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Descent obstruction is equivalent to étale Brauer–Manin obstruction

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Abstract Let X be a projective variety over a number field. Completing earlier work of D. Harari, C. Demarche and M. Stoll, we prove that the obstruction to the Hasse principle and weak approximation on X given by descent on torsors under linear algebraic groups is equivalent to the Brauer–Manin obstruction applied to étale covers of X.

1 Introduction

Let k be a number field, and let X be a smooth projective variety over k. The embedding of k into the ring of adèles $\mathbb{A}_k$ makes the set of k-points $X(k)$ a subset of $X(\mathbb{A}_k) = \prod_v X(k_v)$, where the product is taken over all completions of k. Any torsor $f : Y \to X$ under a linear k-group G gives rise to a subset of $X(\mathbb{A}_k)$ containing $X(k)$, namely,

$$X(\mathbb{A}_k)^f = \bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k)),$$

where $f^\sigma : Y^\sigma \to X$ is the twist of $f : Y \to X$ by the 1-cocycle σ. The set $X(\mathbb{A}_k)^f$ consists of the adelic points (P_v) on X such that the collection of classes of k_v-torsors $f^{-1}(P_v)$ comes from a k-torsor under G, see [10, Def. 4.2], or [14, Def. 5.3.1]. Let

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$$X(\mathbb{A}_k)^{\mathrm{desc}} = \bigcap X(\mathbb{A}_k)^f,$$

where $f : Y \to X$ ranges over all torsors under all linear k-groups. In his paper [15] Stoll introduced a similar set

$$X(\mathbb{A}_k)^{\mathrm{f-cov}} = \bigcap X(\mathbb{A}_k)^f,$$

where $f : Y \to X$ ranges over all torsors under all *finite* k-groups. He proved that this set is well behaved with respect to torsors under finite groups, namely, if $f : Y \to X$ is a torsor under a finite k-group F, then

$$X(\mathbb{A}_k)^{\mathrm{f-cov}} = \bigcup_{[\sigma]\in \mathrm{H}^1(k,F)} f^\sigma\left(Y^\sigma(\mathbb{A}_k)^{\mathrm{f-cov}}\right), \tag{1}$$

see [15, Prop. 5.17]. In this note we use Stoll's method to obtain a similar formula for $X(\mathbb{A}_k)^{\mathrm{desc}}$.

Theorem 1.1 *Let X be a smooth, projective and geometrically integral variety over a number field k. Let $f : Y \to X$ be a torsor under a finite k-group scheme F. Then*

$$X(\mathbb{A}_k)^{\mathrm{desc}} = \bigcup_{[\sigma]\in \mathrm{H}^1(k,F)} f^\sigma\left(Y^\sigma(\mathbb{A}_k)^{\mathrm{desc}}\right). \tag{2}$$

It is well known that a similar property does not hold for the Brauer–Manin set $X(\mathbb{A}_k)^{\mathrm{Br}}$; indeed, $\cup f^\sigma(Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}})$ can be strictly smaller than $X(\mathbb{A}_k)^{\mathrm{Br}}$. This idea is used in the author's counterexample to the Hasse principle not explained by the Brauer–Manin obstruction ([13] or [14, Ch. 8], see also [1, Sect. 4]). In [13, Sect. 3] the author pointed out that this counterexample is explained by the Brauer–Manin obstruction applied to an étale covering of X that is an X-torsor under a finite abelian k-group. Following [12] define

$$X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = \bigcap_f \bigcup_{[\sigma]\in \mathrm{H}^1(k,F)} f^\sigma\left(Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}}\right),$$

where $f : Y \to X$ ranges over all torsors under all finite k-groups. In [12] Poonen constructed a threefold X over any number field (in fact, over any global field of characteristic not equal to 2) such that $X(k) = \emptyset$ but $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} \neq \emptyset$. He asked if such a counterexample to the Hasse principle can be explained by the descent obstruction, that is, whether the set $X(\mathbb{A}_k)^{\mathrm{desc}}$ is empty or not. More generally, he asked [12, Question 3.1] whether one always has the inclusion

$$X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} \subset X(\mathbb{A}_k)^{\mathrm{desc}}. \tag{3}$$

This was proved by Demarche [7] using the results of Harari [9] and Borovoi [3]. As a corollary of our Theorem 1.1 we prove the opposite inclusion.

Corollary 1.2 *Let X be a smooth, projective and geometrically integral variety over a number field k. Then $X(\mathbb{A}_k)^{\mathrm{desc}} \subset X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}$.*

Combining this with Demarche's result we see that the two sets are in fact the same:

$$X(\mathbb{A}_k)^{\mathrm{desc}} = X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}.$$

Theorem 1.1 and Corollary 1.2 are proved in Sect. 2. In Sect. 3 we deduce some easy consequences of these results for the closure of the set of rational points in the adelic points on surfaces of Kodaira dimension zero. The author is grateful to C. Demarche, D. Harari and B. Poonen for a useful discussion. I thank the referee for his careful reading of the paper and many helpful suggestions.

2 Proof of the main theorem

Let k be a field with separable closure $\bar{k}$, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. In this paper a variety over k means a separated k-scheme of finite type, and a k-group means a smooth k-group scheme of finite type. A k-group is linear if its underlying variety is affine. *Unless stated otherwise, the structure groups of all torsors in this paper are linear k-groups.*

If G is a k-group, we denote by $Z^1(k, G)$ the set of continuous 1-cocycles of the profinite group Γ with coefficients in the discrete group $G(\bar{k})$. We agree on the following terminology: if $Y \to X$ is a torsor under G, then a *twist* of Y/X always means the twist of Y by a 1-cocycle from $Z^1(k, G)$. Note that this twist is a torsor under the twisted form of G given by the same cocycle with respect to the action of G on itself by conjugations.

Definition 2.1 Given X-torsors Y and Z under k-groups G and H, respectively, we define an *X-torsor morphism* $Y \to Z$ as an X-morphism compatible with a homomorphism $G \to H$.

An X-torsor morphism $f : Y \to Z$ is *surjective* if f is a surjective morphism; equivalently, if the corresponding homomorphism $\phi : G \to H$ is surjective. In this case $Y \to Z$ is a torsor under $\mathrm{Ker}\,\phi$.

The beginning of the proof of the theorem follows Stoll's proof of his Prop. 5.17 verbatim. His arguments can be summarized as follows.

Lemma 2.2 (Stoll) *Let X be a proper variety over a number field k, and let Y/X be a torsor. For any $(P_v) \in X(\mathbb{A}_k)^{\mathrm{desc}}$ there exists a twist Y'/X of the torsor Y/X with the following property*:
for any surjective X-torsor morphism $Z \to Y'$ there exists a twist Z'/Y' of the torsor Z/Y' such that (P_v) lies in the image of $Z'(\mathbb{A}_k)$.

Proof By [10, Prop. 4.4], or [14, Prop. 5.3.2] (based on the Borel–Serre theorem), there are only finitely many twists of a given torsor which contain adelic points. The finite combinatorics in the first part of the proof of Prop. 5.17 of [15] does not rely on the finiteness of the group schemes involved, and hence can be applied in our more general situation. □

Let Y be an étale X-scheme, and let Z be an affine Y-scheme. Recall that the Weil restriction $R_{Y/X}(Z)$ is an affine X-scheme representing the contravariant functor $S \mapsto \mathrm{Hom}_Y(S \times_X Y, Z)$ from the category of X-schemes to the category of sets. The existence of $R_{Y/X}(Z)$ is proved in [4, Thm. 7.6.4].

Proposition 2.3 *Let k be a field, and let X be a variety over k. Let $Y \to X$ be a torsor under a finite étale k-group, and let $Z \to Y$ be a torsor. Then there exists a torsor $V \to X$ and a surjective X-torsor morphism $h : V \to Y$ such that V, considered as a Y-torsor via h, admits a surjective Y-torsor morphism to Z.*

Proof Define $V = R_{Y/X}(Z) \times_X Y$, and let $h : V \to Y$ be the projection to the second factor. Let F be the finite étale k-group which is the structure group of the torsor $f : Y \to X$, and let G be the linear k-group which is the structure group of the torsor $g : Z \to Y$. A canonical isomorphism $Y \times_X Y = Y \times_k F$ gives rise to canonical isomorphisms

$$V = R_{Y/X}(Z) \times_X Y = R_{Y \times_X Y/Y}(Z \times_X Y) = R_{Y \times_k F/Y}(Z \times_k F). \tag{4}$$

One can also give a geometric description of V. Let L/k be a separable extension which splits F, i.e. such that the scheme $F_L = F \times_k L$ is the disjoint union of $m = |F(\overline{k})|$ copies of $\mathrm{Spec}(L)$, e.g. $L = \overline{k}$. Then $Y_L \to X_L$ is a Galois covering with the Galois group $F(L) = F(\overline{k})$. Writing down the canonical isomorphisms in (4) one sees that V_L is the fibred product of m copies of Z_L over Y_L with respect to the morphisms $\sigma g : Z_L \to Y_L$ for all $\sigma \in F(\overline{k})$.

Since Z is a Y-torsor under $G_Y = G \times_k Y$ we see that $R_{Y/X}(Z)$ is an X-torsor under the X-group scheme $R_{Y/X}(G_Y)$, hence V is a Y-torsor under the Y-group scheme

$$\begin{aligned} R_{Y/X}(G_Y) \times_X Y &= R_{Y \times_X Y/Y}(G_Y \times_X Y) = R_{Y \times_k F/Y}(G \times_k Y \times_k F) \\ &= R_{F/k}(G_F) \times_k Y. \end{aligned}$$

Explicitly, if the scheme F is the disjoint union of $\mathrm{Spec}(k_i)$, where k_i, $i = 1, \ldots, n$, are finite extensions of k, then $V \to Y$ is a torsor under the linear k-group

$$R_{F/k}(G_F) = \prod_{i=1}^{n} R_{k_i/k}(G_{k_i}).$$

It is clear that $R_{F/k}(G_F)_L \simeq (G_L)^m$, so that $R_{F/k}(G_F)$ is an L/k-form of G^m. We can describe this twisted form as follows. Let S_m be the symmetric group acting on the m-element set $F(\overline{k})$ by permutations. Then S_m acts on the coordinate $\overline{k}$-algebra $\overline{k}[F] \simeq \overline{k}^m$ by permutations of factors. The Galois group $\Gamma = \mathrm{Gal}(\overline{k}/k)$ acts on $F(\overline{k})$ by group automorphisms, via a homomorphism $\rho : \Gamma \to S_m$ defined up to conjugation, such that $\rho(\Gamma)$ normalizes $F(\overline{k}) \subset S_m$. Various objects acted on by S_m can be twisted by ρ: the twisted form of k^m is the étale k-algebra $k[F]$, the twisted form of the 'constant' k-group scheme $F(\overline{k})$ is F, and the twisted form of G^m is $R_{F/k}(G_F)$. From the action of $F(\overline{k})$ on G^m by permutations of factors via ρ we obtain a k-action of the twisted form F on the k-group $R_{F/k}(G_F)$, i.e. a morphism of k-group schemes

$$F \times_k R_{F/k}(G_F) \to R_{F/k}(G_F).$$

For $\sigma \in F(\bar{k})$ and $\alpha \in R_{F/k}(G_F)(\bar{k})$ we write ${}^{\sigma}\alpha$ for the result of applying σ to α.

Let us define $\mathcal{G}$ as the semi-direct product $R_{F/k}(G_F) \rtimes F$ with respect to this action. The k-group F acts on $V = R_{Y/X}(Z) \times_X Y$ via the second factor, so that F and $R_{F/k}(G_F)$ both act freely on V. For any $\sigma \in F(\bar{k})$ and any $\alpha \in R_{F/k}(G_F)(\bar{k})$ the element $\sigma\alpha\sigma^{-1}$ acts on $V \times_k \bar{k}$ as ${}^{\sigma}\alpha \in R_{F/k}(G_F)(\bar{k})$. We thus obtain a k-action of $\mathcal{G}$ on V which extends that of $R_{F/k}(G_F)$, and makes V an X-torsor under $\mathcal{G}$. Moreover, the X-morphism $h : V \to Y$ is compatible with the natural surjection $\mathcal{G} \to F$, and so is a surjective X-torsor morphism.

The inclusion $\mathrm{Spec}(k) \hookrightarrow F$ of the origin of the group law on F defines a surjective homomorphism of k-group schemes $\phi : R_{F/k}(G_F) \to G$. It also gives rise to a section $Y \to Y \times_k F$ of $Y \times_k F/Y$, and hence to a morphism of Y-schemes

$$V = R_{Y \times_k F/Y}(Z \times_k F) \to R_{Y/Y}(Z) = Z,$$

which is compatible with ϕ, and hence is a surjective Y-torsor morphism. □

Remark The easiest case of this proposition is when $F = \mathbf{Z}/2$. Then V is simply the fibred product of $Z \to Y$ and the same morphism followed by the automorphism of Y given by the non-trivial element of F. In this case $R_{F/k}(G_F) = G \times_k G$, and the structure group of V/X is the semi-direct product $G^2 \rtimes \mathbf{Z}/2$. This situation arises, for example, for a K3 covering Y of an Enriques surface X. A torsor over Y may or may not be a torsor over X, but irrespective of this there exists a bigger X-torsor V that factors through Z.

The following proposition fills a minor gap in the argument of [14, Prop. 5.3.3], where it is not shown that the collection of local points $P_v \in Y(k_v)$ is in fact an adelic point on Y.

Proposition 2.4 *Let X be a (not necessarily proper) variety over a number field k, and let $f : Y \to X$ be a torsor under a k-group. Then*

$$f(Y(\mathbb{A}_k)) = X(\mathbb{A}_k) \cap \prod_v f(Y(k_v)) \subset \prod_v X(k_v).$$

Proof For a prime v of k we denote by $\mathcal{O}_v$ the ring of integers of k_v, and by $\mathbb{F}_v$ the residue field of $\mathcal{O}_v$. Let G be the structure group of the torsor $f : Y \to X$, and let G^0 be the connected component of G. Then $F = G/G^0$ is a finite étale k-group scheme. There is a finite set S of primes of k such that G^0, G, F extend to group schemes $\mathcal{G}^0$, $\mathcal{G}$, $\mathcal{F}$ over the ring of S-integers $\mathcal{O}_S \subset k$, where $\mathcal{F}$ is finite and étale, and $\mathcal{G}^0$ has connected fibres. We can assume that $f : Y \to X$ comes from a torsor $f : \mathcal{Y} \to \mathcal{X}$ under $\mathcal{G}$, where $\mathcal{X}$ and $\mathcal{Y}$ are faithfully flat $\mathcal{O}_S$-schemes. Let $(P_v) \in X(\mathbb{A}_k)$ be an adelic point. We enlarge S, if necessary, to ensure that $P_v \in \mathcal{X}(\mathcal{O}_v)$ for all primes $v \notin S$.

If $(P_v) \in \prod_v f(Y(k_v))$, then the k_v-torsor $f^{-1}(P_v)$ under G is trivial for every place v of k. To show that Y has an adelic point over (P_v) we need to prove that the

$\mathcal{O}_v$-torsor $f^{-1}(P_v)$ under $\mathcal{G}$ is trivial for every prime $v \notin S$. Its quotient by the action of the group scheme $\mathcal{G}^0 \times_{\mathcal{O}_S} \mathcal{O}_v$ is an $\mathcal{O}_v$-torsor under the finite group scheme $\mathcal{F}$, which acquires a point over k_v. By the valuative criterion of properness this point extends to a point over $\mathcal{O}_v$. The inverse image of this point in $f^{-1}(P_v)$ is an $\mathcal{O}_v$-torsor under $\mathcal{G}^0 \times_{\mathcal{O}_S} \mathcal{O}_v$. But such a torsor is trivial: indeed, by Lang's theorem every torsor under a connected $\mathbb{F}_v$-group is trivial, and by Hensel's lemma an $\mathbb{F}_v$-point extends to an $\mathcal{O}_v$-point. This proves the desired triviality of the $\mathcal{O}_v$-torsor $f^{-1}(P_v)$. □

Corollary 2.5 *Let X be a (not necessarily proper) variety over a number field k, and let $f : Y \to X$ be a torsor under a k-group. Then $f(Y(\mathbb{A}_k))$ is closed in $X(\mathbb{A}_k)$.*

Proof By Proposition 2.4 it is enough to show that $\prod_v f(Y(k_v))$ is a closed subset of $\prod_v X(k_v)$, and for this we need to prove that $f(Y(k_v))$ is closed in $X(k_v)$ for every place v of k. This is a well known consequence of the implicit function theorem, see [10, Lemma 4.6]. □

Corollary 2.6 *Let X be a proper variety over a number field k, and let $f : Y \to X$ be a torsor under a k-group. Then $f(Y(\mathbb{A}_k)) = \prod_v f(Y(k_v))$.*

Proof Since X is proper we have $X(\mathbb{A}_k) = \prod_v X(k_v)$. □

Corollary 2.7 *Let X be a proper variety over a number field k, and let $f : Y \to X$ be a torsor under a linear k-group. Then the set $X(\mathbb{A}_k)^f$ is closed in $X(\mathbb{A}_k)$.*

Proof By Corollary 2.5, $f(Y(\mathbb{A}_k))$ is closed in $X(\mathbb{A}_k)$, and the same is true for every twist of the torsor Y/X. Since X is proper, only finitely many of these twists contain adelic points [14, Prop. 5.3.2], hence $X(\mathbb{A}_k)^f$ is a closed subset of $X(\mathbb{A}_k)$. □

End of proof of Theorem 1.1 The inclusion of the right hand side of (2) into the left hand side is obvious: indeed, for any morphism $f : Y \to X$ we have $f(Y(\mathbb{A}_k)^{\mathrm{desc}}) \subset X(\mathbb{A}_k)^{\mathrm{desc}}$, because torsors can be pulled back from X to Y. To prove the opposite inclusion in (2) we go back to Stoll's arguments in the proof of his Prop. 5.17. Let $(P_v) \in X(\mathbb{A}_k)^{\mathrm{desc}}$, and let $X' \to X$ be a torsor under a finite k-group scheme. Let $f : Y \to X$ be a twist of $X' \to X$ satisfying the conclusions of Lemma 2.2. It is enough to prove that (P_v) lifts to a point in $Y(\mathbb{A}_k)^{\mathrm{desc}}$. Note that since X is proper and f is étale, Y is proper too.

If (P_v) does not lift to a point in $Y(\mathbb{A}_k)^{\mathrm{desc}}$, then the set $f^{-1}((P_v))$ has a covering by its intersections with $Y(\mathbb{A}_k)\backslash Y(\mathbb{A}_k)^g$, for all torsors $g : Z \to Y$. This is an open covering since $Y(\mathbb{A}_k)^g$ is closed in $Y(\mathbb{A}_k)$ by Corollary 2.7. The set $f^{-1}((P_v))$ is a product of finite sets, and hence is compact by Tikhonov's theorem. Thus there are torsors $g_i : Z_i \to Y$, $i = 1, \ldots, n$, such that $f^{-1}((P_v))$ is contained in the union of the open sets $Y(\mathbb{A}_k)\backslash Y(\mathbb{A}_k)^{g_i}$ for $i = 1, \ldots, n$. Let $g : Z \to Y$ be the fibred product of the torsors $g_i : Z_i \to Y$ over Y. We denote by G the structure group of Z/Y. By construction we have $f^{-1}((P_v)) \cap Y(\mathbb{A}_k)^g = \emptyset$.

Let $h : V \to Y$ be a surjective X-torsor morphism satisfying the conclusion of Proposition 2.3. We call H the structure group of the torsor V/Y. The morphism $fh : V \to X$ is a torsor, hence by Lemma 2.2 there exists a 1-cocycle $\sigma \in Z^1(k, H)$ and a point $(M_v) \in V^\sigma(\mathbb{A}_k)$ whose image in X is (P_v). Let ρ be the image of σ

in $Z^1(k, G)$ under the homomorphism $H \to G$. Then $V^\sigma \to Y$ factors through $Z^\rho \to Y$. The image of (M_v) in Z^ρ is an adelic point whose projection to $Y(\mathbb{A}_k)$ is in $f^{-1}((P_v)) \cap Y(\mathbb{A}_k)^g$, hence this set is non-empty. This contradiction completes the proof. □

The following lemma is a well known corollary of a theorem of Gabber (also proved by de Jong [6]), see [14, Prop. 5.3.4] .

Lemma 2.8 *For any smooth projective variety Y over a number field k we have*

$$Y(\mathbb{A}_k)^{\mathrm{desc}} \subset Y(\mathbb{A}_k)^{\mathrm{Br}}.$$

Proof By the aforementioned theorem of Gabber for any $A \in \mathrm{Br}\, Y$ there exists a torsor $Z_A \to Y$ under PGL_m for some m such that A is the image of the class of Z_A/Y in $\mathrm{H}^1(Y, \mathrm{PGL}_m)$ under the connecting map $\mathrm{H}^1(Y, \mathrm{PGL}_m) \to \mathrm{H}^2(Y, \mathbf{G}_m)$. It is easy to see that an adelic point $(Q_v) \in Y(\mathbb{A}_k)$ satisfies the Brauer–Manin condition given by A if and only if (Q_v) lifts to an adelic point on a twist of the torsor $Z_A \to Y$ (see [10, Thm. 4.10], or [14, Prop. 5.3.4]). This implies the lemma. □

Proof of Corollary 1.2 It follows from Theorem 1.1 by Lemma 2.8. □

3 Remarks on surfaces of Kodaira dimension zero

Let k be a field of characteristic 0, and let X be a surface over k, that is, a smooth, projective, geometrically integral variety of dimension 2. The Kodaira dimension of X is defined as

$$\kappa(X) = \mathrm{tr.deg.}_k \left(\bigoplus_{n=0}^{\infty} \mathrm{H}^0(X, \mathcal{O}(K_X)^{\otimes n}) \right) - 1,$$

where K_X is the canonical class. The classification of surfaces of Kodaira dimension zero over an algebraically closed field of characteristic 0 is well known, see [2, Ch. VI]. It follows from this classification that a surface X over k with $\kappa(X) = 0$ is isomorphic to one of the following surfaces:

(1) A k-torsor P of an abelian surface A.
(2) An étale quotient B of P by a free action of μ_n for $n = 2,\ 3,\ 4$ or 6, where μ_n does not contain translations by non-zero elements of A. Then B is a bielliptic surface (see [1, Prop. 1]). The canonical class K_B has order n.
(3) A K3 surface, with $K_X = 0$.
(4) An étale quotient Y of a K3 surface by μ_2, called an Enriques surface. Then $2K_Y = 0$, $K_Y \neq 0$.

If B is a bielliptic surface such that K_B has order n, then for P one can take any B-torsor under μ_n whose type is the homomorphism of Γ-modules $\mathbf{Z}/n \to \mathrm{Pic}\overline{B}$ sending 1 to K_B, see *loc. cit.* We shall refer to P as a B-torsor of *canonical* type. B-torsors of canonical type always exist, and are twists of each other by cocycles from $Z^1(k, \mu_n)$. Each of them is a k-torsor under an abelian surface, namely its Albanese variety. All these abelian surfaces are twists of one of them, say A, by cocycles from $Z^1(k, \mu_n)$, with μ_n acting on A by automorphisms of an abelian variety.

Now let k be a number field. Recall that if $f : Y \to X$ is a torsor under a linear k-group G, and k_v is archimedean, then the function $X(k_v) \to \mathrm{H}^1(k_v, G)$ sending a point P to the class of the fibre $f^{-1}(P)$, is constant on every connected component of $X(k_v)$.

Let $A(\mathbb{A}_k)_0$ be the connected component of 1. It is well known (see, e.g., [15, Cor. 6.2], [14, Prop. 6.2.4]) that if the Tate–Shafarevich group of A is finite, then

$$P(\mathbb{A}_k)^{\mathrm{f-cov}} = P(\mathbb{A}_k)^{\mathrm{Br}} = A(\mathbb{A}_k)_0 \cdot \overline{P(k)}, \tag{5}$$

where $\overline{P(k)}$ is the topological closure of $P(k)$ in $P(\mathbb{A}_k)$. Since $P(\mathbb{A}_k)^{\mathrm{desc}}$ contains $A(\mathbb{A}_k)_0 \cdot \overline{P(k)}$ we also have $P(\mathbb{A}_k)^{\mathrm{f-cov}} = P(\mathbb{A}_k)^{\mathrm{desc}}$. In the notation of [15] let $P(\mathbb{A}_k)_\bullet$ be the product of the topological spaces $P(k_v)$ if v is a finite place, and the sets of connected components $\pi_0(P(k_v))$ if v is an infinite place. The remark about real connected components made above allows us to define $P(\mathbb{A}_k)_\bullet^{\mathrm{desc}}$, $P(\mathbb{A}_k)_\bullet^{\mathrm{f-cov}}$ and $P(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$.

Corollary 3.1 *Let B be a bielliptic surface such that K_B has order n. Let A be the abelian surface which is the Albanese variety of a B-torsor of canonical type. Assume that the Tate–Shafarevich groups of the twisted forms of A by cocycles from $Z^1(k, \mu_n)$ are finite. Then $B(k)$ is dense in $B(\mathbb{A}_k)_\bullet^{\mathrm{f-cov}} = B(\mathbb{A}_k)_\bullet^{\mathrm{desc}}$.*

Proof We have $B(k) = \cup f^\sigma(P^\sigma(k))$, where $\sigma \in Z^1(k, \mu_n)$. Now our statement follows from formulae (1) and (5). □

Since only finitely many twists of a given torsor contain adelic points, it is enough to assume the finiteness of the Tate–Shafarevich group for finitely many twisted forms of A. Harari proved that if B is any bielliptic surface with a k-point, then $B(\mathbb{A}_k)_\bullet^{\mathrm{desc}}$ is strictly smaller than $B(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$ ([8, Prop. 6.2], as interpreted in [10, Sect. 5.2]).

We point out the following consequence of Demarche's result.

Corollary 3.2 *Let X be a simply connected, geometrically connected variety. Then $X(\mathbb{A}_k)^{\mathrm{Br}} = X(\mathbb{A}_k)^{\mathrm{desc}}$.*

Proof Let $(P_v) \in X(\mathbb{A}_k)^{\mathrm{Br}}$, and let $Y \to X$ be any torsor under a finite k-group F. Since X is simply connected, Y is isomorphic to $X \times_k D$, where D is a k-torsor under F. Then there exists an inner form F' of F such that the trivial X-torsor $Y' = X \times_k F'$ is a twist of Y (see [14, Sect. 2.1]). The embedding of the origin of the group law $\mathrm{Spec}(k) \hookrightarrow F'$ defines a section $\varphi : X \to Y'$. It is clear that $(\varphi(P_v)) \in Y'(\mathbb{A}_k)^{\mathrm{Br}}$. This proves that $X(\mathbb{A}_k)^{\mathrm{Br}} = X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}$, and by Demarche's formula (3) this set is contained in $X(\mathbb{A}_k)^{\mathrm{desc}}$. The opposite inclusion is proved in Lemma 2.8. □

The following corollary says that if the Brauer–Manin obstruction is the only obstruction to weak approximation on K3 surfaces, the the descent obstruction is the only obstruction to weak approximation on Enriques surfaces.

Corollary 3.3 *Let f be an étale morphism of degree 2 from a K3 surface X to an Enriques surface Y. We have*

$$Y(\mathbb{A}_k)^{\text{desc}} = \bigcup_{[\sigma]\in \mathrm{H}^1(k,\mu_2)} f^\sigma\left(X^\sigma(\mathbb{A}_k)^{\text{Br}}\right). \tag{6}$$

If $X^\sigma(k)$ is dense in $X^\sigma(\mathbb{A}_k)^{\text{Br}}_\bullet$ for every $\sigma \in Z^1(k,\mu_2)$, then $Y(k)$ is dense in $Y(\mathbb{A}_k)^{\text{desc}}_\bullet$.

Proof Formula (6) is a consequence of (2) and Corollary 3.2. The last statement follows from $Y(k) = \cup f^\sigma(X^\sigma(k))$. □

Remarks 1. It is known that $X(\mathbb{A}_k)^{\text{Br}}$ is open in $X(\mathbb{A}_k)$ for any K3 surface X over a number field k (since $\mathrm{Br}\, X/\mathrm{Br}\, k$ is finite, see [16, Cor. 1.4]).

2. For the Enriques surface Y constructed in [11, Sect. 3.3], the set $Y(\mathbb{A}_k)^{\text{desc}}$ is strictly smaller than $Y(\mathbb{A}_k)^{\text{Br}}$.
3. In [5, Ch. 2] Steven Cunnane found an Enriques surface Y over $\mathbb{Q}$ which is a counterexample to weak approximation explained by neither the Brauer–Manin obstruction on Y nor the *algebraic* Brauer–Manin obstruction on a K3 cover X of Y. In his counterexample the obstruction comes from a transcendental element in $\mathrm{Br}\, X$ represented by a quaternion Azumaya algebra A whose class in $\mathrm{Br}\, X$ is fixed by the Enriques involution. It is a natural question whether this counterexample can be accounted for by the descent obstruction associated to a Y-torsor. The methods of [11] do not seem to work in this case, but Corollary 3.3 gives a positive answer to this question. Explicitly, it suffices to consider the X-torsor under PGL_2 defined by A, and to construct a Y-torsor under $(\mathrm{PGL}_2)^2 \rtimes \mathbf{Z}/2$ as in the remark after Proposition 2.3.

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