_0}) \subset \mathrm{H}^1(k, A[2]).$$

Step (2) One alters χ_0 to produce a χ_1 such that $\mathrm{Sel}_2(A^{\chi_1})$ is generated by a.

Step (3) Under the assumption of the finiteness of $\text{Ш}(A^\chi)\{2\}$, the pairing $\langle x, y\rangle_\lambda$ on $\text{Ш}(A^{\chi_1})\{2\}$ is non-degenerate and alternating (by Poonen–Stoll, since $c_\lambda = 0$). This implies that $\text{Ш}(A^{\chi_1})[2]$ cannot have dimension 1, so it has dimension 0. Thus a goes to zero in $\mathrm{H}^1(k, A^{\chi_1})$, which implies that $Y^{\chi_1} \simeq A^{\chi_1}$ so that $X = \mathrm{Kum}(Y^{\chi_1})$ has k-points (in fact, infinitely many).

A. Morgan had to come up with a different strategy because when $c_\lambda \neq 0$ we cannot conclude that $\text{Ш}(A^{\chi_1})[2] = 0$ in the same way as before because $\langle x, y\rangle_\lambda$ may no longer be alternating. The new strategy is to preserve the group $\mathrm{Sel}_2(A^\chi)$ while ensuring that the pairing $\langle x, y\rangle_\lambda$ takes the desired shape.

Let us call a bilinear pairing $P(x, y)$ on $\mathrm{Sel}_2(A)$ with values in $\mathbb{F}_2$ *admissible* if

- $P(c_\lambda, c_\lambda) = 0$ if and only if $\dim(\mathrm{Sel}_2(A)) + \mathrm{rk}(A(K))$ is even,
- $P(x, x) = P(x, c_\lambda)$ for all $x \in \mathrm{Sel}_2(A)$.

The pairing $\langle x, y\rangle_\lambda$ is admissible by [PS99, Theorems 5 and 8]. The following is the key technical result.

Theorem 4.3 (A. Morgan) *Let A be an abelian variety of dimension $g \geq 2$ with principal polarisation λ defined over a number field k satisfying conditions* (a), (b), (c). *Let P be an admissible pairing on* $\mathrm{Sel}_2(A)$. *Then there is a quadratic character $\chi\colon \Gamma_k \to \mu_2$ such that* $\mathrm{Sel}_2(A^\chi) = \mathrm{Sel}_2(A) \subset \mathrm{H}^1(k, A[2])$, *the ranks of A and A^χ have the same parity, and P is the Cassels–Tate pairing on* $\mathrm{Sel}_2(A^\chi)$.

The new strategy is roughly as follows.

Step (1) The same as before but in addition ensure that 2^∞-Selmer rank is odd.

Step (2) One alters χ_0 to produce many χ_1 such that $\mathrm{Sel}_2(A^{\chi_1})$ is the same as $\mathrm{Sel}_2(A^{\chi_0})$ while ensuring that 2^∞-Selmer rank is odd.

Step (3) Show that varying the character as in Step 2, one can arrange that the Cassels–Tate pairing takes any required form subject to the usual constraints (Theorem 4.3). Then it is easy to find a paring with only a in the kernel. This implies that a is the unique element lifting to the 4-Selmer group of A^χ. Since the 2^∞-Selmer rank is odd, the image of a in the Tate–Shafarevich group $Ш(A^\chi)$ is infinitely divisible. Under the finiteness assumption, a must have trivial image in $Ш(A^\chi)\{2\}$, so we are done.

References

[Amb21] E. Ambrosi. Uniform boundedness for Brauer groups of forms in positive characteristic. *Math. Res. Lett.* **28** (2021) 315–329.

[Bog80] F.A. Bogomolov. Points of finite order on abelian varieties (Russian). *Izv. Akad. Nauk SSSR Ser. Mat.* **44** (1980) 782–804.

[BZ09] F.A. Bogomolov and Yu.G. Zarhin. Ordinary reduction of K3 surfaces. *Cent. Eur. J. Math.* **7** (2009) 206–213.

[BvH09] M. Borovoi and J. van Hamel. Extended Picard complexes and linear algebraic groups. *J. reine angew. Math.* **627** (2009) 53–82.

[BK86] S. Bloch and K. Kato. p-adic étale cohomology. *Inst. Hautes Études Sci. Publ. Math.* **63** (1986) 107–152.

[BN23] M. Bright and R. Newton. Evaluating the wild Brauer group. *Invent. math.*, to appear. `arXiv:2009.03282`

[CM20] A. Cadoret and B. Moonen. Integral and adelic aspects of the Mumford–Tate conjecture. *J. Inst. Math. Jussieu* **19** (2020) 869–890.

[CHT] A. Cadoret, C.Y. Hui, and A. Tamagawa. $\mathbb{Q}_\ell$- versus $\mathbb{F}_\ell$-coefficients in the Grothendieck–Serre–Tate conjectures. *Israel J. Math.*, to appear.

[CTS13b] J.-L. Colliot-Thélène and A.N. Skorobogatov. Descente galoisienne sur le groupe de Brauer. *J. reine angew. Math.* **682** (2013) 141–165.

[CTS21] J.-L. Colliot-Thélène and A.N. Skorobogatov. *The Brauer–Grothendieck group.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 71, Springer, 2021.

[CTS] J.-L. Colliot-Thélène and A.N. Skorobogatov. Low degree unramified cohomology of generic diagonal hypersurfaces. `arXiv:2308.06809`

[D'Ad] M. D'Addezio. Boundedness of the p-primary torsion of the Brauer group of an abelian variety. *Compos. Math.*, to appear. `arXiv:2201.07526`

[Fal83] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. math.* **73** (1983) 349–366; Erratum **75** (1984) 381.

[Fal86] G. Faltings. Complements to Mordell. *Rational points*, G. Faltings, G. Wüstholz et al., 2nd ed., F. Viehweg & Sohn, 1986.

[GS] D. Gvirtz-Chen and A.N. Skorobogatov. Surfaces defined by pairs of polynomials. `https://arxiv.org/abs/2305.08632`

[Hen82] G. Henniart. Représentations ℓ-adiques abéliennes. *Séminaire de théorie des nombres, Paris 1980–81*, Progr. Math. 22, pp. 107–126. Birkhäuser Boston, Boston, MA, 1982.

[HS16] Y. Harpaz and A.N. Skorobogatov. Hasse principle for Kummer varieties. *Algebra Number Theory* **10** (2016) 813–841.

[Ier22] E. Ieronymou. Evaluation of Brauer elements over local fields. *Math. Ann.* **382** (2022) 239–254.

[Ier23] E. Ieronymou. Odd torsion Brauer elements and arithmetic of diagonal quartic surfaces over number fields. *Trans. Amer. Math. Soc.* **376** (2023) 5241–5259.

[IS15] E. Ieronymou and A.N. Skorobogatov. Odd order Brauer–Manin obstruction on diagonal quartic surfaces. *Adv. Math.* **270** (2015), 181–205. Corrigendum: *Adv. Math.* **307** (2017) 1372–1377.

[ISZ11] E. Ieronymou, A.N. Skorobogatov, and Yu.G. Zarhin. On the Brauer group of diagonal quartic surfaces. With an appendix by Peter Swinnerton-Dyer. *J. Lond. Math. Soc.* **83** (2011) 659–672.

[Ito18] K. Ito. Finiteness of Brauer groups of K3 surfaces in characteristic 2. *Int. J. Number Theory* **14** (2018) 1813–1825.

[LP95] M. Larsen and R. Pink. Abelian varieties, ℓ-adic representations, and ℓ-independence. *Math. Ann.* **302** (1995) 561–579.

[LP97] M. Larsen and R. Pink. A connectedness criterion for l-adic Galois representations. *Israel J. Math.* **97** (1997) 1–10.

[LS23] C. Lazda and A.N. Skorobogatov. Reduction of Kummer surfaces modulo 2 in the non-supersingular case. *Épijournal Géom. Algébrique* **7** (2023), Art.10.

[Mat] Y. Matsumoto. Supersingular reduction of Kummer surfaces in residue characteristic 2. `arXiv:2302.09535`

[Mil80] J.S. Milne. *Étale cohomology.* Princeton University Press, 1980.

[Mor85] L. Moret-Bailly. Pinceaux de variétés abéliennes. *Astérisque* **129**, 1985.

[Moo17] B. Moonen. Families of motives and the Mumford–Tate conjecture. *Milan J. Math.* **85** (2017) 257–307.

[OS18] M. Orr and A.N. Skorobogatov. Finiteness theorems for K3 surfaces and abelian varieties of CM type. *Compositio Math.* **154** (2018) 1571–1592.

[OSZ21] M. Orr, A.N. Skorobogatov, and Yu.G. Zarhin. On uniformity conjectures for abelian varieties and K3 surfaces. *Amer. J. Math.* **143** (2021) 1665–1702.

[Qin] Y. Qin. On geometric Brauer groups and Tate–Shafarevich groups. `arXiv:2012.01681`

[P22] M. Pagano. An example of a Brauer–Manin obstruction to weak approximation at a prime with good reduction. *Res. Number Theory* **8** (2022) paper no. 63, 15.

[P] M. Pagano. The role of primes of good reduction in the Brauer–Manin obstruction. `arXiv:2307.16030`

[PSk] A. Petrov and A.N. Skorobogatov. Brauer groups of semiabelian varieties. (Paper in preparation.)

[PR11] B. Poonen and E. Rains. Self cup product and the theta characteristic torsor. *Math. Res. Letters* **18** (2011) 1305–1318.

[PS99] B. Poonen and M. Stoll. Cassels–Tate pairing on polarized abelian varieties. *Ann. Math.* **150** (1999) 1109–1149.

[Rem18] G. Rémond. Conjectures uniformes sur les variétés abéliennes. *Quart. J. Math.* **69** (2018) 459–486.

[SZ08] A.N. Skorobogatov and Yu.G. Zarhin. A finiteness theorem for the Brauer group of abelian varieties and K3 surfaces. *J. Alg. Geom.* **17** (2008) 481–502.

[SZ15] A.N. Skorobogatov and Yu.G. Zarhin. A finiteness theorem for the Brauer group of K3 surfaces in odd characteristic. *Int. Math. Res. Not. IMRN* (2015) 11404–11418.

[Ser77] J.-P. Serre. Représentations ℓ-adiques. *Algebraic number theory.* (Kyoto Internat. Sympos., RIMS, Univ. Kyoto, 1976), pp. 177–193. Japan Soc. Promotion Sci., Tokyo, 1977.

[Ser81] J.-P. Serre. Lettre à Ken Ribet, 1/1/1981. *Œuvres*, Vol. IV, pp. 1–17. 1981.

[Ser94] J.-P. Serre. Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques. In *Motives (Seattle, WA, 1991)*, Proc. Symp. Pure Math., 55, part 1, pp. 377–400. Amer. Math. Soc., Providence, RI, 1994.

[Sha96a] I.R. Shafarevich. On the arithmetic of singular K3-surfaces. *Algebra and analysis* (Kazan, 1994), De Gruyter, 1996, pp. 103–108.

[Stacks] *The stacks project.* `https://stacks.math.columbia.edu/`

[VA17] A. Várilly-Alvarado. Arithmetic of K3 surfaces. *Geometry over nonclosed fields* (eds. F. Bogomolov, B. Hassett, Y. Tschinkel), Simons Symposia **5**, Springer, 2017, pp. 197–248.

[VAV17] A. Várilly-Alvarado and B. Viray. Abelian n-division fields of elliptic curves and Brauer groups of product Kummer and abelian surfaces. *Forum Math. Sigma* **5** (2017), e26, 42 pp.

[Vas08] A. Vasiu. Some cases of the Mumford–Tate conjecture and Shimura varieties. *Indiana Univ. Math. J.* **57** (2008) 1–75.

[Yua20] X. Yuan. Comparison of arithmetic Brauer groups with geometric Brauer groups. `arXiv:2011.12897`

[Zar75] Yu.G. Zarhin. Endomorphisms of Abelian varieties over fields of finite characteristic. *Izv. Akad. Nauk SSSR Ser. Mat.* **39** (1975) 272–277. English translation: *Math. USSR Izv.* **9** (1975) 255–260.

[Zar76] Yu.G. Zarhin. Abelian varieties in characteristic p. *Mat. Zametki* **19** (1976) 393–400. English translation: *Math. Notes* **19** (1976) 240–244.