Brauer group in arithmetic geometry
with special reference to K3 surfaces and abelian varieties

Alexei Skorobogatov

September 8, 2023

Contents

1 Conjectures of Tate and Mumford–Tate 4
  1.1 Tate conjecture for divisors ................................. 4
  1.2 Mumford–Tate conjecture ................................... 5
  1.3 Finiteness properties of the Brauer group ................... 9
  1.4 Uniformity ..................................................... 13
Introduction

These lectures are a sequel to the book [CTS21]. Our aim was to develop some of the stories that were only touched upon or not even mentioned in this book, with emphasis of the applications of the Brauer group in arithmetic geometry.

Over a field of characteristic zero the geometric Brauer group of a smooth proper variety with $b_2 = \rho$ is finite\(^1\). Varieties with $b_2 > \rho$, for example abelian varieties and K3 surfaces, exhibit a new phenomenon: they have infinite geometric Brauer group whose Tate module carries a Galois representation. Various finiteness properties of the Brauer group can be related to deep conjectures about these Galois representations.

The philosophy of these lectures is based on the assumption that K3 surfaces is a convenient vantage point to explore the arithmetic of the Brauer group and the Brauer–Manin obstruction beyond the realm of geometrically rational or rationally connected varieties. What makes K3 surfaces more tractable is a classical construction of Kuga–Satake that associates to a complex K3 surface an abelian variety of large dimension. Geometry of K3 surfaces was intensively studied since the proof of the global Torelli theorem by Pyatetskii-Shapiro and Shafarevich in 1971: a complex K3 surface is determined by the periods of the unique holomorphic 2-form, leading to an interpretation of the moduli spaces of polarised K3 surfaces in terms of Shimura varieties. Relations to abelian varieties and Shimura varieties make K3 surfaces a natural ‘testing ground’ for fundamental conjectures in arithmetic geometry. Indeed, Deligne proved Weil conjectures for K3 surfaces in 1972 via the Kuga–Satake construction before he proved the Weil conjectures in the general case.

We begin with a tour of Tate and Mumford–Tate conjectures, which are known for K3 surfaces, with proofs crucially based on the Kuga–Satake construction, see Section 1. We explain what these conjectures mean for the finiteness properties of the Brauer group. In Section 1.4 we discuss various uniformity conjectures for abelian varieties and K3 surfaces, and links among them. These conjectures assert the boundedness of certain integer invariants of K3 surfaces and abelian varieties defined over a number field of bounded degree (and bounded dimension in the case of abelian varieties). In Section ?? we discuss the problem of explicit calculation of the Brauer group, which requires understanding differentials in Leray spectral sequences. Section ?? is devoted to explicit determination of the Brauer–Manin set. For this one needs to know the behaviour of the ‘evaluation map’ attached to a Brauer class and a place of the ground number field. One natural question here is when such a map can be non-constant. We explain known results in this direction. Finally, in Section ?? we discuss the existence of rational points in the Brauer–Manin set in the best understood particular case, namely, that of Kummer varieties, using a method of Swinnerton-Dyer.

\(^1\)This fails in characteristic $p > 0$, e.g., for supersingular K3 surfaces, see Section 1.3.
Notation

For an abelian group $G$ we denote by $G[n]$ the subgroup $\{x \in G | nx = 0\}$. We write $G_{\text{odd}}$ for the union of all subgroups $G[n]$ where $n$ is odd. If $\ell$ is a prime, then the $\ell$-primary torsion subgroup $G\{\ell\}$ is the union of $G[\ell^n]$ for $n \geq 1$.

The $\ell$-adic Tate module is defined as

$$T_\ell(G) = \text{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, G) = \lim_{\leftarrow n} G[\ell^n],$$

where the transition maps $G[\ell^{n+1}] \to G[\ell^n]$ are multiplications by $\ell$. It is easy to check that $T_\ell(G)$ is a torsion-free $\mathbb{Z}_\ell$-module. There are natural injective maps $T_\ell(G)/\ell^n \to G[\ell^n]$. If the group $G[\ell]$ is finite, then the $\mathbb{Z}_\ell$-module $T_\ell(G)$ is finitely generated. By Nakayama’s lemma we have $T_\ell(A) \simeq \mathbb{Z}_\ell^r$ where $r \leq \dim_{\mathbb{F}_\ell}(G[\ell])$.

If, moreover, $G$ is an $\ell$-primary torsion abelian group, then $T_\ell(G) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is the divisible subgroup $G_{\text{div}}$ of $A$. Define

$$V_\ell(G) = T_\ell(G) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$ 

This is a $\mathbb{Q}_\ell$-vector space.

For a field $k$ we denote by $\bar{k}$ an algebraic closure of $k$, and by $k_s$ the separable closure of $k$ in $\bar{k}$. Let $\Gamma_k = \text{Gal}(k_s/k)$. We write $X^s = X_{k_s} = X \times_k k_s$.

A field is called \textit{finitely generated} if it is finitely generated over its prime subfield. For a scheme $X$ we denote by $\text{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$ the (cohomological) Brauer group of $X$. When $X$ is a variety over a field $k$, the group $\text{Br}(X^s)$ is called the \textit{geometric} Brauer group of $X$. We use the standard notation

$$\text{Br}_0(X) = \text{Im}[\text{Br}(k) \to \text{Br}(X)], \quad \text{Br}_1(X) = \text{Ker}[\text{Br}(X) \to \text{Br}(X^s)^\Gamma].$$

$\text{Br}_1(X)$ is called the \textit{algebraic} Brauer group. Following D’Adsezio, we introduce the following notation: for a field extension $K/k$ we denote by $\text{Br}(X_K)^k$ the image of the natural map $\text{Br}(X_K) \to \text{Br}(X_K)$. In particular, $\text{Br}(X^s)^k$, that is, the image of $\text{Br}(X) \to \text{Br}(X^s)^\Gamma$, is called the \textit{transcendental} Brauer group of $X$. 

3
1 Conjectures of Tate and Mumford–Tate

References: Moonen’s survey paper [Moo17].

1.1 Tate conjecture for divisors

The Tate conjecture is stated for varieties over finitely generated fields of arbitrary characteristic.

**Tate conjecture for divisors.** The equivalent properties in the following theorem hold when the ground field $k$ is finitely generated.

**Theorem 1.1** Let $X$ be a smooth, projective, geometrically integral variety over a field $k$. Let $\Gamma = \text{Gal}(k_s/k)$. Let $\ell \neq \text{char}(k)$ be a prime. The following conditions are equivalent.

(i) The injective map $c_1 : (\text{NS}(X_s) \otimes \mathbb{Z}_\ell)^\Gamma \to H^2_\text{ét}(X_s, \mathbb{Z}_\ell(1))^\Gamma$ is an isomorphism.

(ii) The injective map $c_1 : (\text{NS}(X_s) \otimes \mathbb{Q}_\ell)^\Gamma \to H^2_\text{ét}(X_s, \mathbb{Q}_\ell(1))^\Gamma$ is an isomorphism.

(iii) $(T_\ell(\text{Br}(X_s))^\Gamma = 0$.

(iv) $(V_\ell(\text{Br}(X_s))^\Gamma = 0$.

(v) $\text{Br}(X_s)^{\{\ell\}}$ is finite.

**Proof.** The Kummer exact sequence gives rise to an exact sequence of finitely generated $\mathbb{Z}_\ell$-modules

$$0 \to \text{NS}(X_s) \otimes \mathbb{Z}_\ell \to H^2_\text{ét}(X_s, \mathbb{Z}_\ell(1)) \to T_\ell(\text{Br}(X_s)) \to 0,$$

with a continuous action of $\Gamma$. When tensored with $\mathbb{Q}_\ell$, it gives the exact sequence

$$0 \to \text{NS}(X_s) \otimes \mathbb{Q}_\ell \to H^2_\text{ét}(X_s, \mathbb{Q}_\ell(1)) \to V_\ell(\text{Br}(X_s)) \to 0. \quad (1)$$

By [CTS21, Thm. 5.3.1 (ii)] this sequence is split as a sequence of $\Gamma$-modules. Thus (ii) is equivalent to (iv). The group $T_\ell(\text{Br}(X_s))$ is torsion-free. Thus (iii) is equivalent to (iv). It is clear that (iii) implies (i). That (i) implies (ii) follows from the simple observation that for any finitely generated $\mathbb{Z}_\ell$-module $M$ with an action of $\Gamma$, the map $M^\Gamma \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to (M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)^\Gamma$ is surjective.

For any abelian group $A$ with an action of a group $\Gamma$, one has a natural isomorphism $T_\ell(A)^\Gamma \cong T_\ell(A^\Gamma)$. The group $\text{Br}(X_s)^{\{\ell\}}$ is of cofinite type [CTS21, Prop. 5.2.9], hence so is $\text{Br}(X_s)^{\{\ell\}}$, that is, $\text{Br}(X_s)^{\{\ell\}} \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^m \oplus B$, where $B$ is a finite abelian group. Then $T_\ell(\text{Br}(X_s)^{\{\ell\}}) \cong \mathbb{Z}_m^n$. It follows that (iii) is equivalent to (v), because both statements are equivalent to $m = 0$. □
Remark 1.2 Let $k$ be a finitely generated field of characteristic $p$. For a smooth, projective, and geometrically integral variety $X$ over $k$, finiteness of $\text{Br}(X^s)\{\ell\}_\Gamma$ for one prime $\ell \neq p$ implies finiteness of the prime-to-$p$ subgroup of $\text{Br}(X^s)_\Gamma$. This is due to Cadoret–Hui–Tamagawa[CHT, Cor. 1.4], see also Yanshuai Qin’s paper [Qin, Thm. 1.2].

Theorem 1.3 (i) Tate conjecture for divisors holds for abelian varieties.

(ii) Tate conjecture for divisors holds for K3 surfaces.

Proof. (i) Over a finite field, this was proved by Tate; over a field finitely generated over the prime field, it was proved by Zarhin in characteristic $p > 2$ [Zar75, Zar76], by Faltings in characteristic zero [Fal83, Fal86], and by Mori in characteristic 2, see [Mor85].

(ii) In general, the case of K3 surfaces is reduced to the case of abelian varieties via the Kuga–Satake construction which associates to a K3 surface $X$ an abelian variety $A$ of large dimension. Deligne observed that $A$ can be defined over a finite extension of the ground field. Over a field of characteristic zero, the Tate conjecture for K3 surfaces is due to Tankeev and, independently, Y. André, see [CTS21, Thm. 16.7.1]. More recently, Tate conjecture was proved for K3 surfaces in finite characteristic in growing generality by Nygaard, Ogus, F. Charles, Maulik, and finally in full generality by Madapusi Pera for $p > 2$ and by Madapusi Pera and Wansu Kim for $p = 2$. □

Corollary 1.4 Let $X$ be an abelian variety or a K3 surface over a field $k$ that is finitely generated over its prime subfield. Then $\text{Br}(X^s)\{\ell\}_\Gamma$ is finite for all primes $\ell$ not equal to $\text{char}(k)$.

Proof. This follows from Theorems 1.1 and 1.3. □

1.2 Mumford–Tate conjecture

Unlike Tate conjecture, the conjecture of Mumford–Tate is stated for varieties over finitely generated fields of characteristic zero.

The original Mumford–Tate conjecture was stated for an abelian variety $A$ in terms of the natural Hodge structure on the first homology group $H_1 = H_1(A_\mathbb{C}, \mathbb{Z})$. The free $\mathbb{Z}_\ell$-module $H_1 \otimes \mathbb{Z}_\ell$ is identified with the $\ell$-adic Tate module $T_\ell(A_\bar{k})$ and so carries a natural Galois representation. The idea is that the image of the Galois group is roughly as large as is allowed by the symmetries of the cohomology of $A$, as reflected in the Hodge structure.

Let $X$ be a smooth, projective and geometrically integral variety over a field $k$ that is finitely generated over $\mathbb{Q}$. Let $i \geq 1$ and $j$ be integers. We choose an embedding $k \hookrightarrow \mathbb{C}$ and define $H$ as the quotient of $H^i(X_\mathbb{C}, \mathbb{Z}(j))$ by the torsion subgroup. We write $H_\mathbb{Q} = H \otimes \mathbb{Q}$, $H_\mathbb{R} = H \otimes \mathbb{R}$, $H_\mathbb{C} = H \otimes \mathbb{C}$, and for a prime $\ell$ write $H_\ell = H \otimes \mathbb{Z}_\ell$. Let $\text{GL}(H)$ be the group $\mathbb{Z}$-scheme such that for any
commutative ring \( R \) we have \( \text{GL}(H)(R) = \text{GL}(H \otimes \mathbb{Z} R) \). The generic fibre \( \text{GL}(H)_\mathbb{Q} \) is the algebraic group \( \text{GL}(H_\mathbb{Q}) \) over \( \mathbb{Q} \).

Let us first discuss the Galois side. The comparison theorems between Betti and étale cohomology give an isomorphism between \( H^i \) and the quotient of \( H^i \ell(X_k, \mathbb{Z}_\ell(j)) \) by the torsion subgroup. Let \( \rho : \Gamma_k \rightarrow \text{GL}(H)(\mathbb{Z}_\ell) \) be the resulting continuous representation. Here the twist \( \mathbb{Z}_\ell(j) \) is understood as the Tate twist; it changes the Galois representation by tensoring it with the \( j \)-th power of the cyclotomic character. Since \( \Gamma_k \) is compact and \( \rho \) is continuous, \( \rho(\Gamma_k) \) is a compact, hence closed, subgroup of the \( \ell \)-adic Lie group \( \text{GL}(H_\mathbb{Q}) \). It is therefore a Lie subgroup. Let \( g_\ell \) be its Lie algebra. (This is a \( \mathbb{Q}_\ell \)-vector space.)

Let \( G_{k,\ell} \) be the algebraic group over \( \mathbb{Z}_\ell \) defined as the Zariski closure of \( \rho_\ell(\Gamma_k) \) in \( \text{GL}(H)_\mathbb{Z}_\ell \). The generic fibre \( G_{K,\ell,\mathbb{Q}_\ell} = G_{k,\ell} \times_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \) is the Zariski closure of the image of the \( \mathbb{Q}_\ell \)-representation \( \Gamma_k \rightarrow \text{GL}(H)_\mathbb{Q}_\ell \). The algebraic \( \mathbb{Q}_\ell \)-group \( G_{K,\ell,\mathbb{Q}_\ell} \) is called the \( \ell \)-adic monodromy group. By theorems of Bogomolov [Bog80], Serre [Ser81] and Henniart [Hen82], the group \( \rho_\ell(\Gamma_k) \) is an open subgroup of \( G_{K,\ell,\mathbb{Q}_\ell}(\mathbb{Q}_\ell) \) with respect to the \( \ell \)-adic topology. Hence \( \rho_\ell(\Gamma_k) \) is open in \( G_{k,\ell}(\mathbb{Z}_\ell) = G_{K,\ell,\mathbb{Q}_\ell}(\mathbb{Q}_\ell) \cap \text{GL}(H_\mathbb{Q}) \), but since the latter group is compact, it has finite index in it. (For an abelian variety, this index is know to be bounded, see Remark 1.3 of Zywina’s paper.)

The \( \ell \)-adic monodromy group \( G_{K,\ell,\mathbb{Q}_\ell} \) is not necessarily connected (for example, for \( i = 2 \) and \( j = 1 \), because the action of \( \Gamma_k \) on the Néron–Severi group \( \text{NS}(X_k) \) is via a finite quotient). By a result of Serre there exists a finite field extension \( k^\text{conn} \) of \( k \) such that for every field \( K \subset \bar{k} \) containing \( k^\text{conn} \) and every prime \( \ell \) the group \( G_{K,\ell,\mathbb{Q}_\ell} \) is connected, see [LP97]. We have \( G_{k^\text{conn},\ell,\mathbb{Q}_\ell} = G_{k,\ell,\mathbb{Q}_\ell} \), where \( G_{k,\ell,\mathbb{Q}_\ell} \) is the connected component of identity in \( G_{k,\ell,\mathbb{Q}_\ell} \).

Now we discuss the Hodge side. The free \( \mathbb{Z} \)-module \( H \) of finite rank carries a natural Hodge structure. Let us recall what this means.

The \( \mathbb{R} \)-torus \( S = R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \) is called the Deligne torus. A \( \mathbb{Q} \)-Hodge structure on \( H_\mathbb{Q} \) of pure weight \( n \) can be described as a representation \( h : S \rightarrow \text{GL}(H)_\mathbb{R} \) whose restriction to \( \mathbb{G}_{m,\mathbb{R}} \subset S \) is \( x \mapsto x^{-n} \). Then \( H_\mathbb{C} \) is a direct sum of the subspaces \( (H_\mathbb{C})^p,q = \mathbb{C}^x \times \mathbb{C}^x \) acting by \((z_1, z_2) \mapsto z_1^{-p} z_2^{-q}\). We have \( (H_\mathbb{C})^{p,q} = (H_\mathbb{C})^{q,p} \). In the context of Hodge structures, \( \mathbb{Z}(j) \) is understood as the Hodge structure on \( (2\pi i)^j \mathbb{Z} \) of pure weight \(-j\); twisting by \( j \) means tensoring with \( \mathbb{Z}(j) \). For example, \( H^2(X_\mathbb{C}, \mathbb{Z}(1)) \) carries a natural Hodge structure of weight \( 0 \).

The Mumford–Tate group \( G_\mathbb{Q} \subset \text{GL}(H_\mathbb{Q}) \) of the Hodge structure on \( H_\mathbb{Q} \) is the smallest algebraic group over \( \mathbb{Q} \) such that \( G_\mathbb{R} \) contains the image of the homomorphism \( h : S \rightarrow \text{GL}(H)_\mathbb{R} \). It follows that \( G_\mathbb{Q} \) is connected. The Mumford–Tate group \( G_\mathbb{Q} \) is known to be reductive if the Hodge structure is polarisable, which we assume from now on.

The key property of the Mumford–Tate group is that an element of the full tensor algebra of \( H_\mathbb{Q} \) and \( H^*_\mathbb{Q} \) is fixed by \( G_\mathbb{Q} \) if and only if it has Hodge type \((0,0)\).

In characteristic zero a reductive group is determined by its tensor invariants, \( G_\mathbb{Q} \).
can be characterised by this property.

Let $G$ be the group $\mathbb{Z}$-scheme which is the Zariski closure of the Mumford–Tate group $G_{\mathbb{Q}}$ in $\text{GL}(H)$.

**Mumford–Tate conjecture at a prime $\ell$.** We have $G_{\mathbb{Q}_\ell} = G_{k,\text{conn},\ell,\mathbb{Q}_\ell} = G_{k,\ell,\mathbb{Q}_\ell}$.

An equivalent form of the conjecture is: $G_{\mathbb{Z}_\ell} = G_{k,\text{conn},\ell,\mathbb{Q}_\ell}$. Another equivalent form is in terms of Lie algebras, namely, the equality of Lie subalgebras $\mathfrak{g}_\ell = \mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ of $\mathfrak{gl}(H_{\mathbb{Q}_\ell})$, where $\mathfrak{h}$ is the Lie algebra of the Mumford–Tate group $G_{\mathbb{Q}}$.

By Tankeev and [LP95, Thm. 4.3], Mumford–Tate conjecture at $\ell$ implies Mumford–Tate conjecture at any other prime.

The Mumford–Tate conjecture $G_{\mathbb{Z}_\ell} = G_{k,\text{conn},\ell,\mathbb{Q}_\ell}$ implies that $\rho_\ell(\Gamma_{k,\text{conn}})$ is an open subgroup of $G(\mathbb{Z}_\ell)$ of finite index. This prompts the following stronger version of the conjecture.

**Serre’s integral Mumford–Tate conjecture.** There is a constant $C$ such that for all primes $\ell$ the image $\rho_\ell(\Gamma_{k,\text{conn}})$ is a subgroup of $G(\mathbb{Z}_\ell)$ of index at most $C$.

This was conjectured by Serre to hold for all varieties, see [Ser77, Conjecture C.3.7] and [Ser94, 10.3].

**Definition 1.5** A (polarisable) $\mathbb{Q}$-Hodge structure $H$ is said to be of CM type if its Mumford–Tate group is a torus. Equivalently, the endomorphism algebra $\text{End}_{\mathbb{Q}}(H)$ contains a commutative semisimple $\mathbb{Q}$-algebra $F$ such that $H$ is free of rank 1 as an $F$-module.

A complex abelian variety $A$ (respectively, a $K3$ surface $X$) has CM type if the Hodge structure on $H^1(A, \mathbb{Q})$ (respectively, on $H^2(X, \mathbb{Q})$) is of CM type.

Mumford–Tate conjecture is known for abelian varieties of CM type. More generally, for arbitrary abelian varieties, it is ‘true for centres’ (Vasiu [Vas08, Thm. 1.3.1], Ullmo–Yafaev). Commelin showed that if Mumford–Tate conjecture holds for two abelian varieties, then it holds for their product. Results of Serre imply that this conjecture holds for elliptic curves; Tankeev proved that it holds for simple abelian varieties of prime dimension. For general abelian varieties (already in dimension 4) Mumford–Tate conjecture conjecture is open.

For $K3$ surfaces, in contrast, the situation is much better understood.

**Theorem 1.6** (Tankeev, Y. André, Cadoret–Moonen) The integral Mumford–Tate conjecture holds for $K3$ surfaces over fields finitely generated over $\mathbb{Q}$.

**Proof.** The Mumford–Tate conjecture for $K3$ surfaces was proved by Tankeev, and independently by Y. André. Cadoret and Moonen [CM20] showed, using that the moduli spaces of $K3$ surfaces are closely related to Shimura varieties, that the usual Mumford–Tate conjecture for $K3$ surfaces implies its integral version. □

Actually, more is true. Let $\rho : \Gamma_k \to \text{GL}(H)(\hat{\mathbb{Z}})$ be the continuous representation of $\Gamma_k$ whose $\ell$-adic component is $\rho_\ell$. Cadoret and Moonen proved in [CM20]
that $\rho(\Gamma_{k,\text{conn}})$ is an open subgroup of $G(\hat{\mathbb{Z}})$ and therefore (since $G(\hat{\mathbb{Z}})$ is compact) has finite index. This adelic version of the Mumford–Tate conjecture can only be expected to hold if the Hodge structure on $H$ is Hodge-maximal [CM20, 2.6], which is the case when $X$ is a K3 surface [CM20, Prop. 6.2].

**Example 1.7** Mumford–Tate conjecture holds for diagonal hypersurfaces. Indeed, in the language of André’s motives, varieties dominated by products of curves give rise to ‘abelian motives’. In particular, this is the case for diagonal hypersurfaces. As explained in [Moo17], Vasiu’s result that Mumford–Tate conjecture for abelian varieties is ‘true for centres’ implies that the same holds for abelian motives. The middle cohomology group of the Fermat hypersurface with coefficients $\mathbb{C}$ is the direct sum of 1-dimensional eigenspaces of $(\mu_d)^n$ acting by automorphisms, so the Mumford–Tate group is a torus. This proves the claim.

**Question 1.8** Do integral and adelic versions of Mumford–Tate conjecture hold for diagonal hypersurfaces? I think that the integral Mumford–Tate conjecture should hold.

The integral Mumford–Tate conjecture has the following strong consequence for the finiteness of the Galois invariant subgroup of the Brauer group. In particular, the integral Mumford–Tate conjecture for $X$ implies the finiteness of $\text{Br}(X_{\bar{k}})_{\Gamma_k}$. 

**Proposition 1.9 (M. Orr, A.S.)** Let $X$ be a smooth, projective and geometrically integral variety defined over a field $k$ which is finitely generated over $\mathbb{Q}$. Assume that the integral Mumford–Tate conjecture is true for $X$. Then for every positive integer $m$ there exists a constant $C = C_{m,X}$ such that for every subgroup $\Gamma' \subset \Gamma_k$ of index at most $m$ we have $|\text{Br}(X_{\bar{k}})_{\Gamma'}| < C$.

**Proof.** See [OS18, §5]. The Kummer sequence for a prime $\ell$ gives that $\text{Br}(X_{\bar{k}})\{\ell\}$ is an extension of the torsion subgroup of $H^3_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$, which is finite and is zero for almost all $\ell$, by the divisible subgroup $\text{Br}(X_{\bar{k}})\{\ell\}_{\text{div}} \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2-\rho}$, see [CTS21, Thm. 5.2.9]. Thus we need to bound the size of $\text{Br}(X_{\bar{k}})_{\text{div}}\{\ell\}$ for each $\ell$ and to prove that $\text{Br}(X_{\bar{k}})_{\text{div}}[\ell] = 0$ for all $\ell > \ell_0$. We have an isomorphism of Galois modules

$$\text{Br}(X_{\bar{k}})_{\text{div}}\{\ell\} \cong T_\ell(\text{Br}(X_{\bar{k}})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

so $\text{Br}(X_{\bar{k}})_{\text{div}}[\ell] \cong T_\ell(\text{Br}(X_{\bar{k}}))/\ell$. The integral Mumford–Tate conjecture allows us to replace the Galois group with the Mumford–Tate group. Let us indicate why $(T_\ell(\text{Br}(X_{\bar{k}}))/\ell)^S = 0$ for almost all $\ell$ if $S \subset G(\mathbb{Z}_\ell)$ is a subgroup of bounded index.

Denote by $N$ the image of $\text{NS}(X_{\mathbb{C}})$ in $H$. From the Kummer sequence and the comparisons theorem between singular and $\ell$-adic étale cohomology we obtain an isomorphism $T_\ell(\text{Br}(X_{\bar{k}}))/\ell \cong (H/N)/\ell$. We arrange that $G \times_{\mathbb{Z}} \mathbb{F}_\ell$ is a connected algebraic group over $\mathbb{F}_\ell$ for all $\ell > \ell_0$. We want to show that no subgroup of $G(\mathbb{F}_\ell)$ of bounded index fixes a non-zero vector in $(H/N)/\ell$ if $\ell_0$ is large enough.
By the key property of the Mumford–Tate group, we have
\[(H_Q)^{G_Q} = H_Q \cap H_C^{(0,0)} = N_Q,\]
where the second equality is by the Lefschetz \((1,1)\)-theorem. Since \(G_Q\) is reductive, \(H_Q\) is a semisimple \(G_Q\)-module, thus \((H_Q/N_Q)^{G_Q} = 0\) and hence \((H_C/N_C)^{G_C} = 0\). It follows that the dimensions of the stabilisers of non-zero points of \(H_C/N_C\) are less than \(d = \dim(G_Q)\). Using semicontinuity, one shows the same for the points of \((H/N)/\ell\), if \(\ell_0\) is large enough. These stabilisers are algebraic groups over \(\mathbb{F}_\ell\). The number of their connected components is bounded by an absolute constant. For an arbitrary connected algebraic group \(G\) over \(\mathbb{F}_\ell\) we have
\[(\ell - 1)^{\dim(G)} \leq |G(\mathbb{F}_\ell)| \leq (\ell + 1)^{\dim(G)}.\]
This shows that the index of a stabiliser of a non-zero point of \((H/N)/\ell\) cannot be bounded as \(\ell\) grows. Thus no subgroup of \(G(\mathbb{F}_\ell)\) of bounded index fixes a non-zero vector in \((H/N)/\ell\), if \(\ell > \ell_0\) is large enough.

Corollary 1.10 Let \(X\) be a smooth, projective and geometrically integral variety defined over a field \(k\) which is finitely generated over \(\mathbb{Q}\). If the integral Mumford–Tate conjecture is true for \(X\), then for each positive integer \(n\) there exists a constant \(C = C_{n,X}\) such that for every \((k/L)\)-form \(Y\) of \(X\) defined over a field extension \(L/k\) of degree \([L:k]\) \(\leq n\) we have \(|\text{Br}(Y)^{\text{et}}| < C\).

See [Amb21] for an analogue of this statement in finite characteristic.

1.3 Finiteness properties of the Brauer group

Let \(X\) be a smooth, projective and geometrically integral variety over a field \(k\). Then \(\text{Br}(X)\) is a torsion group [CTS21, Thm. 3.5.5], thus a direct sum of its \(\ell\)-primary torsion subgroups \(\text{Br}(X)\{\ell}\), for all primes \(\ell\).

For \(\ell \neq \text{char}(k)\), the structure of \(\text{Br}(X)\{\ell\}\) is computed by the Kummer exact sequence: the divisible subgroup of \(\text{Br}(X)\{\ell\}\) is isomorphic to \((\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{h_{2-\rho}}\) and the quotient by the divisible subgroup is isomorphic to the torsion subgroup of \(H^3_{\text{et}}(X^s, \mathbb{Z}_\ell)\), see [CTS21, Thm. 5.2.9]. We also note that the natural map
\[\text{Br}(X^s)\{\ell\} \to \text{Br}(X_k)\{\ell\}\]
is an isomorphism [CTS21, Prop. 5.2.3].

The structure of $\text{Br}(X^s)\{p\}$, where $p = \text{char}(k)$, is more involved. First of all, one needs to distinguish between separably closed fields that are not algebraically closed and algebraically closed fields. One problem here is that $\bar{k}$ is not necessarily finitely generated over $k_s$, so $f: \text{Spec}(\bar{k}) \to \text{Spec}(k_s)$ is not an fppf covering.

**Proposition 1.11** The natural map $\text{Br}(X^s)\{p\} \to \text{Br}(X_{\bar{k}})\{p\}$ is injective if the Picard scheme $\text{Pic}_{X^s/k_s}$ is smooth over $k_s$. This holds if one of the following conditions is satisfied:

- $X$ is an abelian variety;
- $X$ is a K3 surface;
- $H^1(X, \mathcal{O}) = 0$;
- $H^2(X, \mathcal{O}) = 0$.

If $\text{Br}(X^s)\{p\} \to \text{Br}(X_{\bar{k}})\{p\}$ is injective, then $\text{Br}(X^s)^k \to \text{Br}(X_{\bar{k}})^k$.

**Proof.** See [CTS21, Thm. 5.2.5]. We write $k = k_s$. Let $f: X \to \text{Spec}(k)$ be the structure morphism. Consider the spectral sequence

$$H^p_{\text{fppf}}(k, R^q f_* \mathbb{G}_m, X) \Rightarrow H^{p+q}_{\text{et}}(X, \mathbb{G}_m, X).$$

Here the isomorphism is due to the fact that fppf cohomology with coefficients in a smooth group scheme coincides with étale cohomology. For any $k$-scheme $T$ the map $\mathcal{O}_T \to f_* \mathcal{O}_X$ is an isomorphism, hence we have an isomorphism of fppf sheaves $\mathbb{G}_{m,k} \to f_* \mathbb{G}_m, X$. We have $H^1_{\text{fppf}}(k, \mathbb{G}_{m,k}) \cong H^2_{\text{et}}(k, \mathbb{G}_{m,k}) = \text{Br}(k) = 0$ because $k$ is separably closed. Moreover, $R^1 f_* \mathbb{G}_m$ is representable by the Picard group scheme $\text{Pic}_{X/k}$.

Thus the spectral sequence gives an exact sequence

$$0 \to H^1_{\text{fppf}}(k, \text{Pic}_{X/k}) \to \text{Br}(X) \to H^0(k, R^2 f_* \mathbb{G}_m, X).$$

If $\text{Pic}_{X/k}$ is smooth, then $H^1_{\text{fppf}}(k, \text{Pic}_{X/k}) = 0$. With some work one shows that $H^0(k, R^2 f_* \mathbb{G}_m, X)$ embeds into $H^0(\bar{k}, R^2 f_* \mathbb{G}_m, X)$, see [D’Ad, §3]. But the injective map $\text{Br}(X) \to H^0(\bar{k}, R^2 f_* \mathbb{G}_m, X)$ factors through $\text{Br}(X_{\bar{k}})$, hence the result.

It is well known that the Picard scheme of an abelian variety is smooth. □

**Example 1.12** The map $\text{Br}(X^s)\{p\} \to \text{Br}(X_{\bar{k}})\{p\}$ is not always injective. Indeed, in characteristic $p = 2$ there exist Enriques surfaces such that the connected component of the origin in $\text{Pic}_{X^s/k_s}$ is the group $k$-scheme $\alpha_2$. The exact sequence of fppf sheaves

$$0 \to \alpha_p \to \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \to 0,$$

where $F(x) = x^p$, gives $H^1_{\text{fppf}}(k_s, \alpha_p) \cong k_s/k_s^p$, which can be infinite.
Example 1.13 Artin showed that if $X$ is a \textit{supersingular} K3 surface over an algebraically closed field $\bar{k}$ of characteristic $p > 0$ (by definition, this means that $\rho = b_2 = 22$), then $\text{Br}(X_{\bar{k}}) \cong \bar{k}$, so this group is not divisible (we cannot divide by $p$), of exponent $p$, and can be uncountable\footnote{Remarkably, in this case, the fppf sheaf $R^2\pi_{*}\mu_p$ is a smooth group $k$-scheme of finite type with connected component $G_a$. This is due to Artin, Artin–Mazur, and Milne, using the Tate conjecture. Bragg and Olsson prove that $R^i\pi_{*}G$, where $X$ is projective and of finite type, and $G$ is a finite flat group $X$-scheme of finite type, is always representable by a group $k$-scheme of finite type, which is an iterated extension of finite commutative group schemes and $k$-forms of vector bundles.}. In contrast, if $X$ is a non-supersingular K3 surface over an algebraically closed field $\bar{k}$ of characteristic $p > 0$, then $\text{Br}(X_{\bar{k}})$ is countable and divisible. Note that in the last case, assuming $k$ is finitely generated, the group $\text{Br}(X_{\bar{k}})^F$ does not always have finite exponent, see below.

We have the following result over an arbitrary ground field.

Theorem 1.14 Let $X$ be a smooth, projective, geometrically integral variety over a field $k$ of characteristic exponent $p$. Then $\text{Br}(X^s)^F/\text{Br}(X^s)^k$, that is, the cokernel of the natural map $\text{Br}(X) \to \text{Br}(X^s)^F$, is the direct sum of a finite group of order coprime to $p$ and a $p$-torsion group of finite exponent. In particular, if $\text{char}(k) = 0$, then $\text{Br}(X^s)^k$ is finite if and only if $\text{Br}(X^s)^F$ is finite.

This was proved in [CTS13b] when $\text{char}(k) = 0$. As pointed out by Xinyi Yuan [Yua20], the same method works over any $k$.

Building on Corollary 1.4 one strengthens that corollary as follows.

Theorem 1.15 Let $k$ be a finitely generated field. If $X$ is an abelian variety or a K3 surface, then the subgroup of $\text{Br}(X^s)^F$ consisting of the elements of order coprime to $\text{char}(k)$ is finite.

Proof. This is proved by Skorobogatov and Zarhin [SZ08, SZ15], except for the case of K3 surfaces over a field of characteristic 2 where the proof is due to K. Ito [Ito18]. For the case of abelian varieties see also [CTS21, Thm. 16.2.3]. The proof uses results of Faltings and Zarhin who showed that for abelian varieties $A, B$ over a field finitely generated over a prime subfield the natural injective map

$$
\text{Hom}(A, B)/\ell \hookrightarrow \text{Hom}(A[\ell], B[\ell])
$$

is an isomorphism for almost all $\ell$.

For the case of K3 surfaces see [CTS21, Thm. 16.7.2, Remark 16.7.3]. The proof uses the Kuga–Satake construction and the aforementioned results of Faltings and Zarhin. In the finite characteristic case, the proof uses the Tate conjecture for K3 surfaces established by Madapusi Pera, and by W. Kim and Madapusi Pera when $\text{char}(k) = 2$. \hfill $\square$

11
Corollary 1.16 Let $X$ be a K3 surface over a finitely generated field. The subgroup of $\text{Br}(X)/\text{Br}_0(X)$ consisting of the elements of order not divisible by $\text{char}(k)$ is finite.

Proof. We just need to use that $H^1(k, \text{Pic}(X^*))$ is finite. □

A recent result of M. D’Adezzio [D’Ad, Thm. 1.1] concerns $p$-primary torsion of the Brauer group of abelian varieties over finitely generated fields of characteristic $p$.

Theorem 1.17 (D’Adezzio) Let $A$ be an abelian variety over a finitely generated field $k$ of characteristic $p > 0$. Then the transcendental Brauer group $\text{Br}(A)^k \cong \text{Br}(A_{\overline{k}})^k$ is a direct sum of a finite group and a finite exponent $p$-group.

The following general statement appears to be a correct generalisation of Theorem 1.15. (Note that this is about $k_s$, not $\overline{k}$.)

Corollary 1.18 Let $A$ be an abelian variety over a finitely generated field $k$. Then $\text{Br}(A)^F$ has finite exponent.


Question 1.19 Does $\text{Br}(X^*)^F$ have finite exponent if $X$ is a K3 surface? It is easy to show that this holds for Kummer surfaces over a field $k$ of characteristic $p \neq 2$. Indeed, let $X = \text{Kum}(A)$, where $A$ is an abelian surface. By Theorem 1.15 it remains to deal with $p$-primary torsion. The rational map of degree 2 from $A$ to $X$ induces a map $\text{Br}(X) \to \text{Br}(A)$ and a compatible map $\text{Br}(X^*) \to \text{Br}(A^*)$. The standard restriction-corestriction argument shows that both maps are injective on the subgroups of elements of odd order, see, e.g., [CTS21, Prop. 3.8.4]. Hence $\text{Br}(X^*)^k \{p\} \subset \text{Br}(A^*)^k \{p\}$, so $\text{Br}(X^*)^k \{p\}$ has finite exponent by Theorem 1.17. Thus $\text{Br}(X^*)^F \{p\}$ has finite exponent by Theorem 1.14.

Example 1.20 Note that the transcendental Brauer group of $A$ may well be infinite, cf. [D’Ad, Cor. 5.4]. Let $E_1, E_2$ be supersingular elliptic curves over an infinite finitely generated field $k$ of characteristic $p$, and let $A = E_1 \times_k E_2$. Infinitely many elements of $\text{Br}(A)$ that survive in $\text{Br}(A_{\overline{k}})$ are easy to construct. Indeed, multiplication by $p$ map $[p] : E_i \to E_i$ is an fppf-torsor with structure group $E_i[p]$, for $i = 1, 2$. The cup-product of these torsors for the two factors is a class in $H^2_{\text{fppf}}(A, E_i[p] \otimes E_2[p])$. The Weil pairing induces an isomorphism of fppf sheaves over $\text{Spec}(k)$:

$\text{Hom}(E_1[p] \otimes E_2[p], \mu_p) \cong \text{Hom}(E_1[p], E_2[p]).$

This gives a map

$\text{Hom}_{\text{fppf schemes}}(E_1[p], E_2[p]) \to H^2_{\text{fppf}}(A_k, \mu_p),$

which one shows to be injective, and a similar map over $k$. The Kummer exact sequence in fppf topology gives a map

$H^2_{\text{fppf}}(A_k, \mu_p) \to \text{Br}(A_k),$

12
there are only finitely many isomorphism classes among the rings \( \text{End}(A) \) of abelian varieties. Coleman’s conjecture about \( \text{End}(A) \) of definition and the dimension (in the case of abelian varieties) are bounded. That certain invariants take only finitely many values provided the degree of the field of definition and the dimension (in the case of abelian varieties) are bounded.

The aim of this section is to discuss links among several conjectures about K3 surfaces and abelian varieties defined over number fields. These conjectures state that certain invariants take only finitely many values provided the degree of the field of definition and the dimension (in the case of abelian varieties) are bounded.

Example 1.21 When \( p \neq 2 \) the above classes descend to \( X = \text{Kum}(A) \). This gives an example of a K3 surface with an infinite transcendental Brauer group, answering [SZ08, Questions 1, 2] in the negative. Indeed, let \( \bar{A} \) be the surface obtained by blowing up the subscheme \( A[2] \subset A \). By the birational invariance of the Brauer group we have \( \text{Br}(A) \cong \text{Br}(\bar{A}) \), see [CTS21, Cor. 6.2.11]. There is a finite surjective morphism \( f: \bar{A} \to X \) such that \( k(\bar{A}) \) is a Galois extension of \( k(X) \) with Galois group \( G \cong \mathbb{Z}/2 \). By [CTS21, Thm. 3.8.5] we have \( \text{Br}(X)\{p\} \cong \text{Br}(A)\{p\}^G \). It is easy to check that the above classes are \( G \)-invariant. Indeed, the generator of \( G \) multiplies the class of each torsor \( [p]: E_i \to E_i \) by \(-1\), hence the cup-product class in \( H^2_{\text{fppf}}(A, E_i[p] \otimes E_2[p]) \) is fixed by \( G \), and thus so is the resulting Brauer class.

Example 1.22 D’Adlezio gives an example to show that in the case of finite characteristic, \( \text{Br}(A_k) \) does not always have finite exponent, see [D’Ad, Cor. 6.7]. Take \( A = E \times_k E \), where \( E \) is an ordinary elliptic curve over \( k \) with \( \text{End}(E) \cong \mathbb{Z} \). Then \( T_p(\text{Br}(A_k)) \) contains the quotient of \( \text{End}(E_k[p^\infty]) \) by \( \text{End}(E_k) \otimes \mathbb{Z}_p \). Taking Galois invariants we obtain that \( T_p(\text{Br}(A_k)) \) contains the quotient of \( \text{End}(E_k[p^\infty])^\Gamma \) by \( \text{End}(E) \otimes \mathbb{Z}_p \cong \mathbb{Z}_p \), so it is enough to show that the rank of the \( \mathbb{Z}_p \)-module \( \text{End}(E_k[p^\infty])^\Gamma \) is at least 2. Since \( \text{End}(E) \cong \mathbb{Z} \), the elliptic curve \( E \) is not supersingular, so the \( p \)-divisible group \( E[p^\infty] \) has at least two slopes. By the Dieudonné–Manin classification, this implies that \( E[p^\infty] \) is isogenous to the direct sum of two non-zero \( p \)-divisible groups, hence the rank of \( \text{End}(E_k[p^\infty])^\Gamma \) is at least 2.

As in Remark 1.21, these classes also descend to \( X = \text{Kum}(A) \), showing that \( \text{Br}(X_k) \) does not always have finite exponent when \( X \) is a K3 surface.

1.4 Uniformity

The aim of this section is to discuss links among several conjectures about K3 surfaces and abelian varieties defined over number fields. These conjectures state that certain invariants take only finitely many values provided the degree of the field of definition and the dimension (in the case of abelian varieties) are bounded.

Coleman’s conjecture about \( \text{End}(A_k) \). Let \( d \) and \( g \) be positive integers. Consider all abelian varieties \( A \) of dimension \( g \) defined over number fields of degree \( d \). Then there are only finitely many isomorphism classes among the rings \( \text{End}(A_k) \).
This or a closely related conjecture is attributed to Robert Coleman in [Sha96a, Remark 4]. There is a version of this conjecture in which End$(A_{\bar{k}})$ is replaced by the ring End$(A)$ of endomorphisms of $A$ defined over $k$. It is not too hard to show that Coleman’s conjecture about End$(A_{\bar{k}})$ is equivalent to Coleman’s conjecture about End$(A)$, see [OSZ21, Thm. 3.4].

Rémond proved that Coleman’s conjecture implies the uniform boundedness of torsion $A(k)_{\text{tors}}$ and of the minimal degree of an isogeny between isogenous abelian varieties, see [Rem18, Thm. 1.1].

**Shafarevich’s conjecture about** $\text{NS}(X_{\bar{k}})$. *Let $d$ be a positive integer. There are only finitely many lattices $L$, up to isomorphism, for which there exists a K3 surface $X$ defined over a number field of degree $d$ such that $\text{NS}(X_{\bar{k}}) \cong L$.***

It is in this form that Shafarevich has stated his conjecture in [Sha96a]. Since there are only finitely many lattices of bounded rank and discriminant, Shafarevich’s conjecture is equivalent to the boundedness of the discriminant of $\text{NS}(X_{\bar{k}})$. One can also state a variant of Shafarevich’s conjecture in which $\text{NS}(X_{\bar{k}})$ is replaced by its Galois-invariant subgroup $\text{NS}(X_{\bar{k}})^{\Gamma}$, or, alternatively, by $\text{Pic}(X)$. By [OSZ21, Thm. 3.5] all these versions of Shafarevich’s conjecture are equivalent.

Similarly to Shafarevich’s conjecture, Coleman’s conjecture can be restated in terms of lattices. Recall that End$(A)$ is an order in the semisimple $\mathbb{Q}$-algebra End$(A)_{\mathbb{Q}} = \text{End}(A) \otimes \mathbb{Q}$. Let us define $\text{discr}(A)$ as the discriminant of the integral symmetric bilinear form $\text{tr}(xy)$ on End$(A)$, where $\text{tr} : \text{End}(A)_{\mathbb{Q}} \rightarrow \mathbb{Q}$ is the reduced trace. An equivalent form of Coleman’s conjecture says that $\text{discr}(A)$ is uniformly bounded for abelian varieties $A$ of bounded dimension defined over number fields of bounded degree.

**Várilly-Alvarado’s conjecture.** [VA17, Conj. 4.6] *Let $d$ be a positive integer and let $L$ be a primitive sublattice of the K3 lattice $E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. If $X$ is a K3 surface defined over a number field of degree $d$ such that $\text{NS}(X_{\bar{k}}) \cong L$, then the cardinality of $\text{Br}(X)/\text{Br}_0(X)$ is bounded.*

A stronger form of this conjecture omits the reference to the Néron–Severi lattice. It concerns the uniform boundedness of the Galois invariant subgroup of the geometric Brauer group.

It is well known that the cardinality of the finite group $H^1(k, \text{Pic}(X_{\bar{k}}))$, where $X$ is a K3 surface over an arbitrary field $k$ of characteristic zero, is bounded, see e.g. [VAV17, Lemma 6.4]. This is based on the well known lemma of Minkowski that says that the order of finite subgroups of $\text{GL}(\mathbb{Z}, n)$ is bounded by a function of $n$.

**Conjecture Br(K3).** *Let $d$ be a positive integer. There is a constant $C = C(d)$ such that, if $X$ is a K3 surface defined over a number field of degree $d$, then $|\text{Br}(X_{\bar{k}})^{\Gamma}| < C$.***

A similar conjecture can be stated for abelian varieties of given dimension.

**Conjecture Br(AV).** *Let $d$ and $g$ be positive integers. There is a constant $C =
$C(d, g)$ such that, if $A$ is an abelian variety of dimension $g$ defined over a number field of degree $d$, then $|\text{Br}(A)^\Gamma| < C$.

As proved in [OSZ21], these conjectures are logically related as follows:

\[
\begin{align*}
\text{Coleman’s conjecture} & \implies \text{Shafarevich’s conjecture} \\
\implies \text{Br}(AV) & \implies \text{Várilly-Alvarado’s conjecture} \\
\implies \text{Br}(K3)
\end{align*}
\]

**Theorem 1.23** The conjectures featuring in this diagram hold for abelian varieties and K3 surfaces with complex multiplication.

**Proof.** See [OS18]. The proof uses a lower bound for the size of Galois orbits of CM points from work of Tsimerman building on the proof of the averaged Colmez conjecture by Andreatta, Goren, Howard and Madapusi Pera, and by X. Yuan and S. Zhang.

Coleman’s conjecture for elliptic curves follows from the Brauer–Siegel theorem, but in the general case not much is known about these conjectures.

All these conjectures may be stated in the form “in a certain class of moduli spaces, only finitely many spaces in the class have rational points over number fields of degree $d$, excluding points which lie in subvarieties of positive codimension parameterising objects with extra structures”.

Some of the above conjectures are known for the fibres of one-parameter families. In particular, a result of Cadoret and Tamagawa implies Coleman’s conjecture within a one-parameter family of abelian varieties. Cadoret and Charles have proved uniform boundedness of the $\ell$-primary subgroup of the Brauer group for one-parameter families of abelian varieties and K3 surfaces. Várilly-Alvarado and Viray obtained bounds for the Brauer group for one-parameter families of Kummer surfaces attached to products of isogenous elliptic curves [VAV17, Thm. 1.8].
References


