Mainz lectures

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1 Kuga–Satake variety

Setting the scene: preliminaries on K3 surfaces

A complex (algebraic) K3 surface X is a smooth, projective, irreducible surface with $\Omega^2_X \cong \mathcal{O}_X$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. Using Serre duality and Riemann–Roch one finds that the classical (Betti) cohomology group $\mathrm{H}^2(X, \mathbb{Z})$ is a free abelian group of rank 22. We have the cup-product

$$\cup : \mathrm{H}^2(X, \mathbb{Z}) \times \mathrm{H}^2(X, \mathbb{Z}) \longrightarrow \mathrm{H}^4(X, \mathbb{Z}) \cong \mathbb{Z},$$

where the last isomorphism is due to the fact that $\dim(X) = 2$. This is a symmetric bilinear pairing. The Poincaré duality implies that this pairing is a perfect duality, that is, it induces an isomorphism

$$\mathrm{H}^2(X, \mathbb{Z}) \tilde{\longrightarrow} \mathrm{Hom}(\mathrm{H}^2(X, \mathbb{Z}), \mathbb{Z}).$$

In other words, the matrix of this bilinear form with respect to a $\mathbb{Z}$-basis of $\mathrm{H}^2(X, \mathbb{Z})$ has determinant in $\mathbb{Z}^\times = \{\pm 1\}$. Topological arguments (Wu's formula, Thom–Hirzebruch index theorem) give that the associated integral quadratic form is even, i.e. $(x^2) \in 2\mathbb{Z}$ for any $x \in \mathrm{H}^2(X, \mathbb{Z})$, and of signature $(3, 19)$. By the classification of even integral quadratic forms, this implies that $\mathrm{H}^2(X, \mathbb{Z})$ can written as the orthogonal direct sum $L = \mathrm{E}_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. Here E_8 is the (positive definite) root lattice of the root system E_8; the lattice $\mathrm{E}_8(-1)$ is obtained by multiplication of the form on E_8 by -1, and U is the hyperbolic lattice of rank 2 (with quadratic form $2x_1x_2$).

Complex tori and their Hodge structures

Let M be a finitely generated free abelian group. Following Deligne, an integral HS on M is a representation of the 2-dimensional real torus $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ in $\mathrm{GL}(M_\mathbb{R})$. Then we have a Hodge decomposition $M_\mathbb{C} = \oplus_{p,q} M^{p,q}$ such that $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $M^{p,q}$ by $z^p \bar{z}^q$. The space $M^{q,p}$ is the complex conjugate of $M^{p,q}$. If $p + q = n$ for all terms in the decomposition, the HS is said to have weight n.

A complex torus is $\mathbb{C}^g/\Lambda$, where $\Lambda \cong \mathbb{Z}^{2g}$ is a full lattice, i.e. $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}^g$. To give a complex torus is the same as to give an integral Hodge structure of type $\{(1,0),(0,1)\}$ on Λ. An abelian variety is a complex torus with a polarisation, which is an integral skew-symmetric form on Λ satisfying some conditions. (This can be also rephrased by saying that the integral HS is polarisable.) For a curve C of genus g the spaces $\mathrm{H}^{1,0} \cong \mathrm{H}^0(C, \Omega^1_C)$ and $\mathrm{H}^{0,1} \cong \mathrm{H}^1(C, \mathcal{O}_C)$ have dimension g, so the Hodge decomposition

$$\mathrm{H}^1(C,\mathbb{Z})_{\mathbb{C}} = \mathrm{H}^1(C,\mathbb{C}) = \mathrm{H}^{1,0} \oplus \mathrm{H}^{0,1}$$

gives rise to a complex torus. Explicitly, integrating g linearly independent holomorphic 1-forms over $2g$ elements of a $\mathbb{Z}$-basis of $\mathrm{H}_1(C,\mathbb{Z})$ produces a full lattice $\Lambda \subset \mathbb{C}^g$ (defined up to a non-zero multiple). Then one shows that the complex torus $\mathbb{C}^g/\Lambda$ has a polarisation, so is an abelian variety.

Hodge structures of K3 type

In a very rough analogy to the Jacobian of a curve, one would like to associate to a complex K3 surface an abelian variety. So let us look at the Hodge structure of a K3 surface with a polarisation. The Hodge decomposition is

$$\mathrm{H}^2(X,\mathbb{Z})_{\mathbb{C}} = \mathrm{H}^2(X,\mathbb{C}) = \mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2},$$

where $\mathrm{H}^{2,0} \cong \mathrm{H}^0(X,\Omega^2_X)$ and $\mathrm{H}^{0,2} \cong \mathrm{H}^2(X,\mathcal{O}_X)$ are both 1-dimensional vector spaces over $\mathbb{C}$. Choose a non-zero $\omega \in \mathrm{H}^{2,0}$. Since $\mathrm{H}^{4,0} = 0$ we have $(\omega^2) = 0$. The complex conjugate $\overline{\omega}$ is a non-zero element of $\mathrm{H}^{0,2}$. Since the pairing

$$\mathrm{H}^{2,0} \times \mathrm{H}^{0,2} \longrightarrow \mathrm{H}^{2,2} = \mathrm{H}^4(X,\mathbb{C}) \cong \mathbb{C}$$

is non-degenerate and the cup-product is symmetric, $(\omega.\overline{\omega})$ is a positive real number. Since $\mathrm{H}^{3,1} = 0$ we have $\mathrm{H}^{2,0} \perp \mathrm{H}^{1,1}$.

It is convenient to twist the HS on $\mathrm{H}^2(X,\mathbb{Z})$ by 1:

$$\mathrm{H}^2(X,\mathbb{Z}(1))_{\mathbb{C}} = \mathrm{H}^{1,-1} \oplus \mathrm{H}^{0,0} \oplus \mathrm{H}^{-1,1}.$$

This has the advantage that the image of $\mathbb{S}$ lies in $\mathrm{SO}(\mathrm{H}^2(X,\mathbb{Z}))_{\mathbb{R}}$. (This also means rescaling the image of the integral cohomology inside the complex cohomology by $2\pi i$, but we shall ignore this.)

The Picard group of a complex K3 surface is a free abelian gropup. Its rank ρ is called the Picard number. The cycle class map gives an embedding

$$\mathrm{Pic}(X) \hookrightarrow \mathrm{H}^2(X,\mathbb{Z}(1)).$$

By the Lefschetz theorem $\mathrm{Pic}(X) = \mathrm{H}^2(X,\mathbb{Z}(1)) \cap \mathrm{H}^{(0,0)}$. Hence $1 \leq \rho \leq 20$. The orthogonal complement to $\mathrm{Pic}(X)$ in $\mathrm{H}^2(X,\mathbb{Z}(1))$ is called the *transcendental lattice* and is denoted by $T(X)$.

Definition 1.1 *Let M be a finitely generated free abelian group with a non-degenerate integral symmetric bilinear form. An integral HS on M is called a* Hodge structure of K3 type, *if the Hodge decomposition is*

$$M_{\mathbb{C}} = M^{1,-1} \oplus M^{0,0} \oplus M^{-1,1},$$

where $\dim(M^{1,-1}) = 1$, $M^{1,-1} \perp M^{0,0}$, *and for a non-zero* $\omega \in M^{1,-1}$ *we have* $(\omega^2) = 0$, $(\omega.\overline{\omega}) > 0$.

Take a primitive element $\lambda \in L$, $(\lambda^2) = 2d > 0$. It can be proved that the set of such elements is an orbit of $\mathrm{Aut}(L)$, hence the isomorphism class of the orthogonal complement $\lambda^{\perp} \subset L$ depends only on d, e.g. $\lambda^{\perp}$ is isomorphic to

$$L_d = \mathrm{E}_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus (-2d).$$

L_d has signature $(2, 19)$. We thus associate to a K3 surface X with a primitive polarisation of degree $2d$ an integral HS of K3 type on L_d.

Associating to an integral HS on L_d of K3 type the 1-dimensional space $\mathrm{H}^{1,-1}$ defines a point in the *period domain*

$$\Omega_d = \{x \in \mathbb{P}(L_{d,\mathbb{C}}) \mid (x^2) = 0,\ (x, \bar{x}) > 0\} = \mathrm{SO}(2, 19)(\mathbb{R})/\mathrm{SO}(2)(\mathbb{R}) \times \mathrm{SO}(19)(\mathbb{R}).$$

The second formula identifies Ω_d with the Grassmannian of positive definite oriented 2-dimensional real subspaces of $L_d \otimes \mathbb{R} \simeq \mathbb{R}^{21}$, by sending x to the plane spanned by $\mathrm{Re}(x), \mathrm{Im}(x)$ in this order. Ω_d has two isomorphic connected components that are interchanged by the complex conjugation (or reversing the orientation).

Although the difference between HS of curves and K3 surfaces prevents us from constructing an analogue of the Jacobian for K3 surfaces without more work, we nevertheless have the following very important result. The classical Torelli theorem says that the isometry class of the integral HS on $\mathrm{H}^1(C, \mathbb{Z})$ uniquely determines the curve C. Piatetskii-Shapiro and Shafarevich proved that the isometry class of the integral HS on $\mathrm{H}^2(X, \mathbb{Z})$ uniquely determines the K3 surface X.

Another obstacle is that the cup-product pairing on $\mathrm{H}^2(X, \mathbb{Z})$ is symmetric, whereas for an abelian variety one would need a skew-symmetric pairing, such as the one given by the cup-product on $\mathrm{H}^1(C, \mathbb{Z})$. Is there a way to go from the special orthogonal group $\mathrm{SO}(21)$ to a simplectic group $\mathrm{Sp}(m)$ for some m? In the representation theory of Lie algebras one shows that the Lie algebra $\mathfrak{o}(2n + 1)$ has a remarkable irreducible representation of dimension 2^n, called the *spinor representation*. It has a unique invariant bilinear form which is skew-symmetric when n is 1 or 2 mod 4, so there is a way get a simplectic Lie algebra from $\mathfrak{o}(2n + 1)$.

Clifford algebra and spinor group

For Lie groups this is a bit more subtle: one needs to replace $\mathrm{SO}(21)$ (which is not simply connected) by its unramified double cover $\mathrm{Spin}(21) \to \mathrm{SO}(21)$. The spinor group is constructed using the Clifford algebra associated to our quadratic form. Let M be a finitely generated free abelian group with a non-degenerate quadratic form q. Define the Clifford algebra $\mathrm{C}(M)$ as the quotient of the full tensor algebra $\oplus_{n\geq 0}M^{\otimes n}$ by the two-sided ideal I generated by the elements of the form $x\otimes x-q(x)$, for $x\in M$. It is clear that there is an isomorphism of abelian groups $\mathrm{C}(M)\simeq\oplus_{n=0}^{\mathrm{rk}(M)}\wedge^n M$, so the rank of $\mathrm{C}(M)$ is $2^{\mathrm{rk}(M)}$. The multiplication by -1 on M acts on $\oplus_{n\geq 0}M^{\otimes n}$. Let us denote by plus the invariant elements and by minus the anti-invariant elements. Since $x\otimes x-q(x)$ is invariant, we have $I=I^+\oplus I^-$. Thus we can define

$$\mathrm{C}^+(M)=(\oplus_{n\geq 0}M^{\otimes 2n})/I^+,\quad \mathrm{C}^-(M)=(\oplus_{n\geq 0}M^{\otimes 2n+1})/I^-,$$

where the first equality is the quotient of a ring by an ideal, whereas the second one is the quotient of a (left or right) $\oplus_{n\geq 0}M^{\otimes 2n}$-module $\oplus_{n\geq 0}M^{\otimes 2n+1}$ by the submodule I^-. We have a natural homomorphism $M\to\mathrm{C}^-(M)$. Over an algebraically closed field the structure of the Clifford algebra is easy: if $\mathrm{rk}(M)$ is odd, then $\mathrm{C}^+(M_\mathbb{C})$ is isomorphic to a matrix algebra $\mathrm{End}_\mathbb{C}(W)$, where W is the spinor representation of $\mathrm{GSpin}(M_\mathbb{C})$. (Exercise: Prove that $\mathrm{C}(M_\mathbb{C})$ is isomorphic to a matrix algebra if $\mathrm{rk}(M)$ is even. For this assume that q is the orthogonal direct sum of the hyperbolic forms $(e_i^2)=(f_i^2)=0$, $(e_i,f_i)=1$, for $i=1,\ldots,n$, and show that W can be the full exterior algebra of the linear span of $e_1,\ldots,e_n$. Deduce that $\mathrm{C}^+(M_\mathbb{C})$ is isomorphic to a matrix algebra if $\mathrm{rk}(M)$ is odd.)

Define the *spinor group* as

$$\mathrm{GSpin}(M)=\{g\in\mathrm{C}^+(M)^\times|gMg^{-1}=M\}.$$

It acts by conjugation on M preserving the quadratic form. This gives an exact sequence of algebraic groups over $\mathbb{Q}$:

$$1\longrightarrow\mathbb{G}_{m,\mathbb{Q}}\longrightarrow\mathrm{GSpin}(M)_\mathbb{Q}\longrightarrow\mathrm{SO}(M)_\mathbb{Q}\longrightarrow 1.$$

Next, consider the *adjoint* action of $\mathrm{GSpin}(M)$ on $\mathrm{C}^+(M)$, i.e. the action by conjugations. This representation of $\mathrm{GSpin}(M)_\mathbb{Q}$ is isomorphic to the direct sum of $\wedge^{2n}M$ for $n\geq 0$.

Kuga–Satake construction I

Let us apply this to the second cohomology of a K3 surface X. Fix a primitive ample class $\lambda\in\mathrm{H}^2(X,\mathbb{Z}(1))$ and define P as the orthogonal complement to λ in $\mathrm{H}^2(X,\mathbb{Z}(1))$, in particular $\mathrm{rk}(P)=21$. We have

$$P_\mathbb{C}=P^{1,-1}\oplus P^{0,0}\oplus P^{-1,1}.$$

Kuga and Satake showed how to define a canonical complex structure on the real vector space $\mathrm{C}^+(P_\mathbb{R})$. We can normalise $\omega\in P^{1,-1}$ so that $(\omega.\overline{\omega})=2$. Write

$\omega = \omega_1 + i\omega_2$, where $\omega_1, \omega_2 \in \mathrm{H}^2(X, \mathbb{R})$. Then $(\omega_1^2) = (\omega_2^2) = 1$ and $(\omega_1, \omega_2) = 0$. By the definition of the Clifford algebra, the following holds in $\mathrm{C}(P_\mathbb{R})$:

$$\omega_1^2 = \omega_2^2 = 1, \quad \omega_1\omega_2 = -\omega_2\omega_1.$$

Let $I = \omega_1\omega_2 \in \mathrm{C}^+(P_\mathbb{R})$. (Check that I does not depend on ω.) Then $I^2 = -1$, so the left multiplication by I defines a complex structure on the real vector space $\mathrm{C}^+(P_\mathbb{R})$, thus making $\mathrm{C}^+(P_\mathbb{R})/\mathrm{C}^+(P)$ a complex torus.

In Deligne's version one equips $\mathrm{C}^+(P)$ with an integral HS of type $\{(1,0),(0,1)\}$ as follows. Since $\mathbb{S}$ preserves the quadratic form on $P_\mathbb{R}$, we have a homomorphism $h : \mathbb{S} \to \mathrm{SO}(P)_\mathbb{R}$ whose kernel is $\{\pm 1\}$. For any $a, b \in \mathbb{R}$, not both equal to 0, we have $a + bI \in \mathrm{GSpin}(P)(\mathbb{R})$. Deligne points out that this is a canonical lifting of $h : \mathbb{S} \to \mathrm{SO}(P)_\mathbb{R}$ to $\tilde{h} : \mathbb{S} \hookrightarrow \mathrm{GSpin}(P)_\mathbb{R}$. (Exercise. Write $z = a + bi$. Check that $a + bI \in \mathrm{C}^+(P_\mathbb{R})$ and $x \mapsto (a+bI)x(a+bI)^{-1}$ acts on ω via multiplication by $z\bar{z}^{-1}$, on $\overline{\omega}$ via multiplication by $\bar{z}z^{-1}$, and fixes $P^{0,0} \cap P_\mathbb{R}$.) This means that the adjoint action of $\mathrm{GSpin}(P_\mathbb{Q})$ on P induces our original HS on P, so it is a HS of K3 type.

Lemma 1.2 *The left action of* $\mathrm{GSpin}(P)_\mathbb{Q}$ *on* $\mathrm{C}^+(P_\mathbb{Q})$ *induces an integral HS of type* $\{(1,0),(0,1)\}$ *on* $\mathrm{C}^+(P_\mathbb{Q})$.

Proof. The adjoint representation of $\mathrm{GSpin}(P)_\mathbb{Q}$ on $\mathrm{C}^+(P_\mathbb{Q})$ is isomorphic to the direct sum of $\wedge^{2n} P_\mathbb{Q}$ for $n \geq 0$. This implies that the induced HS on $\wedge^{2n} P_\mathbb{Q}$ is of type $\{(1,-1),(0,0),(-1,1)\}$. Hence the adjoint HS on $\mathrm{C}^+(P_\mathbb{Q})$ is also of type $\{(1,-1),(0,0),(-1,1)\}$. Over $\mathbb{C}$ the adjoint representation of $\mathrm{GSpin}(P)_\mathbb{C}$ on the matrix algebra $\mathrm{C}^+(P_\mathbb{C})$ is identified with $\mathrm{End}_\mathbb{C}(W) = W \otimes_\mathbb{C} W^*$, where W is the spinor representation. Thus the action of $\mathrm{GSpin}(P)_\mathbb{C}$ on $\mathrm{C}^+(P_\mathbb{C})$ by left multiplication is isomorphic to $W^{\dim(W)}$ as a representation of $\mathrm{GSpin}(P)_\mathbb{C}$. Therefore, the action of $\mathbb{S} \subset \mathrm{GSpin}(P)_\mathbb{R}$ by left multiplication induces an integral HS on $\mathrm{C}^+(P)$; its type must be $\{(1,0),(0,1)\}$ or $\{(-1,0),(0,-1)\}$ otherwise the HS on $W \otimes_\mathbb{C} W^*$ cannot be of type $\{(1,-1),(0,0),(-1,1)\}$. But $\mathbb{R}^\times \subset \mathbb{C}^\times$ acts on $\mathrm{C}^+(P)$ tautologically, so the weight of W is 1 and the type is $\{(1,0),(0,1)\}$. $\square$

It can be shown that this HS is polarisable, so we actually obtain an abelian variety and not just a complex torus. It is called the *Kuga–Satake variety* attached to (X, λ). Let us denote it by KS_X. Here is a crucial property of KS_X.

Lemma 1.3 *We have a natural isomorphism of abelian groups* $\mathrm{H}^1(KS_X, \mathbb{Z}) = \mathrm{C}^+(P)$. *The left action of* $\mathrm{C}^+(P)$ *on* $\mathrm{C}^+(P)$ *commutes with the right action of the opposite algebra* $C = \mathrm{C}^+(P)^{\mathrm{op}}$. *This gives an isomorphism of algebras*

$$\mathrm{C}^+(P) = \mathrm{End}_C(\mathrm{H}^1(KS_X, \mathbb{Z})),$$

which is also an isomorphism of Hodge structures of weight 0. *This is also an isomorphism of* $\mathrm{GSpin}(P)$*-modules (with respect to the adjoint action on* $\mathrm{C}^+(P)$ *and the standard action on the endomorphisms), and it is a unique isomorphism with this property.*

Proof The left multiplication gives an injective map $\mathrm{C}^+(P) \to \mathrm{End}(\mathrm{C}^+(P))$ because $\mathrm{C}^+(P)$ contains 1. Its image lies in $\mathrm{End}_C(\mathrm{C}^+(P))$ and coincides with it, because an element of $\mathrm{End}_C(\mathrm{C}^+(P))$ is determined by the image of 1. It remains to show that any automorphism of $\mathrm{C}^+(P)$ commuting with the adjoint action of $\mathrm{GSpin}(P)$ must be the identity. This can be proved over $\mathbb{C}$. Since $\mathrm{C}^+(P_{\mathbb{C}})$ is a matrix algebra, by the Skolem–Noether theorem any automorphism is $x \to axa^{-1}$, where $a \in \mathrm{C}^+(P_{\mathbb{C}})$. This implies that the commutator $aga^{-1}g^{-1}$ is in the centre of the matrix algebra $\mathrm{C}^+(P_{\mathbb{C}})$. Since $\det(aga^{-1}g^{-1}) = 1$, this commutator is a root of 1. But $\mathrm{GSpin}(P)_{\mathbb{C}}$ is connected, and $aga^{-1}g^{-1} = 1$ for $g = 1$, so $aga^{-1}g^{-1} = 1$ for any $g \in \mathrm{GSpin}(P)_{\mathbb{C}}$. Hence a commutes with $\mathrm{GSpin}(P)_{\mathbb{C}}$, so a is scalar, but then $axa^{-1} = x$. $\square$

2 Moduli spaces as Shimura varieties

Moduli space of polarised K3 surfaces

Let $O(L_d)$ be the orthogonal group of L_d. Define

$$\widetilde{O}(L_d) = \{g \in O(L_d) \mid g \text{ acts trivially on the discriminant group } L_d^*/L_d \simeq \mathbb{Z}/2d\}.$$

This is the stabiliser of λ in $O(L)$. The key (difficult) facts are:

(1) $\widetilde{O}(L_d)\backslash\Omega_d$ is a quasi-projective irreducible variety over $\mathbb{C}$;

(2) there is a coarse moduli space M_d of K3 surfaces with a primitive polarisation of degree $2d$;

(3) M_d is a Zariski open subset of $\widetilde{O}(L_d)\backslash\Omega_d$.

Note that M_d is not smooth, though it is smooth as an orbifold (or Deligne–Mumford stack). This description is a K3 analogue of the coarse moduli space of elliptic curves $\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}$ or the moduli space of dimension g ppav's $\mathcal{A}_g = \mathrm{Sp}(2g,\mathbb{Z})\backslash\mathcal{H}_g$, where $\mathcal{H}$ is the usual upper half-plane and $\mathcal{H}_g$ is the Siegel upper half-plane. (In fact, Ω_d, $\mathcal{H}$, $\mathcal{H}_g$ are Hermitian symmetric domains, so property (1) follows from the Baily–Borel theorem about quotients of Hermitian symmetric domains by torsion-free arithmetic sugroups of their automorphism groups.)

Replacing $O(L_d)$ by the index 2 subgroup $\mathrm{SO}(L_d)$ replaces M_d by an unramified double cover $\widetilde{M}_d \to M_d$.

In modern language $\mathcal{A}_g$ and $\widetilde{M}_d$ are the sets of $\mathbb{C}$-points of Shimura varieties. We have seen that a point in Ω_d is a homomorphism $\mathbb{S} \to \mathrm{SO}(L_d)_{\mathbb{R}}$. The action of $\mathrm{SO}(L_d)(\mathbb{R})$ on Ω_d is transitive, so Ω_d can be identified with the conjugacy class of h in $\mathrm{Hom}(\mathbb{S}, \mathrm{SO}(L_d)(\mathbb{R}))$. This is similar to the classical identification of $\mathcal{H} = \mathrm{SL}(2)(\mathbb{R})/\mathrm{SO}(2)(\mathbb{R})$ with the conjugacy class of $\mathbb{S} \subset \mathrm{GL}(2)^+_{\mathbb{R}}$. (Here $\mathrm{GL}(2)^+_{\mathbb{R}}$ is given by the condition $\det(x) > 0$; note that $\mathrm{GL}(2)(\mathbb{R})/\mathbb{S} = \mathcal{H}^{\pm}$.)

A *Shimura datum* is a pair (G, X), where G is a connected reductive algebraic group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class in $\mathrm{Hom}(\mathbb{S}, G(\mathbb{R}))$ satisfying certain axioms ensuring that X is an Hermitian symmetric domain. Morphisms of Shimura data are defined in the obvious way. In the K3 case $(\mathrm{SO}(L_d), \Omega_d)$ is a Shimura datum. In the case of ppav's the Shimura datum is $(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$.

A *congruence subgroup* is a subgroup of $G(\mathbb{Q})$ cut out by a compact open subgroup $K \subset G(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$, where $\mathbf{A}_{\mathbb{Q},\mathrm{f}}$ is the ring of finite adèles. (Equivalently, a subgroup whose subgroup of finite index preserves a lattice modulo some integer N in a rational representation.) Deligne's definition of the Shimura variety defined by the Shimura datum (G, X) and a compact open subgroup $K \subset G(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$ is

$$\mathrm{Sh}_K(G, X)_{\mathbb{C}} = G(\mathbb{Q}) \backslash X \times G(\mathbf{A}_{\mathbb{Q},\mathrm{f}})/K,$$

where $G(\mathbb{Q})$ acts diagonally on both factors on the left, whereas K acts on $G(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$ on the right. The crucial fact is that any Shimura variety $\mathrm{Sh}_K(G, X)$ is defined over a number field, it is a so called *canonical model*. The set $\mathrm{Sh}_K(G, X)(\mathbb{C})$ is a disjoint union of the quotients $\Gamma \backslash X_0$, where X_0 is a connected component of X and Γ is a congruence subgroup of the stabiliser $G(\mathbb{Q})_0 \subset G(\mathbb{Q})$ of X_0. Rizov proved that M_d is defined over $\mathbb{Q}$, similarly to $\mathcal{A}_g$.

We have $\widetilde{M}_d \hookrightarrow \mathrm{Sh}_{K_0}(\mathrm{SO}(L_d)_{\mathbb{Q}}, \Omega_d)$ for some K_0. One can choose K in such a way that a Zariski open subscheme $\widetilde{M}_{d,K} \hookrightarrow \mathrm{Sh}_K(\mathrm{SO}(L_d)_{\mathbb{Q}}, \Omega_d)$ carries a universal family of K3 surfaces with a level structure given by K. (We shall not need its precise description.)

Kuga–Satake construction II

Recall that $h : \mathbb{S} \to \mathrm{SO}(L_d)_{\mathbb{R}}$ canonically lifts to $\tilde{h} : \mathbb{S} \to \mathrm{GSpin}(L_d)_{\mathbb{R}}$. It follows that the $\mathrm{GSpin}(L_d)(\mathbb{R})$-conjugacy class of $\tilde{h} : \mathbb{S} \to \mathrm{GSpin}(L_d)_{\mathbb{R}}$ maps bijectively to Ω_d, the $\mathrm{SO}(L_d)(\mathbb{R})$-conjugacy class of h. This shows that the homomorphism $\mathrm{GSpin}(L_d) \to \mathrm{SO}(L_d)$ naturally extends to a morphism of Shimura data

$$(\mathrm{GSpin}(L_d), \Omega_d) \longrightarrow (\mathrm{SO}(L_d), \Omega_d).$$

On the other hand, the choice of a polarisation defines a morphism of Shimura data

$$(\mathrm{GSpin}(L_d), \Omega_d) \longrightarrow (\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm}),$$

where $g = 2^{19}$. In fact, one can choose congruence subgroups $K \subset \mathrm{SO}(L_d)_{\mathbb{Q}}$, $K_1 \subset \mathrm{GSpin}(L_d)_{\mathbb{Q}}$ and $K_2 \subset \mathrm{GSp}_{2g,\mathbb{Q}}$ in such a way that there are *finite* morphisms of Shimura varieties defined over some number field

$$\mathrm{Sh}_K(\mathrm{SO}(L_d)_{\mathbb{Q}}, \Omega_d) \xleftarrow{\alpha} \mathrm{Sh}_{K_1}(\mathrm{GSpin}(L_d), \Omega_d) \xrightarrow{\beta} \mathrm{Sh}_{K_2}(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm}). \qquad (1)$$

Moreover, we can choose K, K_1 and K_2 in such a way that a non-empty Zariski open subset of $\mathrm{Sh}_K(\mathrm{SO}(L_d)_{\mathbb{Q}}, \Omega_d)$ carries a universal family of K3 surfaces with polarisation of degree $2d$ and the level structure given by K. (K_1 and K_2 are given by the condition $\rho(g) \equiv 1 \bmod N$, where ρ is the relevant representation, $N \geq 3$.)

Pulling this family back to $\mathrm{Sh}_{K_1}(\mathrm{GSpin}(L_d), \Omega_d)$ we obtain a non-empty Zariski open subset U of $\mathrm{Sh}_{K_1}(\mathrm{GSpin}(L_d), \Omega_d)$ with a universal family of K3 surfaces $\pi : X \to U$ with polarisation of degree $2d$ and the 'spin' level structure given by K_1. Associating to these K3 surfaces their Kuga–Satake abelian varieties we get an abelian scheme $a : A \to U$ with a certain level structure. A generalisation of Lemma 1.3 based on the comparison theorems between étale and Betti cohomology is the following statement.

Proposition 2.1 *There is a unique isomorphism of local systems*

$$\mathrm{C}^+(P^2\pi_*\mathbb{Z}(1)) \cong \mathrm{End}_C(R^1a_*\mathbb{Z}).$$

Here $P^2\pi_*\mathbb{Z}(1)$ is the relative orthogonal complement to the ample primitive class $\lambda \in R^2\pi_*\mathbb{Z}(1)$, and $C = \mathrm{C}^+(L_d)^{\mathrm{op}}$. For any prime ℓ there is a unique isomorphism of étale $\mathbb{Z}_\ell$-sheaves

$$\mathrm{C}^+(P^2\pi_*\mathbb{Z}_\ell(1)) \cong \mathrm{End}_C(R^1a_*\mathbb{Z}_\ell).$$

The two isomorphisms are compatible via the comparison isomorphisms between classical and ℓ-adic étale cohomology.

The uniqueness has the following strong consequence for K3 surfaces defined over a non-closed field k of characteristic zero. Let $\bar{k}$ be an algebraic closure of k. Write $\Gamma = \mathrm{Gal}(\bar{k}/k)$ and $\overline{X} = X \times_k \bar{k}$. Note that if $k \subset \mathbb{C}$, then $\mathrm{Pic}(\overline{X})$ is naturally isomorphic to $\mathrm{Pic}(X_{\mathbb{C}})$.

Corollary 2.2 *Let X be a projective K3 surface over a field k of characteristic zero. There exists a finite extension k'/k and an abelian variety A over k' with complex multiplication by C such that there is an isomorphism of* $\mathrm{Gal}(\bar{k}/k')$*-modules*

$$\mathrm{C}^+(P^2(\overline{X}, \mathbb{Z}_\ell(1))) \cong \mathrm{End}_C(\mathrm{H}^1(A \times_{k'} \bar{k}, \mathbb{Z}_\ell))$$

for any prime ℓ.

Proof. Any projective variety X over k of characteristic zero is actually defined over a field finitely generated over $\mathbb{Q}$. (Join to $\mathbb{Q}$ the coefficients of the equations that define X is some projective space.) So we can assume without loss of generality that k can be embedded into $\mathbb{C}$. Then we have the previous constructions giving us the morphisms α and β in (1), together with the universal families $\pi : X \to U$ and $a : A \to U$. After a finite extension of k we can assume that the morphisms α and β in (1) are defined over k' and X has a spin level structure defined over k'. Hence the specialisation of the isomorphism of Proposition 2.1 gives a *unique* isomorphism. Since it is unique, it must be compatible with the natural Galois action on both sides. $\square$

An easy variant of this result gives an abelian variety A (isogenous to the abelian variety of Corollary 2.2) defined over a finite extension k' and an embedding of $\mathrm{Gal}(\bar{k}/k')$-modules

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)) \hookrightarrow \mathrm{End}(\mathrm{H}^1(A \times_{k'} \bar{k}, \mathbb{Q}_\ell)). \tag{2}$$

When $k \subset \mathbb{C}$, this map is compatible, via the comparison theorem between Betti and étale ℓ-adic cohomology, with the embedding of integral Hodge structures of weight 0:

$$\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \hookrightarrow \mathrm{End}(\mathrm{H}^1(A_{\mathbb{C}}, \mathbb{Z})).$$

Similarly, by avoiding the finitely many primes dividing the discriminant of the intersection pairing on $\mathrm{Pic}(\overline{X})$, one obtains an embedding of $\mathrm{Gal}(\bar{k}/k')$-modules

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_\ell) \hookrightarrow \mathrm{End}_{\mathbb{F}_\ell}(A[\ell]). \tag{3}$$

Deligne used this to prove the Weil onjectures for K3 surfaces over finite fields (before he proved the general case), but this theory has many other nice applications. First of all, the semisimplicity of the Galois module $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$ for a K3 surface X follows from (2) and the semisemplicity for abelian varieties.

Theorem 2.3 *Let X be a K3 surface over a field k finitely generated over $\mathbb{Q}$. Then the Tate conjecture holds for X, that is, we have*

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))^\Gamma = \mathrm{Pic}(\overline{X})^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}_\ell.$$

Note that the profinite, hence compact group Γ acts continuously on the discrete group $\mathrm{Pic}(\overline{X})$, so this action factors through a finite quotient of Γ, i.e. $\mathrm{Gal}(k'/k)$ for a finite Galois extension k' of k.

Proof of Theorem 2.3. This can be done over a finite extension of k, so we can assume that Γ acts trivially on $\mathrm{Pic}(\overline{X})$ and on $\mathrm{End}(\overline{A})$. We need to show that the Γ-invariant subspace of $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$ is $\mathrm{Pic}(\overline{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$. By Faltings, the Tate conjecture holds for A. Thus the Γ-invariant subspace of $\mathrm{End}(\mathrm{H}^1(\overline{A}, \mathbb{Q}_\ell))$ is $\mathrm{End}(\overline{A}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$. Hence the image of $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))^\Gamma$ in $\mathrm{End}(\mathrm{H}^1(\overline{A}, \mathbb{Q}_\ell))$ belongs to the $\mathbb{Q}_\ell$-span of the intersection of the image of $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Q}(1))$ in $\mathrm{End}(\mathrm{H}^1(A_{\mathbb{C}}, \mathbb{Q}))$ with $\mathrm{End}(\overline{A}) \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathrm{End}(\mathrm{H}^1(A_{\mathbb{C}}, \mathbb{Q}))$. But such elements of $\mathrm{End}(\mathrm{H}^1(A_{\mathbb{C}}, \mathbb{Q}))$ have Hodge type $(0, 0)$. Hence every element of $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))^\Gamma$ is a $\mathbb{Q}_\ell$-linear combination of classes of type $(0, 0)$. By the Lefschetz theorem, each such class is algebraic. □

Theorem 2.4 *Let X be a K3 surface over a field k finitely generated over $\mathbb{Q}$. Then $\mathrm{Br}(\overline{X})^\Gamma$ is finite.*

Proof. The ℓ-primary torsion subgroup $\mathrm{Br}(\overline{X})^\Gamma\{\ell\}$ is finite for all primes ℓ. This follows from the Tate conjecture for divisors and the semisimplicity of the Galois module $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$ in exactly the same way as for abelian varieties.

To prove that $\mathrm{Br}(\overline{X})^\Gamma[\ell] = 0$ for almost all ℓ we use the same strategy as for abelian varieties. The argument reduces to proving that for almost all ℓ we have $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_\ell)^\Gamma = (\mathrm{Pic}(\overline{X})/\ell)^\Gamma$. For this it is enough to show that $(T(X)/\ell)^\Gamma = 0$ for almost all ℓ, where $T(X) = \mathrm{Pic}(\overline{X})^\perp$ is the transcendental lattice of X. By the Lefschetz theorem and the non-degeneracy of the intersection pairing on $\mathrm{Pic}(\overline{X})$, the transcendental lattice $T(X)$ does not contain non-zero elements of Hodge type $(0,0)$. Hence the image of $T(X)$ in $\mathrm{End}(\mathrm{H}^1(A_\mathbb{C}, \mathbb{Z}))$ has trivial intersection with $\mathrm{End}(\overline{A})$. It follows that the image of $T(X)/\ell$ in $\mathrm{End}_{\mathbb{F}_\ell}(A[\ell])$ intersects trivially with $\mathrm{End}(\overline{A})/\ell = \mathrm{End}(A)/\ell$ for almost all ℓ. By Faltings and Zarhin, for almost all ℓ we have

$$\mathrm{End}_{\mathbb{F}_\ell}(A[\ell])^\Gamma = \mathrm{End}(A)/\ell.$$

Thus $(T(X)/\ell)^\Gamma = 0$ for almost all ℓ. □

3 Complex multiplication for K3 surfaces

CM points of Shimura varieties

Let H be an integral HS defined by a homomorphism $h : \mathbb{S} \to \mathrm{GL}(H)_\mathbb{R}$. The *Mumford–Tate* group is the smallest connected algebraic group $MT \subset \mathrm{GL}(H)_\mathbb{Q}$ such that $h(\mathbb{S}) \subset MT_\mathbb{R}$. Note that MT is defined over $\mathbb{Q}$, whereas $h(\mathbb{S})$ is defined over $\mathbb{R}$. One says that H has *CM type* if MT is commutative (then MT is an algebraic torus).

A point in a Shimura variety $\mathrm{Sh}_K(G, X)$ is called a CM point if it comes from a homomorphism $h : \mathbb{S} \to G$ whose image is contained in $T_\mathbb{R}$, where $T \subset G$ is an algebraic torus defined over $\mathbb{Q}$. Each CM point of a Shimura variety is defined over a number field; in fact, canonical models of Shimura varieties are defined precisely in order for this to hold. The existence of CM points is thus the fundamental reason for the existence of canonical models of Shimura varieties.

Consider the example of $\mathrm{Sh}_K(\mathrm{GL}(2), \mathcal{H}^\pm)$, where $K = \mathrm{GL}(2, \hat{\mathbb{Z}})$. This is the coarse moduli space of elliptic curves parametrised by their j-invariant, that is, the affine line $\mathbb{A}^1(\mathbb{C}) = \mathrm{SL}(\mathbb{Z})\backslash\mathcal{H}$ defined over $\mathbb{Q}$. An elliptic curve E has CM when $\mathrm{End}(\overline{E})_\mathbb{Q}$ is an imaginary quadratic field K. In this case $\mathbb{S} \subset \mathrm{GL}(2)_\mathbb{R}$ is in fact an algebraic torus MT_E defined over $\mathbb{Q}$ and isomorphic to $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_{m,K})$. The point of $\mathbb{A}^1_\mathbb{Q}$ corresponding to an elliptic curve with CM is defined over a number field. (Class field theory interpretes it as the ring class field of the order $\mathrm{End}(\overline{E}) \subset \mathrm{End}(\overline{E})_\mathbb{Q} = K$.) It is known that then $j \in \overline{\mathbb{Q}}$ (in fact, j is an algebraic integer).

Another example is $\mathcal{A}_g = \mathrm{Sh}_K(\mathrm{GSp}_{2g}, \mathcal{H}^\pm)$, where $K = \mathrm{GSp}_{2g}(\hat{\mathbb{Z}})$, which is the coarse moduli space of principally polarised abelian varieties of dimension g. The

CM-points of $\mathcal{A}_g$ correspond to ppav's with CM in the classical sense, that is, to abelian varieties A such that $\overline{A}$ is isogenous to a product of powers of simple abelian varieties A_i (over $\bar{k}$) for which $\mathrm{End}(A_i)_{\mathbb{Q}}$ is a CM field. (Recall that a CM field is a totally complex quadratic extension of a totally real field.)

Mumford–Tate groups of complex K3 surfaces were classified by Zarhin. He proved that the Hodge structure on the transcendental lattice $T(X)$ is irreducible. As a consequence, the ring of Hodge endomorphisms $\mathrm{End}_{\mathrm{Hdg}}(T(X)_{\mathbb{Q}})$ is a field. Zarhin shows that this field K is either totally real or a CM field. The K3 surface X has CM type if and only if K is a CM field and $\dim_K(T(X)_{\mathbb{Q}}) = 1$. The Mumford–Tate group has the following description. Let $K_+ \subset K$ be the maximal totally real subfield, $[K : K_+] = 2$. Let $T^1 = \mathrm{Res}^1_{K/K_+}(\mathbb{G}_{m,K})$ be the norm 1 torus. This is a non-split 1-dimensional torus over K_+. Then $MT = \mathrm{Res}_{K_+/\mathbb{Q}}(T^1)$.

Taelman showed that all CM fields of even degree between 2 and 20 can be realised by complex K3 surfaces. The nicest case is when K is an imaginary quadratic field: any K3 surface of Picard rank 20 has CM by $K = \mathbb{Q}(\sqrt{\delta})$, $\delta = b^2 - 4ac$, where

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is the matrix of the restriction of the cup-product on $\mathrm{H}^2(X, \mathbb{Z}(1))$ to $T(X)$. In this case $K_+ = \mathbb{Q}$ and M is given by $x^2 - \delta y^2 = 1$.

Brauer groups of K3 surfaces

Let X be a complex K3 surface. Let $T(X)$ be the transcendental lattice of X.

The Brauer group is defined as $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. The Kummer sequence

$$0 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_m \xrightarrow{[\ell^n]} \mathbb{G}_m \longrightarrow 0$$

gives rise to an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X)/\ell^n \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \longrightarrow \mathrm{Br}(X)[\ell^n] \longrightarrow 0.$$

Passing to the limit as $n \to \infty$ we get

$$0 \longrightarrow \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1)) \longrightarrow T_\ell(\mathrm{Br}(X)) \longrightarrow 0,$$

where T_ℓ is the Tate module of the Brauer group, defined as the inverse limit of $\mathrm{Br}(X)[\ell^n]$ as $n \to \infty$. This sequence shows that $T_\ell(\mathrm{Br}(X))$ is dual to $T(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$, where $T(X)$ is the transcendental lattice of X. Thus we have canonical isomorphisms

$$\mathrm{Br}(X)\{\ell\} = T_\ell(\mathrm{Br}(X)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell = \mathrm{Hom}(T(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Taking the direct sum over all primes ℓ we finally get a canonical isomorphism of abelian groups $\mathrm{Br}(X) = \mathrm{Hom}(T(X), \mathbb{Q}/\mathbb{Z})$.

Uniform bounds

J. Tsimerman (following the work of many other people) recently proved the following result (the average Colmez conjecture). Let g be a positive integer. There exist constants $b_g, C_g > 0$ such that, for every principally polarised abelian variety A of dimension g defined over a number field k, if A is of CM type, then

$$|\mathrm{discr}\,(Z(\mathrm{End}(\overline{A})))| < C_g\,[k:\mathbb{Q}]^{b_g}.$$

From this, using the interpretation of the Kuga–Satake construction in terms of Shimura varieties explained above, together with a K3 analogue of Zarhin's quaternion trick for abelian varieties, one can deduce the following results.

Theorem 3.1 (M. Orr, A.S.) (a) *There are only finitely many $\overline{\mathbb{Q}}$-isomorphism classes of abelian varieties of CM type of given dimension which can be defined over number fields of given degree.*

(b) *There are only finitely many $\overline{\mathbb{Q}}$-isomorphism classes of K3 surfaces of CM type which can be defined over number fields of given degree.*

As a consequence one obtains the boundedness of the discriminant (or the isomorphism class) of the lattice $\mathrm{Pic}(\overline{X})$ for K3 surfaces X of CM type defined over number fields of bounded degree. Shafarevich conjectured this in 1994 for arbitrary K3 surfaces defined over number fields of bounded degree.

Another application is the following uniform boundedness result for the Brauer groups.

Theorem 3.2 (M. Orr, A.S.) *For any positive integer n there is a constant C_n such that $|\mathrm{Br}(\overline{X})^\Gamma| < C_n$ for any K3 surface X of CM type defined over a number field of degree at most n. The same is true for abelian varieties of dimension g for a constant $C_{n,g}$ depending on g and n.*

Várilly-Alvarado conjectured this to hold without the CM assumption, as an analogue of Merel's theorem on the universal boundedness of torsion of elliptic curves defined over number fields of bounded degree. We shall see that the analogy between the torsion subgroup of an elliptic curve and the Brauer group of a K3 surfaces can be made more explicit in the CM case.

A crucial ingredient necessary to deduce Theorem 3.2 from Theorem 3.1 is the *integral Mumford–Tate conjecture*, known for abelian varieties of CM type and arbitrary K3 surfaces (Pohlmann; Tankeev, Y. André; Cadoret–Moonen). Let k be a field finitely generated over $\mathbb{Q}$. Then, after an appropriate finite extension of k, the image of the Galois representation in $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))$ is a subgroup of bounded index in $G(\mathbb{Z}_\ell)$, where G is the Mumford–Tate group. Serre conjectured this to hold for any smooth projective variety X

Class field theory

Let k be a number field. Let $\mathbf{A}_k$ be the ring of adèles of k and let $\mathbf{A}_k^\times$ be the group of idèles of k. Let k_∞ be the product of archimedean completions of k. Write $k_{\infty+}$ for the connected component of k_∞. Class field theory gives an exact sequence

$$1 \longrightarrow \overline{k^\times k_{\infty+}^\times} \longrightarrow \mathbf{A}_k^\times \longrightarrow \mathrm{Gal}(k_{\mathrm{ab}}/k) \longrightarrow 1, \tag{4}$$

where k_{ab} is the maximal abelian extension of k. The image of an idèle $s \in \mathbf{A}_k^\times$ in $\mathrm{Gal}(k_{\mathrm{ab}}/k)$ is denoted by $[s, k]$.

Suppose that $\Lambda \subset k$ is a *lattice*, i.e. a free abelian group of rank $[k : \mathbb{Q}]$. (Then $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = k$.) An idèle $s \in \mathbf{A}_k^\times$ associates to Λ another lattice, which we denote by $s\Lambda$. It is defined as the unique lattice in k such that $s\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is the subgroup of $k_p := k \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{v|p} k_v$ equal to $s_p(\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p)$, where $s_p := (s_v)_{v|p} \in \prod_{v|p} k_v^\times$.

Next, s defines a homomorphism $k/\Lambda \to k/s\Lambda$, as follows. The quotient k/Λ is naturally isomorphic to the direct sum of $k_p/(\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ for all primes p. The homomorphism $k/\Lambda \to k/s\Lambda$ is defined as the direct sum of maps

$$k_p/(\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p) \longrightarrow k_p/s_p(\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p)$$

given by multiplication by s_p. Similarly, s defines a homomorphism

$$\mathrm{Hom}(\Lambda, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}(s\Lambda, \mathbb{Q}/\mathbb{Z})$$

sending $f(x)$ to $f(s^{-1}x)$.

A subring R of a number field k is called an *order* in k if R contains 1 and is a lattice in k. To a lattice Λ we associate its order R defined as the set of $x \in k$ for which $x\Lambda \subset \Lambda$. Then $\Lambda \subset k$ is a *proper fractional ideal* of R, which means that R is exactly the set of $x \in k$ for which $x\Lambda \subset \Lambda$.

Elliptic curves

Let E be an elliptic curve over $\mathbb{C}$ such that there exists an isomorphism $\theta : K \tilde{\longrightarrow} \mathrm{End}(E)_{\mathbb{Q}}$, where K is an imaginary quadratic field. We assume that θ is *normalised* in such a way that $\theta(a)$ acts on the holomorphic 1-form $\frac{dx}{y}$ as multiplication by $a \in K \subset \mathbb{C}$. Let $\Lambda \subset K$ be a lattice for which there is an isomorphism $\xi : \mathbb{C}/\Lambda \tilde{\longrightarrow} E$. Then $\mathrm{End}(E)$ is an order in K and Λ is a proper fractional ideal of $\mathrm{End}(E)$. We note that ξ identifies K/Λ with the torsion subgroup of E.

The main theorem of complex multiplication of elliptic curves is the following statement.

Theorem 3.3 (CM of EC) *Let $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$. Let $s \in \mathbf{A}_K^\times$ be an idèle such that $[s^{-1}, K]$ is the image of σ in $\mathrm{Gal}(K_{\mathrm{ab}}/K)$. Let (E, θ) be an elliptic curve with CM by $\theta : K \to \mathbb{C}$. Let E^σ be the elliptic curve over $\mathbb{C}$ which is the conjugate of E*

by σ. Then there is a unique isomorphism $\xi' : \mathbb{C}/s\Lambda \tilde{\longrightarrow} E^\sigma$ such that the diagram commutes:

$$\begin{tikzcd} K/\Lambda \arrow[r, "\xi"] \arrow[d, "s"'] & E \arrow[d, "\sigma"] \\ K/s\Lambda \arrow[r, "\xi'"] & E^\sigma \end{tikzcd}$$

Proof. [12, Thm. 5.4]. □

Since Λ is well defined up to a multiple in $K^\times$, we have the following corollary. Here 'the field of moduli' of (E,θ) means 'the relative field of moduli over k', defined as the subfield of $\mathbb{C}$ invariant under the largest subgroup of $\mathrm{Aut}(\mathbb{C}/k)$ sending (E,θ) to an isomorphic pair.

Corollary 3.4 *The field of moduli of (E,θ) corresponds to the subgroup $k^\times S \subset \mathbf{A}_k^\times$, where*

$$S = \{s \in \mathbf{A}_k^\times | s\mathcal{O} = \mathcal{O}\}.$$

This field is the ring class field of $\mathcal{O}$.

Proof. Theorem 3.3 gives us the condition $s\Lambda = \Lambda$. We have $[K : \mathbb{Q}] = 2$, and in this case any proper fractional $\mathcal{O}$-ideal is invertible, i.e. $\Lambda \cdot \Lambda^{-1} = \mathcal{O}$ for some proper fractional $\mathcal{O}$-ideal Λ^{-1}. (See [4, Prop. 7.4] or [3, Cor. 4.4].) Multiplying by Λ^{-1} we obtain that $s \in S$ if and only if $s\mathcal{O} = \mathcal{O}$, which says that $s_v \in \mathcal{O}_v^\times$ for each finite place v. This implies that $\mathbf{A}_K^\times/K^\times S$ is the ideal class group of the order $\mathcal{O}$. By [4, Prop. 7.22] it is isomorphic to the ring class group of conductor f of $\mathcal{O}$, so that the field of moduli of (E,θ) is the ring class field of the order $\mathcal{O}$, see [4, Section 9,A]. □

CM for K3

Let X be a K3 surface such that there is an isomorphism $\theta : K \tilde{\longrightarrow} \mathrm{End}_{\mathrm{Hdg}}(T(X)_\mathbb{Q})$, where K is a CM field, such that $\dim_K(T(X)_\mathbb{Q}) = 1$. We assume that θ is normalised in such a way that $\theta(a)$ acts on $\mathrm{H}^{1,-1} \simeq \mathbb{C}$ (that is, on the holomorphic 2-form) as multiplication by $a \in K \subset \mathbb{C}$.

Let us assume that $\mathrm{End}_{\mathrm{Hdg}}(T(X))$ (which a priori is an order in K) is the full ring of integers $\mathcal{O}_K$. (When $[K : \mathbb{Q}] > 2$, a lattice is not necessarily an invertible ideal of its order, see [3] for a detailed discussion and counterexamples.) Then $T(X)$ can be identified with a fractional ideal in K uniquely up to multiplication by $K^\times$. We shall refer to X as a K3 surface with CM by $(K,\theta,\mathcal{O}_K)$.

The main theorem of complex multiplication of Brauer groups of K3 surfaces can be stated as follows.

Theorem 3.5 (CM of K3) *Let $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$. Let $s \in \mathbf{A}_K^\times$ be an idèle such that $[s^{-1}, K]$ is the image of σ in $\mathrm{Gal}(K_{\mathrm{ab}}/K)$. Let X be a complex K3 surface with CM* *by $(K,\theta,\mathcal{O}_K)$ and let X^σ be the complex K3 surface which is the conjugate of X by σ. Identify $T(X)$ with a lattice in K. Then $T(X^\sigma)$ is a lattice in $K^\sigma = K$ such that $\sigma_* : T(X) \to T(X^\sigma)$ tensored with $\mathbf{A}_{\mathbb{Q},\mathrm{f}}$ is the multiplication by $q\frac{s}{\bar{s}}$ for some $q \in K^\times$. The following diagram commutes*

$$\begin{CD} \mathrm{Hom}(T(X),\mathbb{Q}/\mathbb{Z}) @>>> \mathrm{Br}(X) \\ @VVV @VV{\sigma_*}V \\ \mathrm{Hom}(q\frac{s}{\bar{s}}T(X),\mathbb{Q}/\mathbb{Z}) @>>> \mathrm{Br}(X^\sigma) \end{CD}$$

Proof. This follows from Rizov's main theorem of complex multiplication of K3 surfaces, see Valloni's preprint. □

In the following corollary 'the field of moduli' means 'the relative field of moduli over K'.

Corollary 3.6 (Valloni) *The field of moduli of the integral HS on $T(X)$ with its integral quadratic form, where X is a K3 surface with CM by $(K,\theta,\mathcal{O}_K)$, is the subfield of K_{ab} corresponding to the subgroup*

$$S = \{s \in \mathbf{A}_K^\times | qs\bar{s}^{-1}\mathcal{O}_K = \mathcal{O}_K \textit{ for some } q \in K^\times,\ q\bar{q} = 1\}.$$

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