Corrigendum to "The Brauer group and the Brauer–Manin set of products of varieties" *J. Eur. Math. Soc.* **16** (2014) 749–768.

Alexei N. Skorobogatov and Yuri G. Zarhin

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Abstract

In his review [F] of our paper [SZ14], Faltings pointed out that he could not follow the proof of Proposition 2.2. In this corrigendum we rectify this and other mistakes in [SZ14].

The main results of [SZ14], Theorems A, B and C, are correct as stated. However, the version of the Künneth formula in degree 2 with coefficients in an arbitrary ring mentioned on p. 750 of [SZ14], with reference to Proposition 2.2, is not true in this generality (see Remark 1.2 for a counterexample). A similar correction needs to be made to Theorem 2.6.

1 Correction to Proposition 2.2

Proposition 1.1 *Let X and Y be non-empty path-connected CW-complexes such that $\mathrm{H}_1(X,\mathbb{Z})$ and $\mathrm{H}_1(Y,\mathbb{Z})$ are finitely generated abelian groups (which holds when X and Y are finite CW-complexes). For any abelian group G we have a canonical isomorphism*

$$\mathrm{H}^1(X\times Y,G)\cong \mathrm{H}^1(X,G)\oplus \mathrm{H}^1(Y,G).$$

If $G=\mathbb{Z}$ or $G=\mathbb{Z}/n$, where n is a positive integer, then there is a canonical isomorphism

$$\mathrm{H}^2(X\times Y,G)\cong \mathrm{H}^2(X,G)\oplus \mathrm{H}^2(Y,G)\oplus \mathrm{Hom}\big(\mathrm{H}^1(X,G)^\vee,\mathrm{H}^1(Y,G)\big),$$

where for a G-module M we write $M^\vee=\mathrm{Hom}(M,G)$.

Proof. We write $\mathrm{H}_n(X)=\mathrm{H}_n(X,\mathbb{Z})$. Since X is non-empty and path-connected we have $\mathrm{H}_0(X)=\mathbb{Z}$, see [Hat02, Prop. 2.7]. The Künneth formula for homology [Hat02, Thm. 3.B.6] gives a split exact sequence of abelian groups

$$0\to\bigoplus_{i=0}^{n}\big(\mathrm{H}_i(X)\otimes \mathrm{H}_{n-i}(Y)\big)\to \mathrm{H}_n(X\times Y)\to\bigoplus_{i=0}^{n-1}\mathrm{Tor}(\mathrm{H}_i(X),\mathrm{H}_{n-1-i}(Y))\to 0.$$

Since $\mathrm{H}_0(X) = \mathbb{Z}$, in degrees 1 and 2 this gives canonical isomorphisms

$$\mathrm{H}_1(X \times Y) \cong \mathrm{H}_1(X) \oplus \mathrm{H}_1(Y) \tag{1}$$

and

$$\mathrm{H}_2(X \times Y) \cong \mathrm{H}_2(X) \oplus \mathrm{H}_2(Y) \oplus \big(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y)\big). \tag{2}$$

For any abelian group G, the universal coefficients theorem [Hat02, Thm. 3.2] gives the following (split) exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Ext}(\mathrm{H}_{n-1}(X), G) \longrightarrow \mathrm{H}^n(X, G) \longrightarrow \mathrm{Hom}(\mathrm{H}_n(X), G) \longrightarrow 0, \tag{3}$$

where the third map evaluates a cocycle on a cycle. This gives a canonical isomorphism

$$\mathrm{H}^1(X, G) \cong \mathrm{Hom}(\mathrm{H}_1(X), G). \tag{4}$$

The desired isomorphism for H^1 now follows from (1).

Using the functoriality of the universal coefficients formula (3) with respect to the projections of $X \times Y$ to X and Y, together with the isomorphisms (1) and (2), we obtain a split short exact sequence

$$0 \to \mathrm{H}^2(X, G) \oplus \mathrm{H}^2(Y, G) \to \mathrm{H}^2(X \times Y, G) \to \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), G) \to 0. \tag{5}$$

The second map here has a retraction induced by the embedding of $X \times y_0$ and $x_0 \times Y$, for some base points x_0 and y_0. The third map in (5) is given by evaluating a cocycle on $X \times Y$ on the product of a cycle on X and a cycle on Y. A similar map with $G = G_1 \otimes G_2$ fits into the following commutative diagram with the natural right-hand vertical map:

$$\begin{array}{ccc} \mathrm{H}^2(X \times Y, G_1 \otimes G_2) & \longrightarrow & \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), G_1 \otimes G_2) \\ \cup \big\uparrow & & \big\uparrow \\ \mathrm{H}^1(X, G_1) \otimes \mathrm{H}^1(Y, G_2) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{H}_1(X), G_1) \otimes \mathrm{Hom}(\mathrm{H}_1(Y), G_2) \end{array} \tag{6}$$

Let $G = \mathbb{Z}$. By assumption, $\mathrm{H}_1(X)$ and $\mathrm{H}_1(Y)$ are finitely generated abelian groups. Let M and N be their respective quotients by the torsion subgroups. The map induced by multiplication in $\mathbb{Z}$

$$\mathrm{Hom}(\mathrm{H}_1(X), \mathbb{Z}) \otimes \mathrm{Hom}(\mathrm{H}_1(Y), \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), \mathbb{Z})$$

coincides with $\mathrm{Hom}(M, \mathbb{Z}) \otimes \mathrm{Hom}(N, \mathbb{Z}) \to \mathrm{Hom}(M \otimes N, \mathbb{Z})$, which is clearly an isomorphism, so the displayed map is also an isomorphism. Using (4) we rewrite it as

$$\mathrm{H}^1(X, \mathbb{Z}) \otimes \mathrm{H}^1(Y, \mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), \mathbb{Z}).$$

Now (5) gives a canonical isomorphism

$$\mathrm{H}^2(X \times Y, \mathbb{Z}) \cong \mathrm{H}^2(X, \mathbb{Z}) \oplus \mathrm{H}^2(Y, \mathbb{Z}) \oplus \big(\mathrm{H}^1(X, \mathbb{Z}) \otimes \mathrm{H}^1(Y, \mathbb{Z})\big). \tag{7}$$

In view of the diagram (6) the last summand is embedded into $\mathrm{H}^2(X \times Y, \mathbb{Z})$ via the cup-product map. Since $\mathrm{H}^1(X, \mathbb{Z})$ is a free abelian group of finite rank, we can rewrite (7) and obtain the desired isomorphism for $\mathrm{H}^2(X \times Y, \mathbb{Z})$.

Now let $G = \mathbb{Z}/n$. Then $\mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), \mathbb{Z}/n)$ is canonically isomorphic to

$$\mathrm{Hom}(\mathrm{H}_1(X), \mathrm{Hom}(\mathrm{H}_1(Y), \mathbb{Z}/n)) \cong \mathrm{Hom}(\mathrm{H}_1(X)/n, \mathrm{H}^1(Y, \mathbb{Z}/n)).$$

Since $\mathrm{Hom}(\mathrm{H}_1(X)/n, \mathbb{Z}/n) \cong \mathrm{H}^1(X, \mathbb{Z}/n)$, we have $\mathrm{H}^1(X, \mathbb{Z}/n)^\vee \cong \mathrm{H}_1(X)/n$. Now (5) produces the required isomorphism for $\mathrm{H}^2(X \times Y, \mathbb{Z}/n)$. □

Remark 1.2 For $X = Y = \mathbb{RP}^2$ we have $\mathrm{H}_1(X) = \mathbb{Z}/2$, so in this case the map induced by multiplication in $\mathbb{Z}/n$ with $n = 4$

$$\mathrm{Hom}(\mathrm{H}_1(X), \mathbb{Z}/n) \otimes \mathrm{Hom}(\mathrm{H}_1(Y), \mathbb{Z}/n) \longrightarrow \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), \mathbb{Z}/n)$$

is zero. From diagram (6) we see that in this case the cup-product map

$$\mathrm{H}^1(X, \mathbb{Z}/n) \otimes \mathrm{H}^1(Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^2(X \times Y, \mathbb{Z}/n)$$

is zero.

2 Correction to Theorem 2.6

Let k be a separably closed field. Let G be a finite commutative group k-scheme of order not divisible by $\mathrm{char}(k)$. The Cartier dual of G is defined as $\widehat{G} = \mathrm{Hom}(G, \mathbb{G}_{m,k})$ in the category of commutative group k-schemes.

For a proper and geometrically integral variety X over k, the natural pairing

$$\mathrm{H}^1_{\text{ét}}(X, G) \times \widehat{G} \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_{m,X}) = \mathrm{Pic}(X)$$

gives rise to a canonical isomorphism

$$\mathrm{H}^1_{\text{ét}}(X, G) \tilde{\longrightarrow} \mathrm{Hom}(\widehat{G}, \mathrm{Pic}(X)). \tag{8}$$

The map in (8) associates to a class of a G-torsor $\mathcal{T} \to X$ its 'type', see [Sko01, Thm. 2.3.6].

Let n be a positive integer not divisible by $\mathrm{char}(k)$. Define S_X as the finite commutative group k-scheme whose Cartier dual is

$$\widehat{S}_X = \mathrm{H}^1_{\text{ét}}(X, \mu_n) \cong \mathrm{Pic}(X)[n]. \tag{9}$$

We shall often consider the Tate twist $\widehat{S}_X(-1)$. So for a finite commutative group k-scheme G such that $nG = 0$ we introduce the notation

$$G^\vee = \mathrm{Hom}(G, \mathbb{Z}/n).$$

In particular, we have $S_X^\vee = \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)$. The pairing $G \times G^\vee \to \mathbb{Z}/n$ gives rise to a canonical isomorphism $G \xrightarrow{\sim} (G^\vee)^\vee$.

Let $\mathcal{T}_X \to X$ be an S_X-torsor whose type is the natural inclusion

$$\widehat{S}_X = \mathrm{Pic}(X)[n] \hookrightarrow \mathrm{Pic}(X);$$

it is unique up to isomorphism. The natural pairing

$$\mathrm{H}^1_{\text{ét}}(X, S_X) \times S_X^\vee \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)$$

with the class $[\mathcal{T}_X] \in \mathrm{H}^1_{\text{ét}}(X, S_X)$ induces the identity map on $S_X^\vee = \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)$. In other words, the image of $[\mathcal{T}_X]$ with respect to the map induced by $a\colon S_X \to \mathbb{Z}/n$ equals $a \in S_X^\vee$.

Suppose that Y is also a proper and geometrically integral variety over k. The image of $[\mathcal{T}_X] \otimes [\mathcal{T}_Y]$ under the map

$$\mathrm{H}^1_{\text{ét}}(X, S_X) \otimes \mathrm{H}^1_{\text{ét}}(Y, S_Y) \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)$$

induced by $a\colon S_X \to \mathbb{Z}/n$ and $b\colon S_Y \to \mathbb{Z}/n$, equals $a \otimes b \in S_X^\vee \otimes S_Y^\vee$.

We refer to [Mil80, Prop. V.1.16] for the existence and properties of the cup-product. Thus we can consider $[\mathcal{T}_X] \cup [\mathcal{T}_Y] \in \mathrm{H}^2_{\text{ét}}(X \times_k Y, S_X \otimes S_Y)$ and

$$a \cup b \in \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n \otimes \mathbb{Z}/n) \cong \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n).$$

The cup-product is functorial, so the image of $[\mathcal{T}_X] \cup [\mathcal{T}_Y]$ under the map induced by $a \otimes b$ is $a \cup b$. This can be rephrased by saying that the natural pairing

$$\mathrm{H}^2_{\text{ét}}(X \times_k Y, S_X \otimes S_Y) \ \times \ S_X^\vee \otimes S_Y^\vee \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \tag{10}$$

with $[\mathcal{T}_X] \cup [\mathcal{T}_Y]$ gives rise to the cup-product map

$$S_X^\vee \otimes S_Y^\vee = \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/n).$$

It is important to note that (10) factors through the pairing

$$\mathrm{H}^2_{\text{ét}}(X \times_k Y, S_X \otimes S_Y) \ \times \ \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n). \tag{11}$$

The pairing (11) with $[\mathcal{T}_X] \cup [\mathcal{T}_Y]$ induces a map

$$\varepsilon\colon \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n).$$

We thus have a commutative diagram, where ξ is induced by multiplication in $\mathbb{Z}/n$:

$$\begin{array}{ccc} S_X^\vee \otimes S_Y^\vee & \xrightarrow{\ \xi\ } & \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n) & \xrightarrow{\ \cup\ } & \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \end{array} \tag{12}$$

The canonical isomorphism $\mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \cong \mathrm{Hom}(S_X, S_Y^\vee)$ allows us to rewrite ε as the map sending $\varphi \in \mathrm{Hom}(S_X, S_Y^\vee)$ to $\varepsilon(\varphi) = \varphi_*[\mathcal{T}_X] \cup [\mathcal{T}_Y]$, where $\cup$ stands for the cup-product pairing

$$\mathrm{H}^1_{\text{ét}}(X, S_Y^\vee) \times \mathrm{H}^1_{\text{ét}}(Y, S_Y) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times Y, S_Y^\vee \otimes S_Y) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n).$$

We write $p_X \colon X \times_k Y \to X$ and $p_Y \colon X \times_k Y \to Y$ for the natural projections. Since X and Y are geometrically integral over the separably closed field k, we can choose base points $x_0 \in X(k)$ and $y_0 \in Y(k)$. We have the induced map

$$(\mathrm{id}_X, y_0)^* \colon\ \mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n)$$

and a similar map for Y. Using these maps we see that

$$(p_X^*, p_Y^*)\colon \quad \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n) \oplus \mathrm{H}^i_{\text{ét}}(Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \tag{13}$$

is split injective, so we have an isomorphism

$$\mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \cong \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n) \oplus \mathrm{H}^i_{\text{ét}}(Y, \mathbb{Z}/n) \oplus \mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)_{\text{prim}}, \tag{14}$$

where $\mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)_{\text{prim}}$ is the intersection of kernels of $(\mathrm{id}_X, y_0)^*$ and $(x_0, \mathrm{id}_Y)^*$. Since k is separably closed, we have $\mathrm{H}^i(k, M) = 0$ for any abelian group M and any $i \geq 1$. Thus $[\mathcal{T}_X] \cup [\mathcal{T}_Y]$ goes to zero under the maps induced by the restrictions to $x_0 \times Y$ and to $X \times y_0$. This implies that $\mathrm{Im}(\varepsilon) \subset \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)_{\text{prim}}$.

The following is a corrected version of [SZ14, Thm. 2.6].

Theorem 2.1 *Let X and Y be proper and geometrically integral varieties over a separably closed field k. Let n be a positive integer not divisible by* $\mathrm{char}(k)$. *Then we have the following statements.*

(i) *Write $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)^\vee = \mathrm{Hom}(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n), \mathbb{Z}/n)$ and similarly for Y. The maps ε and ξ defined above fit into the following commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n) & \xrightarrow{\ \xi\ } & \mathrm{Hom}(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)^\vee, \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)) \\ {\scriptstyle \cup}\downarrow & & {\scriptstyle \cong}\downarrow{\scriptstyle \varepsilon} \\ \mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/n) & \hookleftarrow & \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)_{\text{prim}} \end{array} \tag{15}$$

where ε is an isomorphism.

(ii) *If $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)$ is a free $\mathbb{Z}/n$-module (which holds if* $\mathrm{NS}\,(X)[n] = 0$*), then ξ is an isomorphism, so we have*

$$\mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/n) \cong \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}/n) \oplus \mathrm{H}^2_{\text{ét}}(Y, \mathbb{Z}/n) \oplus \left(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)\right).$$

Proof. Part (ii) is the degree 2 case of [Mil80, Cor. VI.8.13].

Let us prove (i). Diagram (15) is obtained from diagram (12) since $\mathrm{Im}(\varepsilon)$ is a subset of $\mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)_{\text{prim}}$, as explained above. It remains to show that ε is an isomorphism. From the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_{\text{ét}}(X, \mathrm{H}^q_{\text{ét}}(Y, \mathbb{Z}/n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)$$

we get an isomorphism $\mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)_{\text{prim}} \cong \mathrm{H}^1_{\text{ét}}(X, \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n))$. As a particular case of (8) we get an isomorphism

$$\mathrm{H}^1_{\text{ét}}(X, \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)) \cong \mathrm{Hom}(S_Y, S_X^\vee) \cong \mathrm{Hom}(S_X, S_Y^\vee).$$

Thus the source and the target of ε are isomorphic finite abelian groups. One can finish the proof following the original arguments in [SZ14] with small adjustments; see [CTS21, pp. 161–162] for this revised proof.

Here we give a short proof communicated to us by Yang Cao. Since the source and the target of ε have the same cardinality, it is enough to show that

$$\varepsilon\colon \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)_{\text{prim}}$$

is *injective.* More generally, for an integer $m|n$ consider the map

$$\varepsilon_m\colon \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/m) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/m)_{\text{prim}}$$

defined via pairing with $[\mathcal{T}_X] \cup [\mathcal{T}_Y]$. We prove that ε_m is injective by induction on $m|n$. If p is a prime, the usual Künneth formula [Mil80, Cor. VI.8.13] for the field $\mathbb{F}_p$ implies that the cup-product map

$$\cup\colon \mathrm{H}^1_{\text{ét}}(X, \mathbb{F}_p) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_p) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{F}_p)_{\text{prim}}$$

is an isomorphism. We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(S_X, \mathbb{F}_p) \otimes \mathrm{Hom}(S_Y, \mathbb{F}_p) & \xrightarrow{\ \xi\ } & \mathrm{Hom}(S_X \otimes S_Y, \mathbb{F}_p) \\
\cong \downarrow & & \downarrow \varepsilon_p \\
\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/p) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/p) & \xrightarrow{\ \cup\ } & \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/p)_{\text{prim}}
\end{array} \tag{16}$$

In this case ξ is an isomorphism, hence ε_p is also an isomorphism.

Now for a positive integer $m|n$ assume that ε_a is injective for all $a|m$, $a \neq m$. Write $m = ab$. The exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}/a \longrightarrow \mathbb{Z}/m \longrightarrow \mathbb{Z}/b \longrightarrow 0$$

gives rise to the long exact sequences of étale cohomology groups of X, Y and $X \times Y$, which are linked by the split injective maps (13). Using (14) and the well-known fact that $\mathrm{H}^1_{\text{ét}}(X \times Y, \mathbb{Z}/b)_{\text{prim}} = 0$ (see [SZ14, Cor. 1.8] or [CTS21, Thm. 5.7.7 (i)]) we obtain that the top row of the following commutative diagram is exact (see [CTS21, p. 160] for an alternative argument):

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/a)_{\text{prim}} & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/m)_{\text{prim}} & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/b)_{\text{prim}} \\ & \big\uparrow \varepsilon_a & & \big\uparrow \varepsilon_m & & \big\uparrow \varepsilon_b \\ 0 \longrightarrow & \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/a) & \longrightarrow & \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/m) & \longrightarrow & \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/b) \end{array}$$

The bottom row is obviously exact. The diagram implies that the middle map is injective too. We conclude that $\varepsilon = \varepsilon_n$ is injective, hence an isomorphism.

References

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