The Brauer–Grothendieck group

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Introduction

A less known event of 1968 in Paris was the publication of “Dix exposés sur la cohomologie des schémas” by J. Giraud, A. Grothendieck, S.L. Kleiman, M. Raynaud and J. Tate. Included there, with a kind permission of N. Bourbaki, were two talks by Grothendieck in the Bourbaki seminar, entitled “Le groupe de Brauer I” and “Le groupe de Brauer II”, followed by a 100 pages long “Le groupe de Brauer III”. More than fifty years later, it remains the principal source on Grothendieck’s generalisation of the Brauer group of fields to the Brauer group of schemes, in the language of étale cohomology. Masterfully written, with a fresh appeal of a newly designed theory, Grothendieck’s two seminar talks and a long paper are hardly a textbook.

Our first motivation for writing this book was to complement Grothendieck’s foundational text with a more accessible modern exposition, and to give proofs of some results not easily found in the literature. Our second motivation was to describe recent developments in the theory of the Brauer–Manin obstruction and local-to-global principles, as well as new geometric applications of the Brauer group.

Let us give a brief sketch of the history of the Brauer–Grothendieck group.

Soon after the publication of “Le groupe de Brauer I, II, III” it became clear that this is a very useful tool. In his 1970 ICM address, Manin defined a natural pairing between the Brauer group of a variety X over a number field k and the space of its adelic points $X(\mathbf{A}_k)$. He pointed out that this pairing generalises pairings in the theory of abelian varieties (Cassels–Tate pairing on the Tate–Shafarevich group, maps in the Cassels–Tate dual sequence) and in the theory of algebraic tori (Voskresenskiĭ). He also showed how several known counter-examples to the Hasse principle building on reciprocity laws could be interpreted in terms of this pairing. The Brauer–Manin obstruction revolutionised the theory of Diophantine equations by enabling one to study local-to-global principles for rational points beyond the narrow confines of varieties satisfying the Hasse principle and weak approximation.

In a separate development, in 1972 Artin and Mumford used the birational invariance of the Brauer group to construct examples of unirational but not rational varieties over complex numbers. This gave a negative answer to the Lüroth problem in dimension at least 3, by a method different from those of Clemens–Griffiths and Iskovskikh-Manin, found about the same time. In 1984 the unramified Brauer group was used by Saltman who found examples of finite subgroups $G \subset \mathrm{GL}(n, \mathbb{C})$ such that the quotient $\mathrm{GL}(n, \mathbb{C})/G$ is not rational. This gives a negative answer to a problem of Emmy Noether motivated by the inverse Galois problem.

In the 1970s and 1980s, Colliot-Thélène and Sansuc developed the theory of descent and universal torsors, and linked it to the Brauer–Manin obstruction. Jointly with Swinnerton-Dyer they proved that the Brauer–Manin obstruction correctly describes the closure of the set of rational points $X(k)$ in $X(\mathbf{A}_k)$ for some intersections of quadrics. Important results for conic bundles were obtained by Salberger, who also studied analogous results for zero-cycles. In contrast to these developments, in 1997 Skorobogatov constructed a bielliptic surface X over $\mathbb{Q}$ which is a counter-example to the Hasse principle that cannot be explained by the Brauer–Manin obstruction. Stronger versions of the Brauer–Manin obstruction were soon proposed by Harari and Skorobogatov, but a more radical counter-example found by Poonen in 2010 shows that these obstructions are insufficient too.

Very recently, the birational invariance of the Brauer group has become one of the ingredients of the specialisation method discovered by Voisin and developed by Colliot-Thélène and Pirutka, and later by Schreieder. This method was used by Hassett, Pirutka and Tschinkel to give examples of algebraic families of smooth projective varieties over complex numbers some of which are rational whereas some others are not even stably rational.

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Let us give a brief outline of the contents of this book. We refer to the introductions to individual chapters for more details.

The first two chapters contain preliminary material on Galois and étale cohomology. For obvious reasons many results here are stated without proofs, though we give a proof of compatibility of two definitions of the residue map for the Brauer group of a discretely valued field.

Chapter 3 starts with definitions of the two Brauer groups of a scheme: the Brauer group defined in terms of Azumaya algebras, which we call the Brauer–Azumaya group, and the cohomological Brauer group, which we call the Brauer–Grothendieck group. We reproduce de Jong's proof of a theorem of Gabber which says that a natural homomorphism from the Brauer–Azumaya group to the torsion subgroup of the Brauer–Grothendieck group is an isomorphism for a quasi-projective scheme over an affine scheme. Other fundamental subjects discussed in this chapter are localisation and the purity theorem for the Brauer group.

In Chapters 4, 5 and 6 we focus on the Brauer group of a smooth variety over a field. In Chapter 4 we describe the structure of this group and methods to compute it, both in the general case and for classes of varieties satisfying additional geometric assumptions. In Chapter 5 we define the unramified Brauer group and prove that the Brauer group of a smooth and proper variety is a birational invariant. Chapter 6 deals with Severi–Brauer varieties, quadrics, and, more generally, projective and affine hypersurfaces. Here we give a proof that the Severi–Brauer variety associated to a cyclic algebra is birationally equivalent to the affine hypersurface given by a norm equation, a result which seems hard to find in the literature.

Chapter 7 contains various results on the Brauer group of singular varieties, which in particular provide counterexamples to the familiar properties of the Brauer group in the smooth case.

In Chapter 8 we collect results on the Brauer group and on the unramified Brauer group of a variety equipped with an action of a linear algebraic group, such as a torsor or a homogeneous space. We discuss theorems of Saltman and of Bogomolov that can be used to give negative answers to Noether's problem.

Chapters 9, 10 and 11 are devoted to the Brauer group of a family of varieties. The subject of Chapter 9 is schemes over a local ring and varieties over a local field. Here we also discuss split fibres and explore their properties. In Chapter 10, after defining the vertical Brauer group of a morphism, we explain how to compute the Brauer group of a conic bundle over a 1- or 2-dimensional base. We present the Artin–Mumford examples from this birational point of view. Chapter 11 contains an exposition of the specialisation method with applications to the behaviour of stable rationality in a family.

The next group of chapters concerns arithmetic applications. The Brauer–Manin obstruction is introduced and studied in Chapter 12. Chapter 13 contains an exposition of several results stating that for some classes of varieties the Brauer–Manin obstruction correctly describes the closure of the set of rational points inside the topological space of adelic points. We discuss Schinzel's Hypothesis (H), applications of results in additive combinatorics due to Green, Tao and Ziegler to rational points and sketch a proof of a theorem of Harpaz and Wittenberg about families of rationally connected varieties. In this chapter we also give an overview of the theory of obstructions to the local-to-global principles for rational points. Chapter 14 deals with zero-cycle analogues of these themes.

The last chapter concerns finiteness properties of the Brauer group of abelian varieties, K3 surfaces, and varieties dominated by products of curves when the ground field is finitely generated over its prime subfield. The treatment of K3 surfaces necessitates a detour via an interpretation of their moduli spaces as Shimura varieties and the Kuga–Satake contruction. We give complete proofs of the Tate conjecture and the finiteness of the Brauer group for K3 surfaces in the case of characteristic zero.

The reader won't fail to notice that the style of this book varies from chapter to chapter, from a more in-depth treatment to a survey. The authors are aware of these and other imperfections, as well as omissions of a number of important subjects. In this book we only fleetingly discuss descent and torsors, for which we refer to [Sko01]. Other subjects which could have been included but are not included:

unramified cohomology in higher degrees,
the Brauer groups of varieties over finite fields,
Swinnerton-Dyer's method for rational points on a pencil of genus 1 curves,
the integral Brauer–Manin obstruction.

We recommend Poonen's recent book [Po18] as an extremely helpful and comprehensive introduction to rational points. Another book on the Brauer group of varieties was recently published by Gorchinskiy and Shramov [GSh18].

Acknowledgements

This book obviously stems from Grothendieck's original work [Gro68]. We have made extensive use of books by J.-P. Serre [SerCG, SerCL, Ser03], J.S. Milne [Mil80], Ph. Gille and T. Szamuely [GS17], S. Bosch, W. Lütkebohmert and M. Raynaud [BLR90], M. Olsson [Ols16], of the Stacks Project [Stacks], and of Kleiman's survey on the Picard scheme [Kle05]. We have tried to acknowledge our debt to these and other sources whenever possible.

Much of the material for the book was taken from seminars and lecture courses given by the authors over a number of years. The first named author wants to mention his lectures at the following institutions: GAEL (Istanbul, 2007), Emory University (Atlanta, 2008), POSTECH (Pohang, 2010), Kloosterman chair (Leiden, 2011), Summer school (Yaroslavl, 2012), BICMR (Beijing, 2012 and 2015), Arizona Winter School (Tucson, 2015), Lamé chair (Saint Petersburg, 2015), lectures in Santiago and Talca (Chile, 2018), School on birational geometry of surfaces (Gargnano, 2018). The second named author used his lecture notes for his courses "Arithmetic geometry: rational points" at Imperial College London (2013) and "The Brauer–Manin obstruction" at POSTECH (Pohang, 2019), as well as his lecture courses at the following events: Summer school (Yaroslavl, 2014), LMS–CMI Summer school on Diophantine equations (Baskerville Hall, Hay-on-Wye, 2015), Autumn school: topics in arithmetic and algebraic geometry (Mainz, 2017), Summer school–conference on the Brauer groups (Moscow, 2018). We are grateful to the organisers of these meetings for inviting us to speak.

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Notation

For an abelian group A we denote by $A[n]$ the n-torsion subgroup of A, i.e. $A[n] = \{x \in A | nx = 0\}$. If ℓ is a prime number, we denote by $A\{\ell\}$ the ℓ-primary subgroup of A, i.e. the set of elements $x \in A$ such that $\ell^i x = 0$ for some $i \geq 1$. We denote by A_{tors} the torsion subgroup of A, i.e. the union of $A[n]$ for all $n \geq 1$.

For a field k we write $\bar{k}$ for a fixed algebraic closure of k, and $k_{\mathrm{s}} \subset \bar{k}$ for the separable closure of k in $\bar{k}$. Let

$$\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$$

be the Galois group of k. The characteristic exponent of k is 1 if $\mathrm{char}(k) = 0$ and p if $\mathrm{char}(k)$ is a prime number p.

The p-cohomological dimension of a profinite group G, where p is a prime, is the smallest integer n such that $\mathrm{H}^m(G, M)\{p\} = 0$ for all G-modules M such that $M = M_{\mathrm{tors}}$ and all $m > n$. The cohomological dimension of a profinite group G is the supremum of its p-cohomological dimensions over all primes p. The cohomological dimension of a field k is the cohomological dimension of its Galois group Γ.

For a scheme X over a field k, we write $\overline{X} = X \times_k \bar{k}$ and $X^{\mathrm{s}} = X \times_k k_{\mathrm{s}}$. A variety over k is defined as a separated scheme of finite type over k. In particular, a variety is quasi-compact (i.e., it is a finite union of affine open subsets) and quasi-separated (i.e., the diagonal morphism $X \to X \times_{\mathbb{Z}} X$ is quasi-compact; this implies that the intersections of two affine open subsets of a variety X over k is a finite union of affine open subsets of X).

Chapter 1

Galois cohomology

This chapter begins with a brief introduction to the classical theory of quaternion algebras over a field. After recalling basic facts about central simple algebras, we give the classical definition of the Brauer group of a field in terms of such algebras. We state several standard results about Galois cohomology and descent, and then give the cohomological definition of the Brauer group of a field and construct a natural isomorphism between the resulting groups. For a thorough treatment of central simple algebras and the Brauer groups of fields we refer to the book by P. Gille and T. Szamuely [GS17] from which we borrowed some of the material for this chapter. Various aspects of the theory of simple algebras and the Brauer group can be found in Bourbaki's *Algèbre*, Ch. VIII [BouVIII], and in the books by J.-P. Serre [SerCL, SerCG], A.A. Albert [Alb31], I. Reiner [Rei03] and I.N. Herstein [Her68].

In this chapter we also state several results about cyclic algebras and the vanishing of the Brauer group for specific fields, such as finite fields, function fields in one variable over an algebraically closed field, C_1-fields.

In Section 1.4 we discuss the Brauer group of discretely valued fields and the associated crucial notion of residue. There are several approaches to the definition of the residue; we explain how two of them are related to each other. We finish by proving a theorem of D.K. Faddeev which describes the Brauer group of the field of rational functions $k(t)$, where k is a perfect field.

1.1 Quaternion algebras and conics

In this section k is a field of characteristic not equal to 2.

Quaternions

To $a, b \in k^*$ one can attach a non-commutative associative k-algebra in the following way.

Definition 1.1.1 *A* **quaternion algebra** *over k is a k-algebra isomorphic to the 4-dimensional associative algebra $Q_k(a,b)$ with basis 1, i, j, ij and the multiplication table*

$$i^2 = a,\ j^2 = b,\ ij = -ji,$$

where $a, b \in k^*$.

For a field extension $k \subset K$ there is a natural isomorphism

$$Q_k(a,b) \otimes_k K \tilde{\longrightarrow} Q_K(a,b).$$

Exercise 1.1.2 *The map $k \to Q_k(a,b)$ sending x to $x \cdot 1$ identifies k with the centre of $Q_k(a,b)$. The two-sided ideals of $Q_k(a,b)$ are 0 and $Q_k(a,b)$.*

For example, $Q_{\mathbb{R}}(-1,-1)$ is the algebra of Hamilton's quaternions $\mathbb{H}$. This is a division algebra: every non-zero element of $\mathbb{H}$ is invertible.

A natural question is: for which $a, b \in k^*$ is $Q_k(a,b)$ a division algebra?

Definition 1.1.3 *Let Q be a quaternion algebra. A* **pure quaternion** *in Q is 0 or an element $q \in Q$ such that $q \notin k$ but $q^2 \in k$.*

It follows that if $Q \cong Q_k(a,b)$, then the pure quaternions are precisely the elements of the form $yi + zj + wij$. (To see this, square $x + yi + zj + wij$, then there are some cancellations, and if $x \neq 0$, then $y = z = w = 0$). Thus each quaternion $q \in Q$ is uniquely written as $q = q_0 + q_1$, where $q_0 \in k$ and q_1 is a pure quaternion.

Definition 1.1.4 *The* **conjugate** *of $q = q_0 + q_1 \in Q$ is $\bar{q} = q_0 - q_1$. The* **norm** *of $q \in Q$ is $\mathrm{N}(q) = q\bar{q} \in k$. The* **trace** *of $q \in Q$ is $\mathrm{Tr}(q) = q + \bar{q} \in k$.*

For any $q_1, q_2 \in Q$ we have

$$\overline{q_1 . q_2} = \bar{q}_2 . \bar{q}_1, \quad \mathrm{N}(q_1 q_{2)} = \mathrm{N}(q_1)\mathrm{N}(q_2), \quad \mathrm{Tr}(q_1 + q_2) = \mathrm{Tr}(q_1) + \mathrm{Tr}(q_2).$$

Exercise 1.1.5 *If $q \in Q$ is a pure quaternion such that q^2 is not a square in k, then $1, q$ span a quadratic field which is a maximal subfield of Q.*

The quaternion k-algebras $Q_k(a,b)$ and $Q_k(c,d)$ are isomorphic if and only if there exist anti-commuting pure quaternions $I, J \in Q(a,b)$ such that $I^2 = c$, $J^2 = d$. Then $1, I, J, IJ$ is a basis of the k-vector space $Q_k(a,b)$. Thus for any $u, v \in k^*$ we have $Q_k(au^2, bv^2) \cong Q_k(a,b)$.

Lemma 1.1.6 *If $c \in k^*$ is a norm from $k(\sqrt{a})^*$, then $Q_k(a,b) \cong Q_k(a,bc)$.*

Proof. Write $c = x^2 - ay^2$ with $x, y \in k$. Set $J = xj + yij \in Q_k(a,b)$. One checks $Ji = -iJ$ and $J^2 = -\mathrm{N}(J) = bc$. □

If $z \in Q_k(a,b)$ is an invertible element, then $\mathrm{N}(q) \in k^*$. If $\mathrm{N}(q) = 0$, then $q\bar{q} = 0$, so q is a zero divisor. Thus the invertible elements are exactly the elements with non-zero norm. The norm on $Q_k(a,b)$ is the diagonal quadratic form $\langle 1, -a, -b, ab \rangle$, and this leads us to the following criterion.

We write $M_n(k)$ for the k-algebra of $n \times n$-matrices with entries in k. In fact, $M_2(k)$ can be seen as a quaternion algebra of a special kind.

Proposition 1.1.7 *Let $Q = Q_k(a, b)$, where $a, b \in k^*$. The following statements are equivalent:*
(i) *Q is not a division algebra;*
(ii) *Q is isomorphic to the matrix algebra $M_2(k)$;*
(iii) *the diagonal quadratic form $\langle 1, -a, -b\rangle$ represents zero in k;*
(iv) *the norm form $\mathrm{N} = \langle 1, -a, -b, ab\rangle$ represents zero in k;*
(v) *b is in the image of the norm homomorphism $k(\sqrt{a})^* \to k^*$.*

Proof. First assume $a \in k^{*2}$. The equivalence of all statements but (ii) is clear. To prove the equivalence with (ii) we can assume that $a = 1$. The matrix algebra $M_2(k)$ is spanned by

$$1 = \mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}, \quad ij = \begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix},$$

and so is isomorphic to $Q_k(1, b)$.

Now assume that $a \in k^*$ is not a square. Then (i) is equivalent to (iv) since $\mathrm{N}(q) = q\bar{q}$. Next, (iv) implies (v) because the ratio of two non-zero norms is a norm. It is clear that (v) implies (iii) which implies (iv), since N is the diagonal quadratic form $\langle 1, -a, -b, ab\rangle$. So (iii), (iv) and (v) are all equivalent to (i). Lemma 1.1.6 shows that under the assumption of (v) the algebra $Q_k(a, b)$ is isomorphic to $Q_k(a, b^2) \cong Q_k(a, 1)$, so we use the result of the first part of the proof to prove the equivalence with (ii). □

If the conditions of this theorem are satisfied one says that $Q_k(a, b)$ is *split*. If K is a field extension of k such that $Q_K(a, b) \cong Q_k(a, b) \otimes_k K$ is split, then one says that K *splits* $Q_k(a, b)$.

Since any quaternion algebra $Q_k(a, b)$ is split by k_{s}, we see that $Q_k(a, b)$ is a (k_{s}/k)-*form* of the 2×2-matrix algebra, which means that

$$Q_k(a, b) \otimes_k k_{\mathrm{s}} \cong M_2(k_{\mathrm{s}}).$$

For example, $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$.

It is an easy exercise to show that the pure quaternions in $M_2(k)$ are precisely the traceless matrices.

Proposition 1.1.8 *Any quaternion algebra Q split by $k(\sqrt{a})$ contains this field and can be written as $Q = Q_k(a, c)$ for some $c \in k^*$. Conversely, if Q contains $k(\sqrt{a})$, then Q is split by $k(\sqrt{a})$.*

Proof. If the algebra Q is split, take $c = 1$. Assume Q is not split, hence is a division algebra. In particular, a is not a square in k. There exist $q_0, q_1 \in Q$, not both equal to 0, such that $\mathrm{N}(q_0 + q_1\sqrt{a}) = 0$. Since Q is a division algebra, we have $q_0 \neq 0$ and $q_1 \neq 0$. We have

$$\mathrm{N}(q_0 + q_1\sqrt{a}) = \mathrm{N}(q_0) + a\mathrm{N}(q_1) + \sqrt{a}(q_0\bar{q}_1 + q_1\bar{q}_0) = 0,$$

hence $\mathrm{N}(q_0) + a\mathrm{N}(q_1) = 0$ and $q_0\bar{q}_1 + q_1\bar{q}_0 = 0$. Set $q_2 = q_0.\bar{q}_1$. We have

$$q_2^2 = q_0.\bar{q}_1.q_0.\bar{q}_1 = -\bar{q}_0.\bar{q}_1.q_1.\bar{q}_0 = -\mathrm{N}(q_0)\mathrm{N}(q_1) = a\mathrm{N}(q_1)^2.$$

Let $I = q_2/\mathrm{N}(q_1)$. Then $I^2 = a$. Since a is not a square in k, we see that $I \notin k$. The conjugation by I is a k-linear transformation of Q. It preserves the subspace of pure quaternions, since it preserves the condition $z^2 \in k$. The order of this linear transformation is 2 because $I \notin k$, hence I is not in the centre of Q. Thus the -1-eigenspace is non-zero, so we can find a non-zero pure quaternion $J \in Q$ such that $IJ + JI = 0$. We have $J^2 = c \in k$, since J is pure. This is enough to conclude that $Q \cong Q_k(a, c)$.

The converse follows from the fact that $k(\sqrt{a}) \otimes k(\sqrt{a})$ contains zero divisors (the norm form $x^2 - ay^2$ represents zero in $k(\sqrt{a})$). Hence the same is true for $Q \otimes_k k(\sqrt{a})$. □

Corollary 1.1.9 *The quadratic fields that split a quaternion division algebra are exactly the quadratic subfields of this algebra.*

Conics

Definition 1.1.10 *Let Q be a quaternion algebra over k. Let $Q_1 \subset Q$ be the 3-dimensional subspace of pure quaternions. The norm form on Q induces a non-degenerate quadratic form on Q_1. The conic attached to Q is the conic $C(Q)$ defined by this quadratic form in the projective plane $\mathbb{P}^2_k = \mathbb{P}(Q_1)$.*

Thus the conic attached to the quaternion algebra $Q_k(a, b)$ is the plane algebraic curve $C(a, b) \subset \mathbb{P}^2_k$ given by the equation

$$-ax^2 - by^2 + abz^2 = 0.$$

Up to a change of variables, this conic is also given by the equation

$$z^2 - ax^2 - by^2 = 0.$$

By Proposition 1.1.7 the conic $C(Q)$ has a k-point if and only if the quaternion algebra Q is split.

Remark 1.1.11 1. Since the characteristic of k is not 2, every conic can be given by a diagonal quadratic form, and so is attached to some quaternion algebra.

2. The projective line is isomorphic to the conic $xz - y^2 = 0$ via the map $(X : Y) \mapsto (X^2 : XY : Y^2)$.

3. If a conic C has a k-point, then $C \cong \mathbb{P}^1_k$. (The projection from a k-point gives rise to a rational parameterisation of C, which is an isomorphism.)

4. Thus the function field $k(C)$ of a conic C is a purely transcendental extension of k if and only if C has a k-point.

Exercise 1.1.12 1. Check that $Q_k(a, 1 - a)$ and $Q_k(a, -a)$ are split.

2. Check that if $k = \mathbb{F}_q$ is a finite field, then all quaternion k-algebras are split. (By assumption q is not a power of 2. Write $ax^2 = 1 - by^2$ and use a counting argument for x and y to prove the existence of a solution in $\mathbb{F}_q$.)

3. Let Q be a quaternion algebra over k and let C be the associated conic. Then Q is split over k if and only if $Q_{k(t)}$ is split over $k(t)$. (Take a $k(t)$-point on $C \subset \mathbb{P}^2_k$ represented by three polynomials not all divisible by t, and reduce modulo t.)

4. Q is split over $k(C(Q))$. (Consider the generic point of the conic.)

The following theorem of Max Noether [Noe70] is a special case of Tsen's theorem (Theorem 1.2.12 below). It plays an important rôle in the classification of complex algebraic surfaces. The proof given here is due to Tsen.

Theorem 1.1.13 (M. Noether) *Let k be an algebraically closed field. Then all quaternion $k(t)$-algebras are split.*

Proof. It is enough to show that any conic over $k(t)$ has a point (this is Max Noether's statement). We can assume that the coefficients of the corresponding quadratic form are polynomials in t of degree at most m. We look for a solution (X, Y, Z), where X, Y and Z are polynomials in t (not all of them zero) of degree n for some large integer n. The coefficients of these polynomials can be thought of as points of the projective space $\mathbb{P}^{3n+2}$. The solutions bijectively correspond to the points of a closed subset of $\mathbb{P}^{3n+2}$ given by $2n + m + 1$ homogeneous quadratic equations. Since k is algebraically closed this set is non-empty when $3n + 2 \geq 2n + m + 1$, by a standard result from algebraic geometry. (If an irreducible variety X is not contained in a hypersurface H, then $\dim(X \cap H) = \dim(X) - 1$. This implies that on intersecting X with r hypersurfaces the dimension drops at most by r, see [Sha74, Ch. 1]). □

The following theorem is due to Witt [Wit35, §2]

Theorem 1.1.14 (Witt) *Two quaternion algebras are isomorphic if and only if the conics attached to them are isomorphic.*

Proof. We reproduce the proof of [GS17, Thm. 1.4.2]. Recall that $C(Q)$ denotes the conic attached to the quaternion algebra Q. An isomorphism of quaternion algebras $Q \cong Q'$ induces an isomorphism of their vector spaces of pure quaternions respecting the norm form. Hence it induces an isomorphism $C(Q) \cong C(Q')$.

Let us prove that if $C(Q) \cong C(Q')$, then $Q \cong Q'$. If Q is split, then $C(Q)$ has a k-point. Thus $C(Q')$ also has a k-point. But then the norm form of Q' represents zero, and this implies that Q' is split.

Assume from now on that neither algebra is split. Write $Q = Q_k(a, b)$ and write C for the conic $C(Q') \cong C(Q) = C(a, b)$ given by the equation

$$z^2 - ax^2 - by^2 = 0.$$

Let $K = k(\sqrt{a})$ and let $K(C)$ be the function field of the conic $C_K = C \times_k K$. The conic C has a K-point, hence Q' is split by K. By Proposition 1.1.8 we can write $Q' = Q_k(a, c)$ for some $c \in k^*$. By Exercise 4 above Q' is split by the function field $k(C)$. By Proposition 1.1.7 this implies that $c \in k^* \subset k(C)^*$ is contained in the image of the norm map

$$c \in \mathrm{Im}[K(C)^* \longrightarrow k(C)^*].$$

Let $\sigma \in \mathrm{Gal}(K/k) \cong \mathbb{Z}/2$ be the generator. Then we can write $c = f.\sigma(f)$, where f is a rational function on the conic C_K. One can replace f with $f\sigma(g)g^{-1}$ for any $g \in K(C)^*$ without changing c.

The group $\mathrm{Div}(C_K)$ of divisors on $C_K \cong \mathbb{P}^1_K$ is freely generated by the closed points of C_K. This is a module of $\mathbb{Z}/2 = \langle\sigma\rangle$ with a σ-stable basis. The divisors of functions are exactly the divisors of degree 0. The divisor $D = \mathrm{div}(f)$ is an element of $\mathrm{Div}(C_K)$ satisfying $(1+\sigma)D = 0$. By comparing the multiplicities of points in the support of D we deduce that there is $G \in \mathrm{Div}(C_K)$ such that $D = (1-\sigma)G$. Let $P = (1:0:\sqrt{a})$. If $n = \deg(G)$ the divisor $G - nP \in \mathrm{Div}(C_K)$ has degree 0. Since $C_K \cong \mathbb{P}^1_K$, this implies $G - nP = \mathrm{div}(g)$ for some $g \in K(C)^*$. We have

$$\mathrm{div}(f\sigma(g)g^{-1}) = D + \sigma G - G + n(P - \sigma P) = n(P - \sigma P) = n\ \mathrm{div}\left(\frac{z - \sqrt{a}x}{y}\right).$$

It follows that

$$f\sigma(g)g^{-1} = e\left(\frac{z - \sqrt{a}x}{y}\right)^n \in K(C)^*$$

for some $e \in K^*$. Taking norms, we obtain

$$c = f\sigma(f) = N_{K/k}(e)\left(\frac{z^2 - ax^2}{y^2}\right)^n = N_{K/k}(e)b^n \in k(C)^*$$

hence $c = N_{K/k}(e).b^n \in k^*$. Thus $Q' = Q(a,c) = Q(a, N_{k(\sqrt{a})/k}(e).b^n)$ for some integer n. By Lemma 1.1.6, it is isomorphic to $Q(a,b)$ or to $Q(a,1)$. Since Q' is not split, we must have $Q' \cong Q(a,b)$. □

1.2 The language of central simple algebras

1.2.1 Central simple algebras

Quaternion algebras and matrix algebras are particular cases of central simple algebras.

Definition 1.2.1 *An associative k-algebra A is called* **simple** *if the only two-sided ideals of A are 0 and A. An associative k-algebra A is called* **central** *if its centre is k. A* **central simple algebra** *is a finite dimensional k-algebra that is both central and simple.*

Recall that if V and W are vector spaces over k, then $V \otimes_k W$ is the linear span of vectors $v \otimes w$, $v \in V$, $w \in W$, subject to the axioms

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \quad v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2,$$

and

$$c(v \otimes w) = (cv) \otimes w = v \otimes (cw) \quad \text{for any} \quad c \in k.$$

This turns $V \otimes_k W$ into a k-vector space. If (e_i) is a basis of V, and (f_j) is a basis of W, then $(e_i \otimes f_j)$ is a basis of $V \otimes_k W$. The vector spaces $(V \otimes U) \otimes W$ and $V \otimes (U \otimes W)$ are canonically isomorphic.

Given two k-algebras A and B, one defines the structure of a k-algebra on $A \otimes_k B$ by the rule $(x \otimes y) \cdot (x' \otimes y') = (xx') \otimes (yy')$.

Properties. 1. Any central division algebra is a central simple algebra.

2. For any integer $n \geq 1$ the algebra of matrices $M_n(k)$ is a central simple algebra. More generally, if D is a central division algebra, then $M_n(D)$ is a central simple algebra [GS17, Example 2.1.2].

3. $M_m(k) \otimes_k M_n(k) \cong M_{mn}(k)$.

Later we will use the following important property of matrix algebras.

Proposition 1.2.2 *Any automorphism of the k-algebra $M_n(k)$ is induced by conjugation by an invertible matrix. This invertible matrix is well defined up to multiplication by a scalar matrix.*

Proof. [GS17, Lemma 2.4.1, Cor. 2.4.2]. □

The structure of central simple algebras is described by a theorem of Wedderburn.

Theorem 1.2.3 (Wedderburn) *For any central simple algebra A there is a central division algebra D such that $A \cong D \otimes_k M_n(k) = M_n(D)$.*

The integer n is well defined, and the algebra D is well defined up to a non-unique isomorphism. Proofs of this fundamental theorem can be found in [BouVIII, §5, no. 4, Cor. 2], [Her68, Thm. 2.1.6], [GS17, Thm. 2.1.3].

Corollary 1.2.4 *Any central simple algebra over an algebraically closed field k is isomorphic to a matrix algebra $M_n(k)$.*

Proof. We need to prove that a central division k-algebra D coincides with its centre k. Pick any $x \in D$. Let $I \subset k[t]$ be the ideal consisting of polynomials vanishing on x. This is a non-zero ideal, generated by some $f(t) \in k[t]$. Since D is a division algebra, $f(t)$ is irreducible. As k is algebraically closed, $f(t)$ has degree 1, hence $x \in k$. □

Lemma 1.2.5 *Let k be a field and let A be a finite-dimensional k-algebra. Let K/k be a finite field extension. Then A is a central simple k-algebra if and only if $A \otimes_k K$ is a central simple K-algebra.*

Proof. This is [GS17, Lemma 2.2.2]. □

Theorem 1.2.6 *Let k be a field and let A be a finite-dimensional k-algebra. Then A is a central simple algebra if and only if there exists a positive integer n and a finite field extension K/k such that $A \otimes_k K$ is isomorphic to $M_n(K)$. Moreover, if this is so, one can choose K separable over k.*

Proof. See [GS17, Thm. 2.2.1, Thm. 2.2.7]. See also [Alb31, Ch. IV, §7, Thm. 18] and [BouVIII, §10, no. 3, Prop. 4]. □

This theorem and properties 2 and 3 immediately imply that the tensor product $A \otimes_k B$ of two central simple algebras is again a central simple algebra. It also immediately implies that the dimension of a central simple algebra over its centre k is a square of a positive integer d. This integer d is called the *degree* of the algebra.

Two central simple algebras A and B are called *equivalent* if there are n and m such that $A \otimes_k M_n(k) \cong B \otimes_k M_m(k)$. The relation is transitive by property 3. The equivalence class of k consists of the matrix algebras of all sizes.

Theorem 1.2.7 (Brauer) *The tensor product equips the set of equivalence classes of central simple algebras over k with the structure of an abelian group. It is called the* **Brauer group** *of k and is denoted by $\mathrm{Br}(k)$.*

Proof. The neutral element is the class of k. Associativity follows from the associativity of the tensor product. Commutativity follows from the isomorphisms $A \otimes B \xrightarrow{\sim} B \otimes A$ given by $x \otimes y \mapsto y \otimes x$. The inverse element of the class of A is the equivalence class of the *opposite* algebra A^{op}. Indeed, $A \otimes_k A^{\mathrm{op}}$ is a central simple algebra, and there is a non-zero homomorphism $A \otimes_k A^{\mathrm{op}} \to \mathrm{End}_k(A)$ that sends $a \otimes b$ to $x \mapsto axb$. It is injective since a central simple algebra has no two-sided ideals, and hence is an isomorphism by the dimension count. □

We write the group operation in $\mathrm{Br}(k)$ additively.

Theorem 1.2.3 implies that any class $\alpha \in \mathrm{Br}(k)$ is represented by a central division algebra D which is well defined up to a non-unique isomorphism. In particular, the degree of the algebra D is well defined. It is called the *index* of (any algebra in) the class α. The *exponent* of α is the order of α in the group $\mathrm{Br}(k)$.

From Theorem 1.2.3 it follows that two central simple algebras of the same dimension and the same class in $\mathrm{Br}(k)$ are isomorphic. We deduce that cancellation holds: $A \otimes B \cong A \otimes C$ implies $B \cong C$.

By Corollary 1.2.4, the Brauer group of an algebraically closed field is zero. By Theorem 1.2.6 this also holds for a separably closed field. Since $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are the only finite dimensional division $\mathbb{R}$-algebras (and $\mathbb{C}$ is not central), we see from Theorem 1.2.3 that $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2$.

Given a field extension K/k there is a natural *restriction* map

$$\mathrm{res}_{K/k} : \mathrm{Br}(k) \longrightarrow \mathrm{Br}(K)$$

defined by $[A] \mapsto [A \otimes_k K]$. The kernel of $\mathrm{res}_{K/k}$ is denoted by $\mathrm{Br}(K/k)$ and is called the *relative Brauer group.*

In the following lemma we assume that the characteristic of k is not 2.

Lemma 1.2.8 *For any $a, b, b' \in k^*$ we have the following properties:*
(i) $Q_k(a,b) \otimes_k Q_k(a,b') \cong Q_k(a,bb') \otimes_k M_2(k)$.
(ii) $Q_k(a,b) \otimes_k Q_k(a,b) \cong M_4(k)$.

Proof [GS17, Lemma 1.5.2] The vector subspace of $Q_k(a,b)\otimes_k Q_k(a,b')$ spanned by $1\otimes 1$, $i\otimes 1$, $j\otimes j'$, $ij\otimes j'$ is $A_1 = Q_k(a,bb')$. Similarly, the span of $1\otimes 1$, $1\otimes j'$, $i\otimes i'j'$, $-b(i\otimes i')$ is $A_2 = Q_k(b', -a^2b')$. The conic associated to $Q_k(b', -a^2b')$ clearly has a k-point, so $A_2 \cong M_2(k)$. The canonical map

$$A_1 \otimes_k A_2 \longrightarrow Q_k(a,b) \otimes_k Q_k(a,b')$$

defined by the product in $Q_k(a,b) \otimes_k Q_k(a,b')$, is surjective. The kernel of a homomorphism is a two-sided ideal, hence it is zero so that this map is an isomorphism. This proves (i), and (ii) follows. □

Given $a, b \in k^*$ we write (a,b) for the class of $Q_k(a,b)$ in $\mathrm{Br}(k)$. By Lemma 1.2.8 (i) we have $(a,b) \in \mathrm{Br}(k)[2]$. We have already seen that $(au^2, bv^2) = (a,b)$ for any $u, v \in k^*$. Lemma 1.2.8 (ii) shows that associating to $a, b \in k^*$ the class $(a,b) \in \mathrm{Br}(k)[2]$ induces a bilinear map

$$k^*/k^{*2} \times k^*/k^{*2} \longrightarrow \mathrm{Br}(k)[2].$$

By Proposition 1.1.7 we have $(a,b) = 0$ if and only if the conic $z^2 - ax^2 - by^2 = 0$ has a rational point. In particular, we have $(a, -a) = 0$, and $(a,b) = 0$ if $a + b = 1$. Merkurjev proved that the 2-torsion subgroup of $\mathrm{Br}(k)$ is generated by classes (a,b) (see [GS17, §8]).

1.2.2 Cyclic algebras

Quaternion algebras are a special case of the following construction, cf. [GS17, §2.5]. Let K/k be a Galois extension of fields such that the Galois group $G = \mathrm{Gal}(K/k)$ is cyclic of order n. Let σ be a generator of G and let $\chi : G \tilde{\longrightarrow} \mathbb{Z}/n$ be the character sending σ to $1 \in \mathbb{Z}/n$. Let $b \in k^*$.

The *cyclic algebra* $D_k(\chi, b)$ is defined as the k-algebra generated by the field K and a symbol y with the relations $y^n = b$ and $\lambda y = y\sigma(\lambda)$ for any $\lambda \in K$. This is a central simple k-algebra of degree n, which contains K as a maximal subfield. Conversely, any central simple k-algebra of degree n which contains a maximal subfield K which is cyclic of degree n over k is isomorphic to $D_k(\chi, b)$ for some $b \in k^*$.

We write (χ, b) for the class of $D_k(\chi, b)$ in $\mathrm{Br}(k)$.

When $\mathrm{char}(k)$ does not divide n and k contains all n-th roots of 1, one can describe the cyclic algebra $D_k(\chi, b)$ without mentioning the Galois action. Let $\omega \in \mu_n$ be a primitive root. For $a, b \in k^*$ let $(a,b)_\omega$ be the k-algebra with generators x, y and relations $x^n = a$, $y^n = b$, $xy = \omega yx$. One checks that this is a central simple k-algebra. Permuting x and y we find

$$(a, b)_\omega \cong (b, a)_{\omega^{-1}}.$$

Assume that $K = k[t]/(t^n - a)$ is a field. Then K is a cyclic extension of k of degree n. Let $\sqrt[n]{a} \in K$ be the image of t in K. There is a unique element $\sigma \in G = \mathrm{Gal}(K/k)$ such that $\sigma(\sqrt[n]{a}) = \omega\sqrt[n]{a}$. If $\chi : G \to \mathbb{Z}/n$ is the character that sends σ to $1 \in \mathbb{Z}/n$, then the k-algebras $(a, b)_\omega$ and $D_k(\chi, b)$ are isomorphic [GS17, Cor. 2.5.5].

1.2.3 C_1-fields

The point of view of central simple algebras allows one to prove the triviality of the Brauer group of several types of fields which are fundamental for arithmetic and geometry.

Definition 1.2.9 (Lang) *A field k is called a C_1-field if any homogeneous form of degree d in $n > d$ variables with coefficients in k has a non-trivial zero in k.*

One easily checks that any finite field extension of a C_1-field is a C_1-field [GS17, Lemma 6.2.4].

Theorem 1.2.10 *If k is a C_1-field, then* $\mathrm{Br}(k) = 0$.

Proof. A central simple k-algebra A comes equipped with a *reduced norm*, which is a homomorphism $\mathrm{Nrd}_A : A^* \to k^*$. Let d be the degree of A. Choosing a basis of the vector space A over k one can write Nrd_A as a homogeneous form of degree d in d^2 variables with coefficients in k. (By Theorem 1.2.6, after extending the ground field from k to k_s the algebra $A \otimes_k k_s$ can be identified with the matrix algebra $M_d(k_s)$. Under this identification, the reduced norm becomes the determinant.) If $A = D$ is a skewfield such that $D \neq k$, then Nrd_D has no non-trivial zero. (For all this, see [GS17, §2.6, §6.2].) Thus if k is a C_1-field, then $D = k$, so that $\mathrm{Br}(k) = 0$. □

Theorem 1.2.11 *If k is a finite field, then k is a C_1-field and* $\mathrm{Br}(k) = 0$.

Proof. By Wedderburn's Little Theorem every finite ring with no zero divisors is a field. In particular, the only central division k-algebra is k itself. This gives $\mathrm{Br}(k) = 0$. The stronger statement that a finite field is a C_1-field is the Chevalley–Warning theorem [GS17, Thm. 6.2.6]). □

Theorem 1.2.12 (Tsen) *Let k be a field of transcendence degree 1 over an algebraically closed field. Then k is a C_1-field and* $\mathrm{Br}(k) = 0$.

Proof. This is proved in [GS17, Thm. 6.2.8]. The proof is an extension of the proof of Theorem 1.1.13. □

For fields of transcendence degree 1 over a separably closed field, see Proposition 3.8.2.

Recall that a local ring R with maximal ideal $\mathfrak{m}$ and residue field k is *henselian* if for every monic polynomial $f(x) \in R[x]$ every simple root in k of the reduction of $f(x)$ modulo $\mathfrak{m}$ lifts to a root of $f(x)$ in R. Define the *completion* of R at $\mathfrak{m}$ as $\widehat{R} = \varprojlim R/\mathfrak{m}^n$. A local ring R is $\mathfrak{m}$-*adically complete* if the canonical map $R \to \widehat{R}$ is an isomorphism. Using Newton's approximation one proves that any complete local ring is henselian. (See [Stacks, Section 04GE].) If $\mathfrak{m}$ is finitely generated, then the completion $\widehat{R}$ of R at $\mathfrak{m}$ is a complete local ring with maximal ideal $\mathfrak{m}\widehat{R}$ and residue field k, see [Stacks, Section 00M9].

Theorem 1.2.13 *Let R be a henselian discrete valuation ring with algebraically closed residue field k. Let K be the fraction field of R.*

(i) *If R is excellent, for example, if* $\mathrm{char}(K) = 0$ *or R is complete, then K is a C_1-field and* $\mathrm{Br}(K) = 0$.

(ii) *In general, we have* $\mathrm{Br}(K) = 0$.

Proof. (i) See Lang's thesis [Lan52], see also [Shatz, Thm. 27, p. 116]. The excellence property, which is needed to ensure that $\widehat{K}$ is a separable extension of K, is discussed in [BLR90, III, §6].

(ii) There are several other ways to establish $\mathrm{Br}(K) = 0$ under the assumption that R is complete [SerCL, Ch. XII, §1, §2]. As pointed out in [Mil80, Ch. III, Example 2.22 (a)], these proofs also give $\mathrm{Br}(K) = 0$ for R henselian with algebraically closed residue field. □

See Proposition 1.4.3 for the case when the residue field is separably closed but not algebraically closed.

Corollary 1.2.14 *Let R be a complete discrete valuation ring with* **perfect** *residue field k and field of fractions K of characteristic zero. Let K_{nr} be the maximal unramified extension of K. Then K_{nr} is a C_1-field.*

Proof. The field K_{nr} is the field of fractions of a henselian discrete valuation ring with algebraically closed residue field. Since $\mathrm{char}(K) = 0$, the result is a special case of Theorem 1.2.13. □

Remark 1.2.15 Let K a henselian discretely valued field and let $\widehat{K}$ be the completion of K. Then the natural map $\mathrm{Br}(K) \to \mathrm{Br}(\widehat{K})$ is an isomorphism. For a proof see Proposition 6.1.10.

1.3 The language of Galois cohomology

1.3.1 Group cohomology and Galois cohomology

We now assume that the reader is familiar with the cohomology theory of abstract groups, which can be found in many places in the literature, for example in [AW65], [SerCG], [SerCL], [GS17] and [Har17].

Let G be a group and let M be a G-module. The group $\mathrm{H}^0(G,M) := M^G$ is the set of G-invariant elements of M. Higher cohomology groups $\mathrm{H}^n(G,M)$, $n \geq 1$, are the right derived functors of the functor from the category of G-modules to the category of abelian groups that sends M to M^G. They can be computed using the standard projective resolution $P_\bullet \to \mathbb{Z}$ of the trivial G-module $\mathbb{Z}$, as the cohomology groups of the complex $\mathrm{Hom}_G(P_\bullet, M)$. This leads to the definition in terms of homogeneous cocycles, which can be restated as a definition in terms of inhomogeneous cocycles.

We refer to the books mentioned above for the following aspects of the cohomology of groups:

- relation with the cohomology of subgroups: restriction, inflation, and corestriction in the case of a subroup $H \subset G$ of finite index, Shapiro's lemma;
- long exact sequences coming from the Hochschild–Serre spectral sequence;
- cup-products and their properties with respect to boundary maps in exact cohomology sequences;
- cohomology of cyclic groups.

Let G be a group that acts on a not necessarily commutative group A preserving its group structure. We denote the result of applying $\sigma \in G$ to $a \in A$ by ${}^\sigma a$. A 1-*cocycle* is a function $a = \{a_\sigma\} : G \to A$ which satisfies the relation

$$a_{\sigma\tau} = {}^\sigma a_\tau \cdot a_\sigma$$

for all $\sigma, \tau \in G$. The function $G \to A$ whose image is the identity element of A is called the trivial cocycle. Let $Z^1(G,A)$ be the set of 1-cocycles. Two cocycles $\{a_\sigma\}$ and $\{b_\sigma\}$ are called *equivalent* if there exists $c \in A$ such that for any $\sigma \in G$ one has

$$a_\sigma = {}^\sigma c \cdot b_\sigma \cdot c^{-1}.$$

The 1-*cohomology set* $\mathrm{H}^1(G,A)$ is defined as the set of equivalence classes of $Z^1(G,A)$ with respect to this relation. The class of the trivial cocycle is the *distinguished* point of $\mathrm{H}^1(G,A)$, so we can talk about $\mathrm{H}^1(G,A)$ as a pointed set.

Now suppose that G is a profinite group and the action of G on A is *continuous* when A is given the discrete topology. One defines the continuous cohomology pointed set $\mathrm{H}^1(G,A)$ as the direct limit of the pointed sets $\mathrm{H}^1(G/U, A^U)$, where $U \subset G$ ranges over all open normal subgroups – any such subgroup being of finite index in G. Alternatively, one defines $\mathrm{H}^1(G,A)$ as the set of equivalence classes of *continuous* cocycles $G \to A$. Note that for an infinite group G the continuous cohomology set need not coincide with the abstract cohomology set. Unless otherwise mentioned, we shall only use continuous cohomology sets in this book.

Given a short exact sequence of continuous discrete G-groups

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1,$$

where A is normal in B, there is a long exact sequence of pointed sets

$$1\to A^G\to B^G\to C^G\to \mathrm{H}^1(G,A)\to \mathrm{H}^1(G,B)\to \mathrm{H}^1(G,C).$$

If A is central in B, it extends to an exact sequence of pointed sets

$$1\to A^G\to B^G\to C^G\to \mathrm{H}^1(G,A)\to \mathrm{H}^1(G,B)\to \mathrm{H}^1(G,C)\to \mathrm{H}^2(G,A).$$

An important particular case is when k is a field with separable closure k_{s} and absolute Galois group $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$ acting on the group of k_{s}-points of an algebraic group A over k. The pointed set $\mathrm{H}^1(\Gamma, A(k_{\mathrm{s}}))$ does not depend on the choice of k_s; it is well defined up to canonical isomorphism [SerCG, Ch. II, §1, 1.1] and is denoted by $\mathrm{H}^1(k,A)$. The map $K \mapsto \mathrm{H}^1(K, A\times_k K)$ defines a functor from the category of field extensions of k to the category of pointed sets.

We shall mostly deal with the case of the projective linear group, so let us recall its definition. The group $\mathrm{PGL}_n(k)$ is defined by the exact sequence of groups

$$1 \longrightarrow k^* \longrightarrow \mathrm{GL}_n(k) \longrightarrow \mathrm{PGL}_n(k) \longrightarrow 1,$$

where the second map is the embedding of the central subgroup of scalar matrices. The multiplicative group $\mathbb{G}_{m,k}$ represents the functor associating to a commutative k-algebra R the group of invertible elements R^*. The algebraic group $\mathrm{GL}_{n,k}$ represents the functor $\mathrm{GL}_n(R)$. (In particular, $\mathbb{G}_{m,k} = \mathrm{GL}_{1,k}$.) Finally, the algebraic group $\mathrm{PGL}_{n,k}$ is defined by the exact sequence of algebraic k-groups

$$1 \longrightarrow \mathbb{G}_{m,k} \longrightarrow \mathrm{GL}_{n,k} \longrightarrow \mathrm{PGL}_{n,k} \longrightarrow 1. \tag{1.1}$$

If M is a continuous discrete G-module, then the continuous cohomology group $\mathrm{H}^i(G,M)$ is defined for any $i\geq 0$ as the direct limit of $\mathrm{H}^i(G/U,M^U)$ over the set of open normal subgroups $U\subset G$.

If A is a commutative algebraic group over k and $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$, the abelian group $\mathrm{H}^i(\Gamma, A(k_{\mathrm{s}}))$ is well defined for any integer $i \geq 0$, up to canonical isomorphism [SerCG, Chap. II, §1, 1.1]; it is denoted by $\mathrm{H}^i(k,A)$. The map $K \mapsto \mathrm{H}^i(K, A\times_k K)$ defines a functor from the category of field extensions K of k to the category of abelian groups.

1.3.2 Galois descent

A general reference for Galois descent is [BLR90, Section 6.2, Example B], see also [SerCL, Ch. X], [PR91, Section 2.2], [Sko01, Section 2], [GS17, Ch. 2.3], [Ols16, Ch. 4] and [Po18, Ch. 4].

Let K/k be a finite Galois extension of fields with Galois group $\mathrm{Gal}(K/k)$. The descent problem deals with the following question: when can a scheme X' over K be descended to k, that is, is there a scheme X over k such that $X' \cong X\times_k K$? Grothendieck explored the analogy with the classical case, where a topological space or a differentiable manifold can be constructed by glueing together open subsets via transition functions which satisfy a compatibility condition on triple intersections. A 'descent datum' is an analogue of this for schemes. (Descent data can be defined more generally for any category fibred over a category with finite fibred products, see [Ols16, Section 4.2] or Section 2.6.2 below.) In [BLR90, pp. 140-141] it is shown that giving a 'descent datum' on a K-scheme X' with respect to K/k is equivalent to giving an action of $\mathrm{Gal}(K/k)$ on X' that is compatible with the action of $\mathrm{Gal}(K/k)$ on K by automorphisms. This descent problem is 'effective' (that is, there is a scheme X over k such that $X' \cong X \times_k K$) when X' is quasi-separated and the $\mathrm{Gal}(K/k)$-orbit of every point of X' is contained in a quasi-affine open subscheme of X'. In particular, Galois descent is effective for quasi-projective varieties over a field.

Let X be a variety over k. Let K/k be a Galois extension (not necessarily finite) with Galois group $\mathrm{Gal}(K/k)$. A k-variety Y is called a *(K/k)-form of X* if there is an isomorphism $Y \times_k K \cong X \times_k K$ of K-varieties. Using effectivity of Galois descent one shows that if X is a quasi-projective variety over k, then the (K/k)-forms of X are classified, up to isomorphism, by the elements of the Galois cohomology set $\mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{Aut}(X \times_k K))$ in such a way that the isomorphism class of X corresponds to the distinguished point. See [Po18, §4.4, §4.5] for a detailed proof of this classical result.

For example, the (k_{s}/k)-forms of a projective space are called *Severi–Brauer varieties*. It is not hard to see that Severi–Brauer varieties of dimension 1 are precisely the plane projective conics. By a theorem of Châtelet, a Severi–Brauer variety is isomorphic to $\mathbb{P}_k^{n-1}$ if and only if it has a k-point, see Section 6.1 for this and other results on Severi–Brauer varieties. Note that the automorphism functor of $\mathbb{P}_k^{n-1}$ is represented by the group k-scheme $\mathrm{PGL}_{n,k}$.

More generally, suppose that we have a quasi-projective variety X over k endowed with an action of a group k-scheme A. By definition, each cohomology class in $\mathrm{H}^1(k, A)$ contains a 1-cocycle $c : \Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k) \to A(k_{\mathrm{s}})$; it comes from a 1-cocycle $c : \mathrm{Gal}(K/k) \to A(K)$ for some finite Galois extension $k \subset K$. The cocycle c defines a twisted action of $\mathrm{Gal}(K/k)$ on $X \times_k K$ as the composition of the action on $X \times_k K$ via the second factor with the action of $c(g) \in A(K)$. The cocycle condition is equivalent to this being an action of $\mathrm{Gal}(K/k)$ on $X \times_k K$ compatible with the action of $\mathrm{Gal}(K/k)$ on K by automorphisms. By effectivity of Galois descent, there exists a quasi-projective variety X^c over k such that the K-varieties $X \times_k K$ and $X^c \times_k K$ are isomorphic; this isomorphism identifies the action of $\mathrm{Gal}(K/k)$ on $X^c \times_k K$ via the second factor with the twisted by c action of $\mathrm{Gal}(K/k)$ on $X \times_k K$. The variety X^c is called the *twist* of X by c. By construction, it is a (k_{s}/k)-form of X. Replacing c by an equivalent cocycle gives rise to a variety non-canonically isomorphic to X^c. Particular cases of this situation include (see [Sko01, pp. 12–13], [Po18, §4.5]):

(a) Twists of the vector space k^n by a 1-cocycle with coefficients in $A = \mathrm{GL}_{n,k}$ are isomorphic to k^n, cf. [Po18, §1.3].

(b) Twists of the matrix algebra $M_n(k)$ by a 1-cocycle with coefficients in $A = \mathrm{PGL}_{n,k}$ are central simple algebras of degree n. Moreover, by [SerCL, Ch. X, §5, Prop. 8], this gives a bijection between the isomorphism classes of central simple algebras of degree n and the pointed set $\mathrm{H}^1(k, \mathrm{PGL}_{n,k})$.

(c) Torsors of an algebraic k-group A are obtained by twisting A by a 1-cocycle with coefficients in A acting on itself on the left. In this case A represents the automorphism functor of A considered together with its right action on itself, i.e. of A as a right A-torsor. Using effectivity of Galois descent one shows that the isomorphism classes of right A-torsors over k bijectively correspond to the elements of $\mathrm{H}^1(k, A)$. (This is the easy case of [BLR90, §6.5, Thm. 1], see also [Sko01, p. 13].) For example, the affine conic $x^2 - ay^2 = c$ is a torsor for the norm 1 torus given by $x^2 - ay^2 = 1$. Also, a smooth projective curve of genus 1 is a torsor for its Jacobian.

(d) Suppose that an algebraic k-group A acts on an algebraic k-group G by automorphisms. Twisting G by a 1-cocycle $\Gamma \to A$ one obtains a (k_{s}/k)-form of G. For example, the group of invertible elements of a central simple k-algebra of degree n is the group of k-points of a twist of $\mathrm{GL}_{n,k}$ by a 1-cocycle with values in $A = \mathrm{PGL}_{n,k}$. For any commutative algebraic group one defines quadratic twists by taking $A = \{\pm 1\}$, where -1 sends x to x^{-1}. For example, the quadratic twists of $\mathbb{G}_{m,k}$ are the norm tori $x^2 - ay^2 = 1$, where $a \in k^*$. The quadratic twists of an elliptic curve $y^2 = x^3 + ax + b$ are the elliptic curves $cy^2 = x^3 + ax + b$, where $c \in k^*$.

Looking closer at the case of vector spaces one deduces the triviality of 1-cocycles with coefficients in $\mathrm{GL}_{n,k}$.

Theorem 1.3.1 (Speiser) *For any Galois extension of fields K/k with Galois group G we have $\mathrm{H}^1(G, \mathrm{GL}_n(K)) = \{1\}$.*

Proof. The automorphism functor of the n-dimensional vector space is represented by GL_n. The twist of k^n by a 1-cocycle $c : G \to \mathrm{GL}_n(K)$ is a vector space over k of dimension n, so it is isomorphic to k^n. This isomorphism, after tensoring with K, gives a linear transformation $\varphi \in \mathrm{GL}_n(K)$ such that $c(g) = {}^g\varphi \cdot \varphi^{-1}$. Thus c represents the trivial class. See also [GS17, Example 2.3.4] and [Po18, Prop. 1.3.15]. □

This theorem is often proved by a direct cocycle computation, see [SerCL, Ch. X, Prop. 3].

Theorem 1.3.2 (Hilbert's theorem 90) *For any Galois extension of fields K/k with Galois group G we have $\mathrm{H}^1(G, K^*) = 0$.*

This is a particular case of Speiser's theorem for $n = 1$. For later use let us record a corollary of this theorem. Given field extensions $k \subset K \subset L$ with L/k and K/k Galois, there is a short exact sequence

$$0 \longrightarrow \mathrm{H}^2(\mathrm{Gal}(K/k), K^*) \longrightarrow \mathrm{H}^2(\mathrm{Gal}(L/k), L^*) \longrightarrow \mathrm{H}^2(\mathrm{Gal}(L/K), L^*) \quad (1.2)$$

where the first arrow is inflation and the second arrow is restriction.

Applying Hilbert's theorem 90 to (1.1) we see that for any field extension K/k the group of K-points of $\mathrm{PGL}_{n,k}$ is precisely $\mathrm{PGL}_n(K)$. Proposition 1.2.2 shows that the natural map

$$\mathrm{PGL}_n(K) \longrightarrow \mathrm{Aut}_{K-\mathrm{alg}}(M_n(K))$$

is an isomorphism of groups, where K-alg stands for the category of K-algebras. When K is a Galois extension of k, this isomorphism respects the Galois action on both sides. This shows that the automorphism functor of the matrix algebra $M_n(k)$ (which is a functor from the category of field extensions of k to the category of groups) is represented by the algebraic group $\mathrm{PGL}_{n,k}$.

Theorem 1.3.3 (Skolem–Noether) *All automorphisms of a central simple algebra over a field are inner automorphisms.*

Proof. Let A be a central simple algebra over a field k. Pick a finite Galois extension K/k that splits A. The homomorphism $A^*\to\mathrm{Aut}_{k-\mathrm{alg}}(A)$ sending an element to the conjugation by this element extends to a similar map over K. Let $G = \mathrm{Gal}(K/k)$. We then have the exact sequence of G-modules

$$1 \longrightarrow K^* \longrightarrow (A \otimes_k K)^* \longrightarrow \mathrm{Aut}_{K-\mathrm{alg}}(A \otimes_k K) \longrightarrow 1,$$

where surjectivity of the third map follows from Proposition 1.2.2. The long exact cohomology sequence gives an exact sequence of pointed sets

$$1 \longrightarrow k^* \longrightarrow A^* \longrightarrow \mathrm{Aut}_{k-\mathrm{alg}}(A) \longrightarrow \mathrm{H}^1(G, K^*).$$

Since $\mathrm{H}^1(G, K^*) = 0$ by Hilbert's theorem 90, the homomorphism $A^*\to\mathrm{Aut}_{k-\mathrm{alg}}(A)$ is surjective. □

There is actually a more general result.

Theorem 1.3.4 (Skolem–Noether) *Let k be a field, let B be a simple k-algebra and let A be a central simple algebra over k. Then all non-zero k-homomorphisms $B\to A$ are injective and can be obtained from one another by conjugations in A.*

Proof. See [Rei03, Thm. 7.21]. □

1.3.3 Cohomological description of the Brauer group

Let K/k be a finite Galois extension of fields with Galois group G. Recall that a central simple algebra of degree n over k is split by K, i.e., is a (K/k)-form of $M_n(k)$, if and only if there exists an isomorphism of K-algebras $A \otimes_k K \cong M_n(K)$. Let us denote by $\mathrm{Az}_{n,K}$ the set of isomorphism classes of central simple algebras of degree n over k which are split by K. As discussed in the previous section, we have a bijection of pointed sets

$$\mathrm{Az}_{n,K} \xrightarrow{\sim} \mathrm{H}^1(G, \mathrm{PGL}_n(K)).$$

Since $\mathrm{H}^1(G, \mathrm{GL}_n(K)) = \{1\}$ by Theorem 1.3.1, the exact sequence of pointed cohomology sets attached to (1.1)

$$\mathrm{H}^1(G, \mathrm{GL}_n(K)) \longrightarrow \mathrm{H}^1(G, \mathrm{PGL}_n(K)) \longrightarrow \mathrm{H}^2(G, K^*),$$

gives rise to maps

$$\mathrm{Az}_{n,K} \longrightarrow \mathrm{H}^2(G, K^*)$$

with trivial kernel. One easily checks that for given n and r there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & k^* & \to & \mathrm{GL}_n(k) & \to & \mathrm{PGL}_n(k) & \to & 1 \\ & & || & & \downarrow & & \downarrow & & \\ 1 & \to & k^* & \to & \mathrm{GL}_{nr}(k) & \to & \mathrm{PGL}_{nr}(k) & \to & 1 \end{array}$$

where the middle vertical map sends a matrix M to the matrix with r diagonal blocks equal to M and zero elsewhere. Replacing k by K and taking Galois cohomology we obtain commutative diagrams

$$\begin{array}{ccc} \mathrm{H}^1(G, \mathrm{PGL}_n(K)) & \to & \mathrm{H}^2(G, K^*) \\ \downarrow & & || \\ \mathrm{H}^1(G, \mathrm{PGL}_{nr}(K)) & \to & \mathrm{H}^2(G, K^*) \end{array}$$

The left vertical map can be identified with the map $\mathrm{Az}_{n,K} \to \mathrm{Az}_{nr,K}$ sending A to $A \otimes_k M_r(k)$. Passing to the limit over n we obtain a map of pointed sets

$$\mathrm{Br}(K/k) \longrightarrow \mathrm{H}^2(G, K^*)$$

with trivial kernel. Using Theorem 1.2.6 and passing to the limit over finite Galois extensions K/k, we get a map of pointed sets

$$\mathrm{Br}(k) \longrightarrow \mathrm{H}^2(k, k_{\mathrm{s}}^*)$$

with trivial kernel. One then establishes the following properties.

- These maps are homomorphisms of groups, hence they are injective. See [GS17, Prop. 2.7.9].
- These maps are surjective. This is proved by a cocycle computation using the classical construction of crossed products, see [SerCL, Ch. X, §5, Prop. 9]. An elegant cocycle-free proof is given in [GS17, Thm. 4.4.1].

We summarise this as the following theorem.

Theorem 1.3.5 *For a field k and a Galois extension of fields K/k there are natural isomorphisms of abelian groups*

$$\mathrm{Br}(K/k) \xrightarrow{\sim} \mathrm{H}^2(\mathrm{Gal}(K/k), K^*)$$

and

$$\mathrm{Br}(k) \xrightarrow{\sim} \mathrm{H}^2(k, k_{\mathrm{s}}^*).$$

The second isomorphism is functorial with respect to arbitrary field extensions of k, see [SerCL, Ch. 10, §4].

The cohomological description of the Brauer group is very useful. For example, it immediately gives

Corollary 1.3.6 *For any field k the Brauer group $\mathrm{Br}(k)$ is a torsion group.*

Proof. The group $\mathrm{Br}(k)$ is the direct limit of $\mathrm{Br}(K/k) = \mathrm{H}^2(\mathrm{Gal}(K/k), K^*)$, where K/k is a finite Galois extension. But if G is finite, then $\mathrm{H}^i(G, M)$, where M is any G-module and $i \geq 1$, is annihilated by the order of G. (This follows from the fact that the composition of the restriction to a subgroup $H \subset G$ followed by the corestriction is the multiplication by the index $[G : H]$. One applies this to the case when H is the identity element of G.) □

Theorem 1.3.7 *Let k be a perfect field,* $\mathrm{char}(k) = p > 0$. *Then* $\mathrm{Br}(k)\{p\} = 0$.

Proof. Let k_{s} be a separable closure of k. The map $x \mapsto x^p$ is an isomorphism of the Galois module k_{s}^*. Thus multiplication by p is an automorphism of the group $\mathrm{Br}(k) = \mathrm{H}^2(k, k_{\mathrm{s}}^*)$. Since this is a torsion group, we are done. □.

Let $k \subset K$ be an arbitrary field extension. The map

$$\mathrm{res}_{K/k} : \mathrm{Br}(k) \to \mathrm{Br}(K)$$

defined by associating to a central simple k-algebra A the central simple K-algebra $A \otimes_k K$ coincides with the cohomological restriction map

$$\mathrm{H}^2(k, k_{\mathrm{s}}^*) \to \mathrm{H}^2(K, K_{\mathrm{s}}^*).$$

Let us spell out the formalism of corestriction in the special case of the Brauer group and finite separable extensions of fields. For a more general context, which includes not necessarily separable field extensions, see Section 3.8. Let $K \subset k_{\mathrm{s}}$ be a separable finite field extension of k. We have an isomorphism

$$\mathrm{Br}(K) = \mathrm{H}^2(K, k_{\mathrm{s}}^*) = \mathrm{H}^2(k, (k_{\mathrm{s}} \otimes_k K)^*)$$

obtained using Shapiro's lemma and the fact that $(k_{\mathrm{s}} \otimes_k K)^*$ is the direct product of finitely many copies of k_{s}^* indexed by the embeddings of $K \hookrightarrow k_{\mathrm{s}}$, so the $\mathrm{Gal}(k_{\mathrm{s}}/k)$-module $(k_{\mathrm{s}} \otimes_k K)^*$ is induced from the $\mathrm{Gal}(k_{\mathrm{s}}/K)$-module k_{s}^*. The norm $\mathrm{N}_{K/k} : K \to k$ gives rise to a map of Galois modules $(k_{\mathrm{s}} \otimes_k K)^* \to k_{\mathrm{s}}^*$, hence to a homomorphism $\mathrm{H}^2(k, (k_{\mathrm{s}} \otimes_k K)^*) \to \mathrm{H}^2(k, k_{\mathrm{s}}^*)$. This defines a *corestriction* map

$$\mathrm{cores}_{K/k} : \mathrm{Br}(K) = \mathrm{H}^2(K, k_{\mathrm{s}}^*) = \mathrm{H}^2(k, (k_{\mathrm{s}} \otimes_k K)^*) \longrightarrow \mathrm{H}^2(k, k_{\mathrm{s}}^*) = \mathrm{Br}(k).$$

Since $\mathrm{N}_{K/k}(x) = x^n$ for $x \in k$, where $n = [K : k]$, the composition

$$\mathrm{cores}_{K/k} \circ \mathrm{res}_{K/k} : \mathrm{Br}(k) \longrightarrow \mathrm{Br}(K) \longrightarrow \mathrm{Br}(k)$$

is the multiplication by the degree $[K : k]$.

1.3.4 Kummer sequence, cyclic algebras and cup-products

Let k be a field with separable closure $k_{\rm s}$ and Galois group $\Gamma = {\rm Gal}(k_{\rm s}/k)$.

Assume that n is invertible in k. Then the map $x \mapsto x^n$ on $k_{\rm s}^*$ induces an exact sequence of Galois modules

$$1 \longrightarrow \mu_n \longrightarrow k_{\rm s}^* \longrightarrow k_{\rm s}^* \longrightarrow 1, \tag{1.3}$$

called the *Kummer sequence*. Taking Galois cohomology, and using Hilbert's theorem 90, we obtain isomorphisms

$$k^*/k^{*n} \tilde{\longrightarrow} {\rm H}^1(k, \mu_n) \quad \text{and} \quad {\rm H}^2(k, \mu_n) \tilde{\longrightarrow} {\rm Br}(k)[n].$$

The first of these isomorphisms associates to an element $a \in k^*$ the class of the 1-cocycle $g \mapsto g(b)b^{-1} \in \mu_n(k_{\rm s})$, where $b \in k_{\rm s}^*$ is such that $b^n = a$, and $g \in \Gamma$. This is precisely the image of $a \in k^*$ under the connecting map $\delta : k^* \to {\rm H}^1(k, \mu_n)$ in the long exact sequence of Galois cohomology attached to (1.3).

Let G be a cyclic group of order n. Fix a generator σ of G. Let $\chi \in {\rm Hom}(G, \mathbb{Z}/n) = {\rm H}^1(G, \mathbb{Z}/n)$ be the homomorphism sending σ to $1 \in \mathbb{Z}/n$. The exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

induced by multiplication by n on $\mathbb{Z}$ gives rise to an isomorphism

$$\partial : {\rm H}^1(G, \mathbb{Z}/n) \tilde{\longrightarrow} {\rm H}^2(G, \mathbb{Z})[n],$$

and so defines the class $\partial(\chi) \in {\rm H}^2(G, \mathbb{Z})[n]$. For any G-module A the cup-product with $\partial(\chi) \in {\rm H}^2(G, \mathbb{Z})$ defines an isomorphism

$${\rm H}^i(G, A) \tilde{\longrightarrow} {\rm H}^{i+2}(G, A)$$

for $i \geq 1$. For $i = 0$ it induces an isomorphism

$$A^G/N_G A = \hat{\rm H}^0(G, A) \tilde{\longrightarrow} {\rm H}^2(G, A),$$

where $N_G \in \mathbb{Z}[G]$ is the formal sum of all elements of G. The first equality here is the definition of Tate's cohomology group $\hat{\rm H}^0(G, A)$.

For a Galois field extension K/k with cyclic Galois group ${\rm Gal}(K/k) = \mathbb{Z}/n$ with generator σ, the previous considerations give an isomorphism

$$k^*/{\rm N}_{K/k}(K^*) \tilde{\longrightarrow} {\rm H}^2(G, K^*) = {\rm Ker}[{\rm Br}(k) \to {\rm Br}(K)]. \tag{1.4}$$

It is defined by the cup-product with $\partial(\chi) \in {\rm H}^2(G, \mathbb{Z})$, so it depends on the choice of a generator $\sigma \in G$.

Recall that for $a \in k^*$ we denote by $(\chi, a) \in {\rm Br}(k)$ the class of the cyclic algebra $D_k(\chi, a)$, see Section 1.2.2. It is known [GS17, Prop. 4.7.3, Cor. 4.7.4] that

$$(\chi, a) = a \cup \partial(\chi) = \partial(\chi) \cup a = \chi \cup \delta(a) \in {\rm Br}(k). \tag{1.5}$$

Here $\delta(a) \in k^*/k^{*n} = {\rm H}^1(k, \mu_n)$ is the image of a under the connecting map define by the Kummer sequence, and $\chi \cup \delta(a) \in {\rm H}^2(k, \mu_n) \subset {\rm Br}(k)$ is given by the cup-product $\mathrm{H}^1(k,\mathbb{Z}/n)\times \mathrm{H}^1(k,\mu_n)\to\mathrm{H}^2(k,\mu_n)$. From the isomorphism (1.4) we deduce that $(\chi,a)=0$ in $\mathrm{Br}(k)$ if and only if $a\in k^*$ is a norm for the extension K/k.

Now assume $\mu_n(k_{\mathrm{s}})\subset k$, so that μ_n is isomorphic to $\mathbb{Z}/n$ as a Γ-module. Since $\mathrm{H}^1(k,\mu_n)=k^*/k^{*n}$ we see that every cyclic field extension of k of degree n is of the form $k(\sqrt[n]{a})$ for some $a\in k^*$. The cup-product pairing

$$\cup : k^*/k^{*n}\times k^*/k^{*n}=\mathrm{H}^1(k,\mu_n)\times \mathrm{H}^1(k,\mu_n)\longrightarrow \mathrm{H}^2(k,\mu_n^{\otimes 2}).$$

is anticommutative, that is, $a\cup b=-b\cup a$. Choose an isomorphism $\mu_n\tilde{\longrightarrow}\mathbb{Z}/n$, which is equivalent to choosing a primitive root of unity $\omega\in k$ (sent to $1\in\mathbb{Z}/n$). This induces an isomorphism $\mu_n^{\otimes 2}\tilde{\longrightarrow}\mu_n$, hence an isomorphism

$$\mathrm{H}^2(k,\mu_n^{\otimes 2})=\mathrm{H}^2(k,\mu_n)\otimes\mu_n\tilde{\longrightarrow}\mathrm{H}^2(k,\mu_n)=\mathrm{Br}(k)[n].$$

The inverse map sends a class $\alpha\in\mathrm{H}^2(k,\mu_n)$ to $\alpha\otimes\omega$. For $a,b\in k^*$ we denote the image of (a,b) under the composite map

$$k^*\times k^*\longrightarrow k^*/k^{*n}\times k^*/k^{*n}\longrightarrow \mathrm{H}^2(k,\mu_n^{\otimes 2})\longrightarrow \mathrm{H}^2(k,\mu_n)=\mathrm{Br}(k)[n]$$

by $(a,b)_\omega$. Under the isomorphism $\mathrm{H}^2(k,\mu_n^{\otimes 2})=\mathrm{H}^2(k,\mu_n)\otimes\mu_n$ the class $a\cup b$ corresponds to $(a,b)_\omega\otimes\omega$.

The class of $(a,b)_\omega$ is the class of the algebra defined in Section 1.2.2, see [GS17, Prop. 4.7.1]. The equality $(a,b)_\omega=-(b,a)_\omega$ follows from the equality $a\cup b=-b\cup a$.

For any integer $n>1$, by treating separately odd and even integers one checks that both $-a$ and $1-a$ are norms for the extension $k[t]/(t^n-a)$ of k. Thus $a\cup(-a)=0$ and $a\cup(1-a)=0$.

When $n=2$ is invertible in k we recover the case of quaternion algebras. The bilinearity of the cup-product then gives various properties that we proved in a more explicit way in Section 1.1.

1.4 Galois cohomology of discretely valued fields

Let R be a discrete valuation ring with field of fractions K and residue field k. Let ℓ be a prime number invertible in R. The literature contains various constructions of residue maps

$$\partial_R : \mathrm{Br}(K)\{\ell\}\longrightarrow \mathrm{H}^1(k,\mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

When k is *perfect* of characteristic $p>0$, there are constructions of a residue map

$$\partial_R : \mathrm{Br}(K)\longrightarrow \mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z})$$

which also take care of the p-primary subgroup of $\mathrm{Br}(K)$.

One approach that we do not pursue here is via the Merkurjev–Suslin theorem, which gives an isomorphism $\mathrm{K}_2(F)/n\cong\mathrm{H}^2(F,\mu_n^{\otimes 2})$ valid for any field

F and any integer n invertible in F (see, e.g., [GS17, Ch. 8]). When, moreover, $\mu_n \subset F$, we obtain an isomorphism $\mathrm{K}_2(F)/n \tilde{\longrightarrow} \mathrm{Br}(F)[n]$, which depends on the choice of a primitive n-th root of unity in F. Thus if $\mu_n \subset K$ and $(\mathrm{char}(K), n) = 1$ we can combine the Merkurjev–Suslin isomorphism with the tame symbol $\mathrm{K}_2(K)/n \to k^*/k^{*n}$ to obtain a composite map

$$\mathrm{Br}(K)[n] \cong \mathrm{K}_2(K)/n \xrightarrow{\text{tame}} k^*/k^{*n}$$

without assuming that k is perfect or has characteristic coprime to n.

The classical case is that of local fields, i.e. *complete* discretely valued fields K with *finite* (hence perfect) residue field k. Then K is either a finite extension of the p-adic field $\mathbb{Q}_p$ or the field of formal power series in one variable over a finite field. In these cases the local class field theory gives an isomorphism $\mathrm{Br}(K) \tilde{\longrightarrow} \mathbb{Q}/\mathbb{Z}$, often called the *local invariant*. Its construction goes back to the 1930s and is due to Hasse and Witt [Wit37], and so predates Galois cohomology. This approach uses Brauer classes of central simple algebras over local fields and maximal orders in such algebras; the key fact is that a central division ring over K contains a maximal subfield which is unramified over K, see [SerCL, Ch. XII, §2] and [Rei03, Ch. 8]. We do not discuss this here but concentrate instead on the cohomological constructions with finite and infinite coefficients.

1.4.1 Residue with finite coefficients

For this construction we assume that n is coprime to $\mathrm{char}(k)$. The goal is to define a residue map

$$\mathrm{H}^2(K, \mu_n) \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/n),$$

which can then be composed with the inverse of the isomorphism

$$\mathrm{H}^2(K, \mu_n) \tilde{\longrightarrow} \mathrm{Br}(K)[n]$$

provided by the Kummer sequence (1.3). Our exposition in this section is based on Chapters II and III of [Ser03] and Chapters 6 and 7 of [GS17].

Theorem 1.4.1 *Let G be a profinite group and let N be a closed normal subgroup of G. Let C be a discrete G-module.*

(i) *Suppose that $\mathrm{H}^n(N, C) = 0$ for $n > 1$. Then there is a long exact sequence*

$$\ldots \to \mathrm{H}^i(\Gamma, C^N) \to \mathrm{H}^i(G, C) \to \mathrm{H}^{i-1}(\Gamma, \mathrm{H}^1(N, C)) \to \mathrm{H}^{i+1}(\Gamma, C^N) \to \ldots \quad (1.6)$$

(ii) *Define $\Gamma = G/N$. In addition to the assumptions of (i) assume that N acts trivially on C, so that C can be considered as a Γ-module. If, moreover, the exact sequence*

$$1 \longrightarrow N \longrightarrow G \longrightarrow \Gamma \longrightarrow 1 \quad (1.7)$$

is split, then for each $i \geq 1$ there is a split exact sequence

$$0 \longrightarrow \mathrm{H}^i(\Gamma, C) \longrightarrow \mathrm{H}^i(G, C) \longrightarrow \mathrm{H}^{i-1}(\Gamma, \mathrm{Hom}(N, C)) \longrightarrow 0. \quad (1.8)$$

Here $\mathrm{Hom}(N, C)$ denotes the group of continuous homomorphisms $N \to C$, i.e. homomorphisms with finite image and open kernel.

Proof. (i) There is the Hochschild–Serre spectral sequence

$$\mathrm{H}^p(\Gamma, \mathrm{H}^q(N, C)) \Rightarrow \mathrm{H}^{p+q}(G, C).$$

The assumption of (i) implies that this spectral sequence gives rises to the exact sequence (1.6).

(ii) We have $C^N = C$. Let $\sigma : \Gamma \to G$ be a homomorphism such that the composition $\Gamma \to G \to \Gamma$ is the identity map. The composition of the inflation map $\mathrm{H}^i(\Gamma, C) \to \mathrm{H}^i(G, C)$ from (1.6) with the restriction $\sigma^* : \mathrm{H}^i(G, C) \to \mathrm{H}^i(\Gamma, C)$ is the identity. This implies the injectivity of $\mathrm{H}^i(\Gamma, C) \to \mathrm{H}^i(G, C)$ for $i \geq 0$. Thus we obtain the exact sequences (1.8). The same argument gives that these sequences are split. □

Let R be a *henselian* discrete valuation ring with field of fractions K and residue field k. We have a chain of field extensions

$$K \subset K_{\mathrm{nr}} \subset K_{\mathrm{t}} \subset K_{\mathrm{s}},$$

where K_{s} is a separable closure of K, $K_{\mathrm{nr}} \subset K_{\mathrm{s}}$ is the maximal unramified extension of K, and $K_{\mathrm{t}} \subset K_{\mathrm{s}}$ is the maximal tamely ramified extension of K. Let $G = \mathrm{Gal}(K_{\mathrm{s}}/K)$ and $\Gamma = \mathrm{Gal}(K_{\mathrm{nr}}/K) = \mathrm{Gal}(k_{\mathrm{s}}/k)$. Let $I = \mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{nr}})$ be the inertia group and let $N = \mathrm{Gal}(K_{\mathrm{t}}/K_{\mathrm{nr}})$ be the tame inertia group. By Hensel's lemma, the field K_{nr} contains all n-th roots of 1, for n prime to the characteristic of k.

The field K_{t} is obtained from K_{nr} by adjoining the n-th roots of a fixed uniformiser $\pi \in K_{\mathrm{nr}}$, for all n coprime to $\mathrm{char}(k)$. Indeed, let L be a finite tame extension of K_{nr} and let $e = [L : K_{\mathrm{nr}}]$ be its degree, which is prime to $\mathrm{char}(k)$. Let $\pi_1 \in L$ be a uniformiser. We have $\pi = u\pi_1^e$, where u is a unit in L. By Hensel's lemma, any unit in L is an e-th power. Thus we can choose π_1 such that $\pi = \pi_1^e$. By Eisenstein's criterion, $L = K_{\mathrm{nr}}(\pi^{1/e})$.

Hence $N = \varprojlim \mu_n$, where $(\mathrm{char}(k), n) = 1$. In other words, the profinite group N is isomorphic to $\hat{\mathbb{Z}}$ if $\mathrm{char}(k) = 0$, and is isomorphic to the quotient of $\hat{\mathbb{Z}}$ by its maximal pro-p-subgroup if $\mathrm{char}(k) = p > 0$. It follows that $\mathrm{cd}(N) \leq 1$, that is, for any discrete torsion Galois module C we have $\mathrm{H}^i(N, C) = 0$ for any $i \geq 2$. The wild inertia subgroup $\mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{t}})$ is trivial if $\mathrm{char}(k) = 0$, otherwise it is a pro-p-group. Thus for any continuous discrete torsion G-module C annihilated by an integer coprime to $\mathrm{char}(k)$, one has $\mathrm{H}^i(\mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{t}}), C) = 0$ for $i > 0$. The Hochschild–Serre spectral sequence then gives that $\mathrm{H}^i(I, C) = 0$ for all $i \geq 2$.

For each $n > 1$ coprime to p choose an n-th root π_n of π in K_{s} in such a way that $(\pi_{mn})^m = \pi_n$ for all m, n. Let K' be the extension of K generated by all the roots π_n. It is clear that K_{nr} and K' are linearly disjoint over K, and $K_{\mathrm{t}} = K_{\mathrm{nr}}K'$. This implies that the exact sequence

$$0 \longrightarrow N \longrightarrow G/\mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{t}}) \longrightarrow \Gamma \longrightarrow 1$$

is split. Since the p-cohomological dimension of Γ is at most 1 [SerCG, Ch. 2, §2, Prop. 3], every homomorphism $\Gamma \to G/\mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{t}})$ lifts to a homomorphism $\Gamma \to G$, see [SerCG, Ch. 1, §3, Prop. 16]. Hence the following exact sequence is also split:

$$1 \longrightarrow I \longrightarrow G \longrightarrow \Gamma \longrightarrow 1.$$

Now Theorem 1.4.1 gives rise to split exact sequences for all $i \geq 1$

$$0 \longrightarrow \mathrm{H}^i(k, C) \longrightarrow \mathrm{H}^i(K, C) \xrightarrow{r} \mathrm{H}^{i-1}(k, C(-1)) \longrightarrow 0. \tag{1.9}$$

Definition 1.4.2 *Let C be a Γ-module of exponent coprime to* $\mathrm{char}(k)$*, where* $\Gamma = \mathrm{Gal}(K_{\mathrm{nr}}/K) = \mathrm{Gal}(k_{\mathrm{s}}/k)$*. For $i \geq 1$ the map*

$$r : \mathrm{H}^i(K, C) \longrightarrow \mathrm{H}^{i-1}(k, C(-1))$$

is called the **residue map***. An element $x \in \mathrm{H}^i(K, C)$ is called* **unramified** *if* $r(x) = 0$.

We have a cup-product pairing of Galois cohomology groups of K

$$\cup : \mathrm{H}^1(K, \mu_n) \times \mathrm{H}^{i-1}(K, C(-1)) \longrightarrow \mathrm{H}^i(K, C). \tag{1.10}$$

The exact sequence (1.9) allows one to identify $\mathrm{H}^{j-1}(k, C(-1))$ with a subgroup of $\mathrm{H}^{j-1}(K, C(-1))$. This gives rise to the pairing

$$\cup : \mathrm{H}^1(K, \mu_n) \times \mathrm{H}^{i-1}(k, C(-1)) \longrightarrow \mathrm{H}^i(K, C). \tag{1.11}$$

The pairing (1.10) is functorial in K, so (1.11) is too (see [Ser03, Prop. 8.2]).

Examples Let $C = \mu_n$, where $(n, \mathrm{char}(k)) = 1$.

(1) For $i = 1$ and one obtains a split exact sequence

$$0 \longrightarrow k^*/k^{*n} \longrightarrow K^*/K^{*n} \xrightarrow{r} \mathbb{Z}/n \longrightarrow 0. \tag{1.12}$$

The residue map in this sequence is induced by the valuation $v : K^* \to \mathbb{Z}$. Indeed, the following diagram commutes:

$$\begin{array}{ccc} K^*/K^{*n} & \xrightarrow{r} & \mathbb{Z}/n \\ \uparrow & & \uparrow \\ K^* & \xrightarrow{v} & \mathbb{Z} \end{array}$$

To see this we check that $(\pi)_n \in K^*/K^{*n}$ is sent by r to

$$1 \in \mathbb{Z}/n = \mathrm{Hom}(\mu_n, \mu_n) = \mathrm{Hom}(N, \mu_n) = \mathrm{Hom}(I, \mu_n).$$

By definition, r sends the class of the K-torsor with the equation $x^n = \pi$ to the class of the same torsor over K_{nr}. The smallest extension of K_{nr} over which the points of this torsor are defined, is $K_{\mathrm{nr}}(\sqrt[n]{\pi})$. Thus the inertia group I acts on the points on this torsor through its tame quotient N, more precisely, through $\mathrm{Gal}(K_{\mathrm{nr}}(\sqrt[n]{\pi})/K_{\mathrm{nr}}) = \mu_n$. This is exactly the isomorphism used in the above description of N as the inverse limit of μ_n, for n coprime to char(k), so the action on the points of our torsor corresponds to $1 \in \mathbb{Z}/n$.

(2) For $i = 2$ one obtains a split exact sequence

$$0 \longrightarrow \mathrm{H}^2(k, \mu_n) \longrightarrow \mathrm{H}^2(K, \mu_n) \xrightarrow{r} \mathrm{H}^1(k, \mathbb{Z}/n) \longrightarrow 0, \tag{1.13}$$

which, in view of the Kummer exact sequence (1.3), can be rewritten as follows:

$$0 \longrightarrow \mathrm{Br}(k)[n] \longrightarrow \mathrm{Br}(K)[n] \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/n) \longrightarrow 0. \tag{1.14}$$

Proposition 1.4.3 *Let R be a henselian discrete valuation ring with fraction field K and residue field k. Let Γ be the absolute Galois group of K. Let p be the characteristic exponent of k. If R is strictly henselian, i.e., if k is separably closed, then we have the following statements.*

(i) *For any prime $\ell \neq p$, any ℓ-primary torsion Γ-module C and any integer $i \geq 2$, we have $\mathrm{H}^i(K, C) = 0$. In other words, $\mathrm{cd}_\ell(K) \leq 1$.*

(ii) *For any $i \geq 1$ the group $\mathrm{H}^i(K, \mathbb{G}_m)$ is a p-primary torsion group (so the group is trivial when $p = 1$).*

(iii) *The Brauer group $\mathrm{Br}(K)$ is a p-primary torsion group.*

(iv) *If k is algebraically closed, then $\mathrm{cd}(K) \leq 1$ and $\mathrm{H}^i(K, \mathbb{G}_m) = 0$ for all $i \geq 1$.*

Proof. Part (i) is an immediate consequence of the exact sequence (1.9). Statement (ii) then follows from the Kummer sequence (1.3) and statement (iii) is just the special case $i = 2$.

We owe the following proof of (iv) to L. Moret-Bailly. Quite generally, if R is a discrete valuation ring with field of fractions K and L/K is an arbitrary finite field extension, the integral closure $S \subset L$ (which need not be finite over R if R is not excellent) is a semilocal Dedekind domain, and for each maximal ideal q of S, the quotient S/q if finite over $R/(q \cap R)$. This is a special case of the Krull–Akizuki Theorem [BouAC, Ch. 7, §2, no. 5]. If, moreover, R is henselian, then since S is integral over R and has no zero-divisors, a limit argument shows that it is a henselian local ring [Ray70b, Chap. I, §2, Prop. 2 p. 7]. In the case considered in (iv), the residue fields of R and hence of S are algebraically closed. By Theorem 1.2.13 we thus have $\mathrm{Br}(L) = 0$ for any finite field extension L/K. By [SerCG, Chap. II, §3.1, Prop. 5], this implies $\mathrm{cd}(K) \leq 1$, which in turn implies $\mathrm{H}^i(K, \mathbb{G}_m) = 0$ for all $i \geq 1$. □

Remark 1.4.4 Let R be a complete discrete valuation ring with fraction field K of characteristic 0 and residue field k of characteristic $p > 0$. Assume that K contains the p-th roots of 1. When k is not perfect, Kato [Kat86] has constructed a filtration on $\mathrm{Br}(K)[p]$ whose smallest term is $\mathrm{H}^1(k, \mathbb{Z}/p) \subset \mathrm{Br}(K)[p]$ but whose successive quotients involve the groups of absolute differentials $\Omega^i_{k/\mathbb{Z}}$ of k, i.e. the groups of differentials Ω^i_{k/k^p}. See also [CT99a] and [GO08].

Proposition 1.4.5 *Let R be a henselian discrete valuation ring with field of fractions K and residue field k. Let C be a Γ-module of exponent n invertible* *in R. Let π be a uniformiser of R and let $(\pi)_n$ be the image of π under the map $K^*\to \mathrm{H}^1(K,\mu_n)$ given by the Kummer sequence (1.3). Any $\alpha\in \mathrm{H}^i(K,C)$ is uniquely written as*

$$\alpha=\alpha_0+(\pi)_n\cup\alpha_1,$$

where $\alpha_0\in \mathrm{H}^i(k,C)$ and $\alpha_1\in \mathrm{H}^{i-1}(k,C(-1))$. Moreover, $\alpha_1=r(\alpha)$.

Proof. See [Ser03, Ch. II, Prop. 7.11, p. 18]. □

Using this, one proves the following general formula [Ser03, II.6.5, Exercise 7.12]. Let A,B,C be n-torsion Γ-modules such that there is a Γ-equivariant pairing $A\times B\to C$. It induces the pairing

$$\cup:\mathrm{H}^p(K,A)\times \mathrm{H}^q(K,B)\longrightarrow \mathrm{H}^{p+q}(K,C).$$

For $\alpha\in \mathrm{H}^p(K,A)$ and $\beta\in \mathrm{H}^q(K,B)$, one has

$$r(\alpha\cup\beta)=r(\alpha)\cup\beta+(-1)^p\alpha\cup r(\beta)+r(\alpha)\cup r(\beta)\cup(-1)_n\in \mathrm{H}^{p+q-1}(k,C(-1)),$$

where $(-1)_n\in \mathrm{H}^1(k,\mu_n)$ denotes the class of $-1\in k^*/k^{*n}=\mathrm{H}^1(k,\mu_n)$.

Here are some applications of this formula.

- The cup-product followed by the residue

  $$\mathrm{H}^1(K,\mu_n)\times \mathrm{H}^1(K,\mu_n)\xrightarrow{\cup}\mathrm{H}^2(K,\mu_n^{\otimes 2})\xrightarrow{r}\mathrm{H}^1(k,\mu_n)$$

  gives rise to the skew-symmetric pairing

  $$K^*/K^{*n}\times K^*/K^{*n}\longrightarrow k^*/k^{*n}. \tag{1.15}$$

  The above formula for the residue of the cup-product shows that the value of this pairing on the classes of $a,b\in K^*$ is the image in k^*/k^{*n} of the following element of A^*:

  $$(-1)^{v(a)v(b)}b^{v(a)}/a^{v(b)}\in A^*. \tag{1.16}$$

- If we consider

  $$\mathrm{H}^1(K,\mathbb{Z}/n)\times \mathrm{H}^1(K,\mu_n)\xrightarrow{\cup}\mathrm{H}^2(K,\mu_n)\xrightarrow{r}\mathrm{H}^1(k,\mathbb{Z}/n),$$

  then for any $\chi\in \mathrm{H}^1(k,\mathbb{Z}/n)\subset \mathrm{H}^1(K,\mathbb{Z}/n)$ and any $b\in K^*$ we obtain

  $$r(\chi\cup b)=-v(b)\chi\in \mathrm{H}^1(k,\mathbb{Z}/n).$$

- However if we consider

  $$\mathrm{H}^1(K,\mu_n)\times \mathrm{H}^1(K,\mathbb{Z}/n)\xrightarrow{\cup}\mathrm{H}^2(K,\mu_n)\xrightarrow{r}\mathrm{H}^1(k,\mathbb{Z}/n),$$

  then for any $\chi\in \mathrm{H}^1(k,\mathbb{Z}/n)\subset \mathrm{H}^1(K,\mathbb{Z}/n)$ and any $b\in K^*$ we obtain

  $$r(b\cup\chi)=v(b)\chi\in \mathrm{H}^1(k,\mathbb{Z}/n).$$

  This implies that the map $s:\mathrm{H}^1(k,\mathbb{Z}/n)\to \mathrm{H}^2(K,\mu_n)$ given by

  $$s(\chi)=(\pi)_n\cup\chi, \tag{1.17}$$

  where $(\pi)_n$ is the image of π in K^*/K^{*n}, is a section of the residue r.

1.4.2 Extensions of rings

Let R be a discrete valuation ring with field of fractions K and residue field k. Let L be a finite separable extension of K. Then the integral closure B of R in L is a semi-local Dedekind domain which is a finitely generated R-module [SerCL, Ch. I, §4, Prop. 8]. Let $\mathfrak{m}_i$, for $i = 1, \ldots, n$, be the maximal ideals of B. Let $k_i = B/\mathfrak{m}_i$ be the residue field at $\mathfrak{m}_i$. Let e_i be the ramification index of $\mathfrak{m}_i$ over K.

Proposition 1.4.6 *Let ℓ be a prime invertible in R. Then one has commutative diagrams*

$$\begin{array}{ccc} \mathrm{Br}(L)\{\ell\} & \xrightarrow{r} & \mathrm{H}^1(k_i, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ \uparrow{\scriptstyle \mathrm{res}_{K/L}} & & \uparrow{\scriptstyle e_i \mathrm{res}_{k/k_i}} \\ \mathrm{Br}(K)\{\ell\} & \xrightarrow{r} & \mathrm{H}^1(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \end{array} \qquad \begin{array}{ccc} \mathrm{Br}(L)\{\ell\} & \xrightarrow{r} & \bigoplus_{i=1}^n \mathrm{H}^1(k_i, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ \downarrow{\scriptstyle \mathrm{cores}_{K/L}} & & \downarrow{\scriptstyle \sum \mathrm{cores}_{k/k_i}} \\ \mathrm{Br}(K)\{\ell\} & \xrightarrow{r} & \mathrm{H}^1(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \end{array}$$

Proof. We give a sketch of the proof and refer to [Ser03, §8] for details.

Let $\widehat{R}$ be the completion of R and let $\widehat{K}$ be the completion of K. Let $\widehat{B}_i$ be the completion of B with respect to the discrete valuation defined by $\mathfrak{m}_i$. Similarly, let $\widehat{L}_i$ be the completion of L at $\mathfrak{m}_i$. Clearly, $\widehat{K}$ is the field of fractions of $\widehat{R}$ and $\widehat{L}_i$ is the field of fractions of $\widehat{B}_i$. By [SerCL, Ch. II, §3] we have

$$L \otimes_K \widehat{K} \xrightarrow{\sim} \prod_{i=1}^n \widehat{L}_i, \qquad B \otimes_R \widehat{R} \xrightarrow{\sim} \prod_{i=1}^n \widehat{B}_i.$$

It is enough to prove the proposition in the case when R is complete. For the first diagram, using Proposition 1.4.5, it suffices to check commutativity for $(\pi_R) \cup \chi \in \mathrm{Br}(K)\{\ell\}$, where $\chi \in \mathrm{H}^1(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ and π_R is a uniformiser of R. This follows from the functoriality of the pairing with respect to extensions of the field K.

For the second diagram, one can reduce to the following two cases: L/K unramified, i.e. $e(L/K) = 1$ and the residue field extension k_L/k is separable, and L/K with k_L/k purely inseparable. In the first case, one considers $(\pi_R) \cup \chi$, where $\chi \in \mathrm{H}^1(k_L, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. In the second case it is enough to consider the elements of $\mathrm{Br}(L)\{\ell\}$ of the form $(\pi_B) \cup \chi$, where $\chi \in \mathrm{H}^1(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. The result follows from the standard "projection formulae". □

Proposition 1.4.7 *Let $K \subset L$ be an unramified extension of henselian discretely valued fields with residue fields $k \subset k_L$. Let $\alpha \in \mathrm{Br}(K)\{\ell\}$, where ℓ is invertible in k. Suppose that $\mathrm{res}_{L/K}(\alpha) \in \mathrm{Br}(L)$ is unramified, so that $\mathrm{res}_{L/K}(\alpha)$ is the image of an element $\beta \in \mathrm{Br}(k_L)$ under the injective map $\mathrm{Br}(k_L)\{\ell\} \to \mathrm{Br}(L)\{\ell\}$ from the exact sequence (1.14). Then β is contained in the image of the restriction map $\mathrm{res}_{k_L/k} : \mathrm{Br}(k) \to \mathrm{Br}(k_L)$.*

Proof. Take any n such that $\ell^n \alpha = 0$. By Proposition 1.4.5, α is uniquely written as

$$\alpha = \alpha_0 + (\pi)_{\ell^n} \cup \alpha_1,$$

where $\pi \in K$ is a uniformiser, $(\pi)_{\ell^n} \in \mathrm{H}^1(K, \mu_{\ell^n})$ is the image of $\pi \in K^*$ under the boundary map of the Kummer sequence, $\alpha_0 \in \mathrm{Br}(k)[\ell^n]$ and $\alpha_1 \in \mathrm{H}^1(k, \mathbb{Z}/\ell^n)$. Moreover, $\alpha_1 = r_K(\alpha)$ is the residue of α. By the compatibility of pairings for K and L, the image of $(\pi)_{\ell^n} \cup \alpha_1$ in $\mathrm{Br}(L)$ is $(\pi)_{\ell^n} \cup \mathrm{res}_{k_L/k}(\alpha_1)$, where π is understood as an element of L.

Since $\mathrm{res}_{L/K}(\alpha_0)$ and $\mathrm{res}_{L/K}(\alpha)$ are unramified, $(\pi)_{\ell^n} \cup \mathrm{res}_{k_L/k}(\alpha_1)$ is also unramified. As L is unramified over K, the uniformiser $\pi \in K$ is also a uniformiser of L. Therefore, the residue map $r_B : \mathrm{Br}(L)[\ell^n] \to \mathrm{H}^1(k_L, \mathbb{Z}/\ell^n)$ sends $(\pi)_{\ell^n} \cup \mathrm{res}_{k_L/k}(\alpha_1)$ to $\mathrm{res}_{k_L/k}(\alpha_1) \in \mathrm{H}^1(k_L, \mathbb{Z}/\ell^n)$, so this last element is zero. Hence $(\pi)_{\ell^n} \cup \alpha_1$ goes to zero in $\mathrm{Br}(L)$, so that $\mathrm{res}_{L/K}(\alpha)$ is the image of $\mathrm{res}_{k_L/k}(\alpha_0)$. □

Corollary 1.4.8 *Let $R \subset B$ be an unramified extension of (not necessarily henselian) discrete valuation rings with fraction fields $K \subset L$ and residue fields $\kappa \subset \lambda$. Let $\alpha \in \mathrm{Br}(K)\{\ell\}$, where ℓ is a prime invertible in R. Suppose that the image of α in $\mathrm{Br}(L)$ is unramified, so it is the image of a (well defined) element $\beta \in \mathrm{Br}(B)$. Then the image of β under the natural map $\mathrm{Br}(B) \to \mathrm{Br}(\lambda)$ is contained in the image of the restriction map $\mathrm{Br}(\kappa) \to \mathrm{Br}(\lambda)$.*

Proof. The statement only concerns the value of β at the closed point $\mathrm{Spec}(\lambda)$ of $\mathrm{Spec}(B)$, so we can assume without loss of generality that R and B are henselian. In this case the statement follows from Proposition 1.4.7. □

1.4.3 Witt residue

Let R be a henselian discrete valuation ring with fraction field K and *perfect* residue field k. As above, we have inclusions of discretely valued fields

$$K \subset K_{\mathrm{nr}} \subset K_{\mathrm{t}} \subset K_{\mathrm{s}}.$$

The residue field of any of the fields K_{nr}, K_{t}, K_{s} is the algebraic closure of k. We have $\Gamma = \mathrm{Gal}(K_{\mathrm{nr}}/K) = \mathrm{Gal}(k_{\mathrm{s}}/k)$.

By Theorem 1.2.13 we have $\mathrm{Br}(K_{\mathrm{nr}}) = 0$. By Hilbert's theorem 90 the Hochschild–Serre spectral sequence

$$\mathrm{H}^p(\Gamma, \mathrm{H}^q(K_{\mathrm{nr}}, K_{\mathrm{s}}^*)) \Rightarrow \mathrm{H}^{p+q}(K, K_{\mathrm{s}}^*) \tag{1.18}$$

gives an isomorphism $\mathrm{H}^2(\Gamma, K_{\mathrm{nr}}^*) \xrightarrow{\sim} \mathrm{Br}(K)$. Composing it with the Galois equivariant map $v : K_{\mathrm{nr}}^* \to \mathbb{Z}$ given by the valuation we obtain

$$\mathrm{Br}(K) \xleftarrow{\sim} \mathrm{H}^2(\Gamma, K_{\mathrm{nr}}^*) \xrightarrow{v_*} \mathrm{H}^2(\Gamma, \mathbb{Z}) \xleftarrow{\sim} \mathrm{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{Q}/\mathbb{Z}),$$

where the isomorphism $\mathrm{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(\Gamma, \mathbb{Z})$ comes from Galois cohomology of the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0. \tag{1.19}$$

Definition 1.4.9 *The resulting map*

$$r_W : \mathrm{Br}(K) \longrightarrow \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$$

is called the **Witt residue***.*

We note that the choice of a uniformiser defines a section of the homomorphism $v : K_{\mathrm{nr}}^* \to \mathbb{Z}$, and hence of r_W. In particular, the Witt residue map r_W is surjective (for the kernel of the Witt residue, see Theorem 3.6.2). This section can be described in terms of the cup-product. Since Γ is a quotient of $\mathrm{Gal}(K_{\mathrm{s}}/K)$, we can view a continuous character $\chi : \Gamma \to \mathbb{Q}/\mathbb{Z}$ as a character of $\mathrm{Gal}(K_{\mathrm{s}}/K)$. Applying the differential in the long exact sequence attached to the exact sequence of $\mathrm{Gal}(K_{\mathrm{s}}/K)$-modules (1.19) we obtain $\delta(\chi) \in \mathrm{H}^2(K, \mathbb{Z})$. For any $b \in K^*$ the cup-product $\delta(\chi) \cup b$ under the pairing

$$\mathrm{H}^2(K, \mathbb{Z}) \times \mathrm{H}^0(K, K_{\mathrm{s}}^*) \longrightarrow \mathrm{Br}(K) \tag{1.20}$$

is an element of $\mathrm{Br}(K)$, see also [SerCL, Ch. XIV, §1]. (In Section 1.3.4 this element was denoted by (χ, b).) Thus, if $\pi \in R$ is a uniformiser, then the map

$$s_W(\chi) = \delta(\chi) \cup \pi \tag{1.21}$$

is a section of r_W.

1.4.4 Compatibility of residues

Theorem 1.4.10 *Let R be a henselian discrete valuation ring with fraction field K and perfect residue field k. Let n be an integer invertible in R. The composite map*

$$\mathrm{H}^2(K, \mu_n) \xrightarrow{\sim} \mathrm{Br}(K)[n] \xrightarrow{r_W} \mathrm{H}^1(k, \mathbb{Z}/n),$$

where the first map comes from the Kummer sequence (1.3), coincides with the opposite of the residue $r : \mathrm{H}^2(K, \mu_n) \to \mathrm{H}^1(k, \mathbb{Z}/n)$.

Proof. This was proved by Serre's in his 1991–1992 course at Collège de France, cf. the appendix to the thesis of E. Frossard [Fro95, Lemme A.3.2]. See also [GS17, Prop. 6.8.9].

The idea is to use explicit splittings of the residue maps r and r_W given by their respective sections s and s_W, see (1.17) and (1.21). Let $\chi \in \mathrm{H}^1(k, \mathbb{Z}/n) = \mathrm{Hom}(\Gamma, \mathbb{Z}/n)$. We need to show that the Brauer class given by $s(\chi) = (\pi)_n \cup \chi$ is the opposite of $s_W(\chi) = \delta(\chi) \cup \pi$. The proof of this property works more generally for any field K of characteristic coprime to n. Let $G = \mathrm{Gal}(K_{\mathrm{s}}/K)$. We shall show that for any character $\chi \in \mathrm{Hom}(G, \mathbb{Z}/n)$ and any $a \in K^*$ the image of $(a)_n \cup \chi \in \mathrm{H}^2(G, \mu_n)$ in $\mathrm{H}^2(G, K_{\mathrm{s}}^*)$ equals $-\delta(\chi) \cup a$.

We shall use the following well known properties, see [HS70, Ch. IV, §9] or [BouX, §7.6, Prop. 5]. Suppose we are given an exact sequence of G-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

a G-module M, and a positive integer m.

(a) The differential $\mathrm{Ext}^m_G(M,C)\to\mathrm{Ext}^{m+1}_G(M,A)$ in the *second* argument can be identified with the class of the splicing of an m-fold extension of M by C with the given short exact sequence.

(b) The differential $\mathrm{Ext}^m_G(A,M)\to\mathrm{Ext}^{m+1}_G(C,M)$ in the *first* argument can be identified with the class of the splicing of an m-fold extension of A by M with the given short exact sequence, multiplied by $(-1)^{m+1}$.

We have a canonical isomorphism of functors $\mathrm{Ext}^n_G(\mathbb{Z},\cdot)=\mathrm{H}^n(K,\cdot)$. Thus $\chi\in\mathrm{Hom}(G,\mathbb{Z}/n)=\mathrm{H}^1(K,\mathbb{Z}/n)=\mathrm{Ext}^1_G(\mathbb{Z},\mathbb{Z}/n)$ gives rise to an extension of $\mathbb{Z}$ by $\mathbb{Z}/n$. This gives us the first short exact sequence in

$$0\to\mathbb{Z}/n\to E_\chi\to\mathbb{Z}\to 0,\qquad 0\to\mathbb{Z}\to\mathbb{Z}\to\mathbb{Z}/n\to 0;\tag{1.22}$$

the second short exact sequence is obtained from the multiplication by n map $[n]:\mathbb{Z}\to\mathbb{Z}$. We denote it by M_n, and write $[M_n]$ for the class of M_n in $\mathrm{Ext}^1_G(\mathbb{Z}/n,\mathbb{Z})$. Given $a\in K^*$, we let $f_a:\mathbb{Z}\to K_{\mathrm{s}}^*$ be the map of G-modules sending 1 to a.

We write $M_n\cup E_\chi$ for the 2-fold extension of $\mathbb{Z}$ by $\mathbb{Z}$ obtained by splicing the short exact sequences in (1.22). We write $f_{a*}M_n$ for the extension of $\mathbb{Z}/n$ by K_{s}^* which is the push-forward of M_n via f_a. Similarly, $f_{a*}(M_n\cup E_\chi)=f_{a*}M_n\cup E_\chi$ is the push-forward of $M_n\cup E_\chi$ by f_a. We use square brackets to denote the classes of these extensions in the relevant Ext-groups.

The first two rows in the following diagram of pairings are Yoneda pairings, which are defined by splicing exact sequences:

$$\begin{array}{ccccccc}
\mathrm{Ext}^1_G(\mathbb{Z}/n,\mathbb{Z}) & \times & \mathrm{Ext}^1_G(\mathbb{Z},\mathbb{Z}/n) & \longrightarrow & \mathrm{Ext}^2_G(\mathbb{Z},\mathbb{Z}) & = & \mathrm{H}^2(G,\mathbb{Z})\\
\downarrow & & \| & & \downarrow & & \downarrow\\
\mathrm{Ext}^1_G(\mathbb{Z}/n,K_{\mathrm{s}}^*) & \times & \mathrm{Ext}^1_G(\mathbb{Z},\mathbb{Z}/n) & \longrightarrow & \mathrm{Ext}^2_G(\mathbb{Z},K_{\mathrm{s}}^*) & = & \mathrm{H}^2(G,K_{\mathrm{s}}^*)\\
\epsilon\uparrow\simeq & & \| & & & & \uparrow\\
\mathrm{H}^1(G,\mu_n) & \times & \mathrm{H}^1(G,\mathbb{Z}/n) & & \xrightarrow{\cup} & & \mathrm{H}^2(G,\mu_n)
\end{array}$$

The upper vertical maps denoted by arrows are given by the push-forward via $f_a:\mathbb{Z}\to K_{\mathrm{s}}^*$. It is thus clear that the upper part of this diagram commutes. The map ϵ is the edge map $\mathrm{H}^1(G,\mathrm{Hom}(\mathbb{Z}/n,K_{\mathrm{s}}^*))\to\mathrm{Ext}^1_G(\mathbb{Z}/n,K_{\mathrm{s}}^*)$ from the spectral sequence

$$\mathrm{H}^p(G,\mathrm{Ext}^q(\mathbb{Z}/n,K_{\mathrm{s}}^*))\Rightarrow\mathrm{Ext}^{p+q}_G(\mathbb{Z}/n,K_{\mathrm{s}}^*).$$

In the category of abelian groups we have $\mathrm{Ext}^q(\mathbb{Z}/n,K_{\mathrm{s}}^*)=0$ for $q\geq 1$ since K_{s}^* is divisible by n, hence ϵ is an isomorphism. The pairing in the bottom row is the cup-product pairing, which is defined via the tensor product $\mu_n\otimes_{\mathbb{Z}}\mathbb{Z}/n\xrightarrow{\sim}\mu_n$. The commutativity of the lower part of the diagram, i.e., the equality of the 'internal product' to the 'Yoneda-edge-product', is proved in [GH70, Prop. 4.5].

The upper pairing of the diagram applied to $[M_n]\in\mathrm{Ext}^1_G(\mathbb{Z}/n,\mathbb{Z})$ and $[E_\chi]\in\mathrm{Ext}^1_G(\mathbb{Z},\mathbb{Z}/n)$ gives $[M_n\cup E_\chi]\in\mathrm{Ext}^2_G(\mathbb{Z},\mathbb{Z})$, see [BouX, §7.4, Prop. 3]. By property (a) above this equals the differential of $[E_\chi]$ in the long exact sequence of $\mathrm{Ext}^n_G(\mathbb{Z},\cdot)$'s in the second argument. This sequence is the same as the long exact sequence of $\mathrm{H}^n(G,\cdot)$, so we conclude that $[M_n\cup E_\chi]=\delta(\chi)$. It follows that the middle pairing of the diagram sends $[f_{a*}M_n]$ and $[E_\chi]$ to

$$[f_{a*}M_n\cup E_\chi]=f_{a*}(\delta(\chi))=\delta(\chi)\cup a=a\cup\delta(\chi).$$

The bottom pairing sends $(a)_n$ and χ to $(a)_n\cup\chi$. By definition χ goes to $[E_\chi]$, so to prove that $\delta(\chi)\cup a$ is the image of $-(a)_n\cup\chi$ it remains to show that the edge map ϵ sends $(a)_n$ to $-[f_{a*}M_n]$.

To check this consider the following diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_G(\mathbb{Z},K_{\mathrm{s}}^*) & \longrightarrow & \mathrm{Ext}^1_G(\mathbb{Z}/n,K_{\mathrm{s}}^*)\\
|| & & \epsilon\uparrow\simeq\\
\mathrm{H}^0(G,\mathrm{Hom}(\mathbb{Z},K_{\mathrm{s}}^*)) & \longrightarrow & \mathrm{H}^1(G,\mathrm{Hom}(\mathbb{Z}/n,K_{\mathrm{s}}^*))
\end{array}$$

Here the upper horizontal arrow is the differential in the long exact sequence of Ext's in the first argument associated to the exact sequence M_n. The lower horizontal arrow is the differential in the long exact sequence of cohomology attached to the Kummer exact sequence (1.3). The commutativity of the last diagram is proved by a standard derived category argument based on the representation of the left derived functor $\mathbf{R}\mathrm{Hom}_G(\cdot,K_{\mathrm{s}}^*)$ from the bounded derived category of continuous discrete G-modules to abelian groups as the composition of the derived functors of $\mathrm{Hom}(\cdot,K_{\mathrm{s}}^*)$ and $M\mapsto M^G$. By property (b) above applied to $m=0$ the upper arrow sends a to $-[f_{a*}M_n]$. We conclude that $\epsilon((a)_n)=-[f_{a*}M_n]$. □

1.5 The Faddeev exact sequences

Let k be a *perfect* field with algebraic closure $k_{\mathrm{s}}=\bar{k}$ and Galois group $\Gamma=\Gamma_k=\mathrm{Gal}(\bar{k}/k)$. To a monic irreducible polynomial $P(t)\in k[t]$ we attach a free $\mathbb{Z}$-module $\mathbb{Z}_P$ generated by the roots of $P(t)$ in $\bar{k}$ with a natural action of Γ permuting these generators. It is clear that the Γ-module $\mathbb{Z}_P$ is induced from the trivial $\mathrm{Gal}(\bar{k}/k(P))$-module $\mathbb{Z}$, where $k(P)=k[t]/(P(t))$. In particular, by Shapiro's lemma, we have $\mathrm{H}^n(\Gamma_k,\mathbb{Z}_P)=\mathrm{H}^n(\Gamma_{k(P)},\mathbb{Z})$ for all $n\geq 0$. For $n=2$ we obtain a canonical isomorphism

$$\mathrm{H}^2(\Gamma_k,\mathbb{Z}_P)=\mathrm{Hom}(\Gamma_{k(P)},\mathbb{Q}/\mathbb{Z}).$$

The valuations attached to the roots of $P(t)$ give rise to a map of Γ-modules

$$\bar{k}(t)^*\longrightarrow\mathbb{Z}_P,$$

which has a section sending the generator of $\mathbb{Z}_P$ corresponding to a root $\varepsilon\in\bar{k}$ to $t-\varepsilon$. Using this notation we rewrite the natural exact sequence of Γ-modules

$$0\longrightarrow\bar{k}^*\longrightarrow\bar{k}(t)^*\longrightarrow\mathrm{Div}(\mathbb{A}^1_{\bar{k}})\longrightarrow 0 \tag{1.23}$$

as a split exact sequence of Γ-modules

$$0 \longrightarrow \bar{k}^* \longrightarrow \bar{k}(t)^* \longrightarrow \bigoplus_{P(t)} \mathbb{Z}_P \longrightarrow 0, \tag{1.24}$$

where the sum is over all monic irreducible polynomials $P(t) \in k[t]$.

Proposition 1.5.1 (D.K. Faddeev) *Let k be a perfect field. There is a split exact sequence*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(k(t)) \longrightarrow \bigoplus_{P(t)} \mathrm{Hom}(\Gamma_{k(P)}, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0, \tag{1.25}$$

where the direct sum is over all monic irreducible polynomials $P(t) \in k[t]$. The second arrow is given by the inclusion of fields $k \subset k(t)$. For each $P(t)$, the map $\mathrm{Br}(k(t))\to\mathrm{Hom}(\Gamma_{k(P)}, \mathbb{Q}/\mathbb{Z})$ factors through the Witt residue attached to the valuation given by $P(t)$.

Proof. Applying $\mathrm{H}^2(\Gamma_k, \cdot)$ to (1.24) we obtain (1.25), once we identify the middle term with $\mathrm{Br}(k(t))$. The natural isomorphism $\Gamma_k = \mathrm{Gal}(\bar{k}(t)/k(t))$ gives rise to the inflation map

$$\mathrm{inf} : \mathrm{H}^2(\Gamma_k, \bar{k}(t)^*) \longrightarrow \mathrm{H}^2(\mathrm{Gal}(\overline{k(t)}/k(t)), \overline{k(t)}^*) = \mathrm{Br}(k(t)).$$

It is enough to prove that this map is an isomorphism. Indeed, inf fits into the Hochschild–Serre spectral sequence

$$\mathrm{H}^p(\Gamma_k, \mathrm{H}^q(\mathrm{Gal}(\overline{k(t)}/\bar{k}(t), \overline{k(t)}^*)) \Rightarrow \mathrm{H}^{p+q}(\mathrm{Gal}(\overline{k(t)}/k(t)), \overline{k(t)}^*).$$

We have $\mathrm{H}^1(\mathrm{Gal}(\overline{k(t)}/\bar{k}(t)), \overline{k(t)}^*) = 0$ (Theorem 1.3.2, Hilbert's theorem 90) and $\mathrm{H}^2(\mathrm{Gal}(\overline{k(t)}/\bar{k}(t)), \overline{k(t)}^*) = \mathrm{Br}(\bar{k}(t)) = 0$ (Theorem 1.2.12, Tsen's theorem). The spectral sequence now implies that inf is an isomorphism.

It remains to check the compatibility with the Witt residue. Let $k[t]_P$ be the localisation of $k[t]$ at the principal ideal $(P(t))$, let $k[t]_P^{\mathrm{h}}$ be the henselisation of $k[t]_P$. It is a henselian discrete valuation ring with residue field $k(P)$. The integral closure of k in $k[t]_P^{\mathrm{h}}$ is a field of representatives for $k(P)$ inside $k[t]_P^{\mathrm{h}}$, that is, the reduction map induces an isomorphism between this field and the residue field $k(P)$. Henceforth we denote this field by $k(P)$.

Let $K \subset \overline{k(t)}$ be the fraction field of $k[t]_P^{\mathrm{h}}$. Let K_{nr} be the maximal unramified extension of K. We note that $K_{\mathrm{nr}} = K \otimes_{k(P)} \bar{k}$ and $\mathrm{Gal}(K_{\mathrm{nr}}/K) = \Gamma_{k(P)}$. The map $\mathrm{Br}(k(t))\to\mathrm{Hom}(\Gamma_{k(P)}, \mathbb{Q}/\mathbb{Z})$ comes from $\mathrm{H}^2(\Gamma_k, \bar{k}(t)^*)\to\mathrm{H}^2(\Gamma_k, \mathbb{Z}_P)$ which factors as

$$\mathrm{H}^2(\Gamma_k, \bar{k}(t)^*) \longrightarrow \mathrm{H}^2(\Gamma_k, (K \otimes_k \bar{k})^*) \longrightarrow \mathrm{H}^2(\Gamma_k, \mathbb{Z}_P).$$

Since $\mathrm{H}^2(\Gamma_k, (K\otimes_k\bar{k})^*) = \mathrm{H}^2(\Gamma_{k(P)}, (K\otimes_{k(P)}\bar{k})^*) = \mathrm{H}^2(\Gamma_{k(P)}, K_{\mathrm{nr}}^*)$ by Shapiro's lemma, our map can also be written as

$$\mathrm{H}^2(\Gamma_k, \bar{k}(t)^*) \longrightarrow \mathrm{H}^2(\Gamma_{k(P)}, K_{\mathrm{nr}}^*) \longrightarrow \mathrm{H}^2(\Gamma_{k(P)}, \mathbb{Z}).$$

Here the second map is given by the valuation, so, by definition, it is the Witt residue. □

Theorem 1.5.2 *Let k be a perfect field. There is a split exact sequence*

$$0\to\mathrm{Br}(k)\to\mathrm{Br}(k(t))\to\bigoplus_{x\in(\mathbb{P}^1_k)^{(1)}}\mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z})\to 0, \tag{1.26}$$

where the direct sum is over all closed points of $\mathbb{P}^1_k$. The third map is compatible with the Witt residues. The fourth map is the sum of corestrictions $\mathrm{cores}_{k(x)/k}$ over all closed points of $\mathbb{P}^1_k$, including the point at infinity.

Proof. Instead of (1.23) we now consider the exact sequence of Γ-modules

$$0\longrightarrow \bar{k}^*\longrightarrow \bar{k}(t)^*\longrightarrow \mathrm{Div}(\mathbb{P}^1_{\bar{k}})\longrightarrow \mathbb{Z}\longrightarrow 0, \tag{1.27}$$

where the fourth arrow is given by the degree. This sequence can be obtained by splicing two exact sequences of Γ-modules, both of which are split:

$$0\longrightarrow \bar{k}^*\longrightarrow \bar{k}(t)^*\longrightarrow \mathrm{Div}_0(\mathbb{P}^1_{\bar{k}})\longrightarrow 0,$$

where $\mathrm{Div}_0(\mathbb{P}^1_{\bar{k}})$ is the degree 0 subgroup of $\mathrm{Div}(\mathbb{P}^1_{\bar{k}})$, and

$$0\longrightarrow \mathrm{Div}_0(\mathbb{P}^1_{\bar{k}})\longrightarrow \mathrm{Div}(\mathbb{P}^1_{\bar{k}})\longrightarrow \mathbb{Z}\longrightarrow 0.$$

Applying $\mathrm{H}^2(\Gamma_k,\cdot)$ to (1.27) we obtain (1.26). The compatibility of the third arrow with the Witt residues follows from the last sentence of Theorem 1.5.1. The fourth map is the sum of maps

$$\mathrm{H}^2(\Gamma_k,\mathbb{Z}_{k(x)})\longrightarrow \mathrm{H}^2(\Gamma_k,\mathbb{Z}),$$

each of which is induced by the summation map $\mathbb{Z}_{k(x)}\to\mathbb{Z}$. This implies the final statement of the theorem. □

The exact sequence (1.25) is split and it is instructive to write down an element of $\mathrm{Br}(k(t))$ that lifts a character $\chi\in\mathrm{Hom}(\Gamma_{k(P)},\mathbb{Q}/\mathbb{Z})$ for a given monic irreducible polynomial $P(t)$. Let $\delta(\chi)\in\mathrm{H}^2(k(P),\mathbb{Z})$ be the image of χ under the differential δ in the long exact sequence of cohomology groups attached to the exact sequence of $\Gamma_{k(P)}$-modules (1.19). We also denote by $\delta(\chi)$ the image of this element in $\mathrm{H}^2(k(P)(t),\mathbb{Z})$ under the restriction from $k(P)$ to $k(P)(t)$. Let τ_P be the image of t in $k(P)=k[t]/(P(t))$. Then $t-\tau_P\in k(P)(t)$. Let us denote by $A_\chi\in\mathrm{Br}(k(P)(t))$ the cup-product of $\delta(\chi)$ and $t-\tau_P$ with respect to the pairing (1.20):

$$\mathrm{H}^2(k(P)(t),\mathbb{Z})\times \mathrm{H}^0(k(P)(t),\mathbb{G}_m)\longrightarrow \mathrm{Br}(k(P)(t)).$$

It is clear that A_χ is unramified on $\mathbb{P}^1_{k(P)}$ away from the $k(P)$-point $t=\tau_P$ and the point at infinity, i.e. the residues of A_χ at all other closed points of $\mathbb{P}^1_{k(P)}$ are trivial. In Section 1.4.1 we have seen that the residue of A_χ at $t=\tau_P$ is

$$r(A_\chi)=v(t-\tau_P)\chi=\chi\in\mathrm{H}^1(k(P),\mathbb{Z}/n).$$

The same formula shows that the residue of A_χ at the point at infinity is $-\chi$.

Let us abbreviate the notation for the corestriction map from $k(P)(t)$ to $k(t)$ as $\mathrm{cores}_{k_P/k}$. Using Proposition 1.4.6 we see that $\mathrm{cores}_{k_P/k}(A_\chi)$ is an element of $\mathrm{Br}(k(t))$ unramified away from the closed point P, which is the zero set of $P(t)$, and possibly the point at infinity. More precisely, the residue of $\mathrm{cores}_{k(P)/k}(A_\chi)$ at P is χ and the residue at ∞ is $-\mathrm{cores}_{k(P)/k}(\chi)$.

Let $A \in \mathrm{Br}(k(t))$ be an arbitrary element. Let χ_P be the residue of A at the closed point P of $\mathbb{P}^1_k$. Let S be the set of closed points $P \in \mathbb{A}^1_k$ for which $\chi_P \neq 0$. Then $A - \sum_{P\in S} \mathrm{cores}_{k(P)/k}(A_{\chi_P})$ is unramified over $\mathbb{A}^1_k$. Faddeev's exact sequence (1.25) now shows that

$$A = \sum_{P\in S} \mathrm{cores}_{k(P)/k}(A_{\chi_P}) + A_0,$$

for some $A_0 \in \mathrm{Br}(k)$. In particular, if A is unramified at ∞, then $A_0 = A(\infty)$ is the value of A at ∞.

Chapter 2

Étale cohomology

In the first two sections of this chapter we introduce notation and terminology, and state basic results about étale sheaves and étale cohomology. It would not be realistic to give proofs; instead, we try to help the reader navigate through J.S. Milne's book [Mil80], a main reference for this chapter. We refer to [SGA4$\frac{1}{2}$, Arcata] for a helpful gentle introduction to étale cohomology. See also [Tam94].

The third section reports on purity results for étale cohomology with torsion coefficients and on residues in the étale cohomological context.

In the next two sections we discuss the first cohomology group of the multiplicative group, which is the Picard group, and then the relative Picard group and the Picard scheme. Already for a smooth projective variety over a field, the Brauer group of the ground field appears naturally when one wants to see if a Galois invariant element of the geometric Picard group comes from an element of the Picard group of the variety.

The last section is a very short summary of stacks and gerbes based on M. Olsson's book [Ols16, Ch. 9]. This material will be used in the next chapter.

2.1 Topologies, morphisms and sheaves

2.1.1 Grothendieck topologies on a scheme

We start with the basic definitions [Mil80, II, §1].

Let E be a class of morphisms of schemes which contains all isomorphisms and is closed under composition, such that a base change of any morphism in E is in E.

Let X be a scheme. Let $\mathbf{C}_X$ be a full subcategory of the category of schemes over X such that for any $Y\to X$ in $\mathbf{C}_X$ and any morphism $U\to Y$ in E the composition $U\to X$ is in $\mathbf{C}_X$.

An E-covering of an object Y of $\mathbf{C}_X$ is a family of E-morphisms $\{g_i : U_i\to Y\}$ such that $Y = \cup g_i(U_i)$.

The class of all such coverings of all such objects is called the E-topology on $\mathbf{C}_X$. The category $\mathbf{C}_X$ with the E-topology is the E-site $\mathbf{C}_{X,E}$. A map $V\to W$

in $\mathbf{C}_X$ which is in E is referred to as an open set of W in the E-topology.

A site $\mathbf{C}_{X,E}$ is *small* if the underlying category of schemes $\mathbf{C}_X$ is the category of schemes Y/X such that $Y \to X$ is in E.

A site $\mathbf{C}_{X,E}$ is *big* if the underlying category $\mathbf{C}_X$ is the category of all schemes over X. We recall the definitions of the sites that will be used in this book.

$X_{\text{ét}}$ is the small étale site, i.e. the category of schemes that are étale over X endowed with the étale topology. In other words, an "open set" $U_i \to Y$ is an étale morphism.

$X_{\text{Ét}}$ is the big étale site, i.e. the category of schemes over X endowed with the étale topology.

X_{zar} is the small Zariski site, i.e. the category of open subschemes of X endowed with the Zariski topology.

X_{Zar} is the big Zariski site, i.e. the category of schemes over X endowed with the Zariski topology.

X_{fppf} is the big flat site, so that the category consists of all schemes over X. An "open set" $U \to Y$ is a flat morphism which is locally of finite presentation.

2.1.2 Presheaves and sheaves

A *presheaf* of abelian groups on X is a contravariant functor $\mathcal{P}$ from the underlying category of the site to the category of abelian groups. We refer to $\mathcal{P}(Y)$ as the group of sections over Y. For example,

$\mathbb{G}_{a,X}$ is the presheaf such that $\mathbb{G}_{a,X}(Y) = \mathrm{H}^0(Y, \mathcal{O}_Y)$,
$\mathbb{G}_{m,X}$ is the presheaf such that $\mathbb{G}_{m,X}(Y) = \mathrm{H}^0(Y, \mathcal{O}_Y^*)$.
$\mu_{n,X}$, for $n > 0$, is the presheaf such that $\mu_n(Y) = \{x \in \mathrm{H}^0(Y, \mathcal{O}_Y^*) | x^n = 1\}$.

Presheaves on X form an abelian category, where a sequence of presheaves is exact if and only if the corresponding sequence of sections over Y is an exact sequence of abelian groups, for any Y/X in the underlying category. We denote this abelian category by $\mathrm{P}(X)$, when the topology is clear from the context.

A presheaf $\mathcal{P}$ is a *sheaf* if for any scheme Y/X in our category, and any covering $\{U_i\}$ of Y, any section over Y is uniquely determined by its restrictions to all the U_i, and any family of sections over the U_i which agree on $U_i \times_Y U_j$ come from a section over Y. We denote by $a\mathcal{P}$ the sheaf associated to the presheaf $\mathcal{P}$ [Mil80, Thm. II.2.11]. One can give an explicit construction of $a\mathcal{P}$ in terms of the sheafified 0-th Čech cohomology presheaf $\check{\mathcal{H}}^0(\mathcal{P})$, namely, $a\mathcal{P} = \check{\mathcal{H}}^0(\check{\mathcal{H}}^0(\mathcal{P}))$, see [Mil80, Remark III.2.2 (c)].

Let us define the category of sheaves $\mathrm{S}(X)$ as the full subcategory of $\mathrm{P}(X)$ whose objects are sheaves. Thus, if $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, then a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ in $\mathrm{S}(X)$ is the same as a morphism of presheaves $\mathcal{F} \to \mathcal{G}$. The kernel $\mathrm{Ker}(\varphi)$ in $\mathrm{S}(X)$ is the same as the kernel of the morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ (which is a sheaf). However, the cokernel presheaf $\mathrm{Coker}(\varphi)$ is not always a sheaf (for example, the cokernel of the differentiation on the sheaf of holomorphic functions on $\mathbb{C} \smallsetminus \{0\}$ is not a sheaf). The cokernel in $\mathrm{S}(X)$ is the sheaf associated to the presheaf $\mathrm{Coker}(\varphi)$. This makes $\mathrm{S}(X)$ an abelian category[1]. It follows that the inclusion functor $i : \mathrm{S}(X) \to \mathrm{P}(X)$ is left exact, and $a : \mathrm{P}(X) \to \mathrm{S}(X)$ is the left adjoint of i, so we have an isomorphism of (bi-)functors

$$\mathrm{Hom}_{\mathrm{S}(X)}(a\mathcal{P}, \mathcal{F}) = \mathrm{Hom}_{\mathrm{P}(X)}(\mathcal{P}, i\mathcal{F}),$$

see [Mil80], Remark II.2.14 (a) and Thm. II.2.15. The functor $a : \mathrm{P}(X) \to \mathrm{S}(X)$ is exact [Mil80, Thm. 2.15 (a)].

If G is a commutative group scheme over X, then the functor represented by G, that is, the functor associating to a scheme Y/X the abelian group $\mathrm{Hom}_X(Y, G)$, is not only a presheaf but is actually a sheaf for each of the topologies mentioned above, by [Mil80, Cor. II.1.7]. In particular, the presheaves $\mathbb{G}_{a,X}$ and $\mathbb{G}_{m,X}$ are sheaves because they are represented by the additive group scheme $\mathbb{G}_a = \mathrm{Spec}(\mathbb{Z}[x])$ and the multiplicative group scheme $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[x, x^{-1}])$, respectively. This also holds for $\mu_{n,X}$.

2.1.3 Points and stalks in the étale topology

Let $x = \mathrm{Spec}(k(x))$ be a point of the scheme X. The local ring of X at x is denoted by $\mathcal{O}_{X,x}$. We have

$$\mathcal{O}_{X,x} = \varinjlim \mathcal{O}(U),$$

where the injective limit is taken over all open subsets $U \subset X$ containing x. The analogue of the local ring in the étale topology is

$$\mathcal{O}^{\mathrm{h}}_{X,x} = \varinjlim \mathcal{O}(U),$$

where U is a connected étale X-scheme equipped with a lifting $x \hookrightarrow U$ of $x \hookrightarrow X$. The superscript h says that $\mathcal{O}^{\mathrm{h}}_{X,x}$ is the *henselisation* of the local ring $\mathcal{O}_{X,x}$. The residue field of the local ring $\mathcal{O}^{\mathrm{h}}_{X,x}$ is $k(x)$.

Now let $\bar{x} \to X$ be a geometric point, i.e. a morphism $\mathrm{Spec}(k(\bar{x})) \to X$, where $k(\bar{x})$ is algebraically closed. One says that $\bar{x}$ lies over x if x is the image of $\bar{x}$ in X; then $k(x) \subset k(\bar{x})$. Define

$$\mathcal{O}^{\mathrm{sh}}_{X,x} = \varinjlim \mathcal{O}(U),$$

where U is a connected étale X-scheme equipped with a lifting $\bar{x} \to U$ of $\bar{x} \to X$. The superscript sh says that $\mathcal{O}^{\mathrm{sh}}_{X,x}$ is strictly henselian; it is the *strict henselisation* of the local ring $\mathcal{O}_{X,x}$. The residue field of the local ring $\mathcal{O}^{\mathrm{sh}}_{X,x}$ is the separable closure of $k(x)$ in $k(\bar{x})$. Speaking of "the" strict henselisation is a common abuse of language, which we shall keep in this book. If $k(x)$ is not separably closed, the ring extension $\mathcal{O}^{\mathrm{sh}}_{X,x}$ of $\mathcal{O}_{X,x}$ is defined up to a non-unique isomorphism. Replacing $\bar{x} \to X$ by a different geometric point over x produces a local ring isomorphic to $\mathcal{O}^{\mathrm{sh}}_{X,x}$; this isomorphism is determined by the induced

[1] Thus $\mathrm{S}(X)$ is an abelian category and is also a full subcategory of $\mathrm{P}(X)$, but $\mathrm{S}(X)$ is *not* an abelian subcategory of $\mathrm{P}(X)$, because the notion of cokernel is not the same.

isomorphism of residue fields, which are two separable closures of $k(x)$, hence they are isomorphic but in a non-unique way.

One writes $\mathcal{O}_{X,\bar{x}} = \mathcal{O}^{\mathrm{sh}}_{X,x}$.

For more on henselisation and strict henselisation see [Ray70b, Chap. VIII], [BLR90, §2.3] and [Stacks, Section 0BSK].

The *stalk* of an étale presheaf $\mathcal{P}$ on X at a geometric point $u : \bar{x} \to X$ is defined as

$$\mathcal{P}_{\bar{x}} = \varinjlim \mathcal{P}(U),$$

where U is connected and étale over X such that u factors through $U \to X$, see [Mil80], Section II.2. It is clear from the definition that we have

$$(\mathbb{G}_{a,X})_{\bar{x}} = \mathcal{O}_{X,\bar{x}}, \quad (\mathbb{G}_{m,X})_{\bar{x}} = \mathcal{O}^*_{X,\bar{x}}.$$

2.1.4 Morphisms of sites. Direct and inverse images of sheaves

Let $\pi : X' \to X$ be a morphism of schemes. Suppose that we have a site on X and a site on X'. Then π is *continuous* or, in other words, defines a *morphism of sites*, if the following properties are satisfied:

(a) if Y/X is in the underlying category of X, then $Y \times_X X'/X'$ is in the underlying category of X';

(b) if $U \to Y$ is “an open subset” of Y/X, then $U \times_X X' \to Y \times_X X'$ is “an open subset” of $Y \times_X X'/X'$.

For example, the identity map on X defines morphisms of sites

$$X_{\mathrm{fppf}} \to X_{\text{Ét}} \to X_{\text{ét}} \to X_{\mathrm{zar}}.$$

Here is another example. Let K/k be an arbitrary extension of fields. Let X be a k-scheme and $X_K = X \times_k K$. For each of the above topologies, the morphism of schemes $X_K \to X$ defines a morphism of the associated sites.

A continuous morphism of sites $\pi : X' \to X$ defines a functor $\pi_p : \mathrm{P}(X') \to \mathrm{P}(X)$ which associates to a presheaf $\mathcal{P}$ on X' the presheaf on X which sends Y/X from the underlying category of X to $\mathcal{P}(Y \times_X X')$. It is obvious that π_p is an exact functor.

For a presheaf $\mathcal{P}$ on X and an object Y'/X' of the underlying category of X', define

$$\pi^p(\mathcal{P})(Y') = \varinjlim \mathcal{P}(Y),$$

where Y/X ranges over the objects of the underlying category of X such that the composed map $Y' \to X' \to X$ factors though Y. Then it is easy to check that π^p is a functor $\mathrm{P}(X) \to \mathrm{P}(X')$ which is left adjoint to π_p:

$$\mathrm{Hom}_{\mathrm{P}(X)}(\mathcal{P}_1, \pi_p \mathcal{P}_2) = \mathrm{Hom}_{\mathrm{P}(X')}(\pi^p \mathcal{P}_1, \mathcal{P}_2).$$

In particular, the stalk of an étale presheaf $\mathcal{P}$ at the geometric point $u : \bar{x} \to X$ is the abelian group $\mathcal{P}_{\bar{x}} = u^p \mathcal{P}$. If $\pi : X' \to X$ belongs to the underlying category of X, then $\pi^p(\mathcal{P})$ is easy to describe: this is just the restriction of the presheaf $\mathcal{P}$ to X'.

It is easy to see that π_p sends sheaves to sheaves [Mil80, II.2.7]. In general, this does not hold for π^p. However, for a geometric point $u : \bar{x} \to X$, u^p does send étale sheaves to étale sheaves. This implies that $\mathcal{P}$ and $a\mathcal{P}$ have the same stalks [Mil80, Remark II.2.14 (c)]. A sequence of étale sheaves is exact if and only if the corresponding sequence of stalks is exact for all geometric points $\bar{x}$ of X, see [Mil80, Thm. II.2.15 (c)].

Let $\pi : X' \to X$ be a continuous morphism, and let $\mathcal{F}$ be a sheaf on X'. The direct image $\pi_*\mathcal{F}$ is defined as $\pi_p\mathcal{F}$, that is, $\pi_*\mathcal{F}(Y) = \mathcal{F}(Y \times_X X')$, where Y/X is in the underlying category of the site on X. The inverse image $\pi^*\mathcal{G}$ of a sheaf $\mathcal{G}$ on X is defined as $a\pi^p\mathcal{G}$, so one can write $\pi^* = a\pi^p i$. If the Grothendieck topologies on X and X' are the same and the morphism $\pi : X' \to X$ is in the underlying category of X, then π^* is the restriction of $\mathcal{F}$ to X'.

In particular, if G_X is a sheaf on X represented by a commutative group scheme G over X, then $\pi^* G_X = G_{X'}$ when $\pi : X' \to X$ is in the underlying category of X. For example, this holds for the big étale site. (But $\pi^* G_X \neq G_{X'}$ for the small étale site unless π is étale.)

The functors π_* and π^* are adjoint:

$$\mathrm{Hom}_{\mathrm{S}(X)}(\mathcal{F}, \pi_*\mathcal{F}') = \mathrm{Hom}_{\mathrm{S}(X')}(\pi^*\mathcal{F}, \mathcal{F}'),$$

and this implies that π_* is left exact. Note that since the cokernels in $\mathrm{P}(X)$ and $\mathrm{S}(X)$ are not the same, π_* is not in general an exact functor. (Though π_* is exact if $X' \to X$ is a finite morphism, and the sites on X' and X are the small étale sites, see [Mil80, Cor. II.3.6].) As for π^*, this functor is exact for the small étale or Zariski sites, and also when the underlying category of the site is the category of X-schemes ([Mil80], Prop. II.2.6 (a) and the beginning of Section II.3). Thus π^* is exact for all the sites listed in Section 2.1.1.

Let $\pi : X' \to X$ be a morphism, and let $\mathcal{F}$ be an étale sheaf on X. If x' is a point of X' that maps to $x \in X$, then we can choose a geometric point over x' to be also a geometric point over x, that is, $\bar{x} = \bar{x}'$. This formally implies that we have $(\pi^*\mathcal{F})_{\bar{x}'} = \mathcal{F}_{\bar{x}}$ (see also [Mil80, Thm. II.3.2 (a)]).

Now let π be quasi-compact. Let $x = \mathrm{Spec}(k(x)) \in X$, and let $\bar{x} = \mathrm{Spec}(k(x)_{\mathrm{s}})$. Let $\mathcal{F}$ be an étale sheaf on X'. One proves that the stalk of $\pi_*\mathcal{F}$ at $\bar{x}$ can be computed at the strict henselisation of X at x:

$$(\pi_*\mathcal{F})_{\bar{x}} = \tilde{\mathcal{F}}(X' \times_X \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,\bar{x}})), \tag{2.1}$$

where $\tilde{\mathcal{F}}$ is the inverse image of $\mathcal{F}$ with respect to the first projection

$$X' \times_X \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,\bar{x}}) \longrightarrow X',$$

see [Mil80, Thm. II.3.2 (b)].

2.2 Cohomology

2.2.1 Definition and basic properties

One proves that the category of sheaves on X has enough injectives [Mil80, Prop. III.1.1], which makes it possible to define the cohomology groups $\mathrm{H}^n(X,\mathcal{F})$ as the right derived functors of the sections functor $\mathcal{F}\mapsto\mathcal{F}(X)$. If $\pi : X'\to X$ is a continuous morphism, e.g. a morphism of schemes, then the higher derived image sheaves $(R^n\pi_*)(\mathcal{F})$ are the right derived functors of π_*. One proves that $(R^n\pi_*)(\mathcal{F})$ is the sheaf associated to the presheaf that sends an 'open set' U to the group $\mathrm{H}^n(U\times_X X',\mathcal{F})$.

If $\mathcal{G}$ is a sheaf on X, then the functor $\mathrm{Hom}_X(\mathcal{G},\cdot)$ is left exact; so one defines $\mathrm{Ext}^n_X(\mathcal{G},\cdot)$ as its right derived functors. Since $\mathcal{F}(X)=\mathrm{Hom}_X(\mathbb{Z}_X,\mathcal{F})$, where $\mathbb{Z}_X$ is the constant sheaf defined by $\mathbb{Z}$, we have $\mathrm{Ext}^n_X(\mathbb{Z},\mathcal{F})=\mathrm{H}^n(X,\mathcal{F})$.

Let us consider the small étale site on a scheme X. If $\pi : X'\to X$ is a quasi-compact morphism, then the stalk of $(R^n\pi_*)(\mathcal{F})$ at $\bar{x}$ can be described in the same way as we described $(\pi_*\mathcal{F})_{\bar{x}}$ in (2.1):

$$(R^n\pi_*)(\mathcal{F})_{\bar{x}} = \mathrm{H}^n(X'\times_X \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,\bar{x}}),\tilde{\mathcal{F}}),$$

where $\tilde{\mathcal{F}}$ is as in the end of the previous section. See [Mil80, Thm. III.1.15].

If $\pi : X'\to X$ is a proper morphism, then a corollary of the *proper base change theorem* says that for a torsion sheaf $\mathcal{F}$ on X', the stalk of $(R^n\pi_*)(\mathcal{F})$ at $\bar{x}$ is $\mathrm{H}^n(X'_{\bar{x}},\mathcal{F})$, where $X'_{\bar{x}}=\pi^{-1}(\bar{x})$ is the fibre of π at $\bar{x}$. See [Mil80, VI.2.5].

By a corollary of the *smooth base change theorem* [Mil80, VI.4.2], if X is a connected scheme, $\pi : X'\to X$ is a smooth and proper morphism, and n is prime to the residual characteristics of X, then the groups $\mathrm{H}^n(X'_{\bar{x}},\mathbb{Z}/n)$ are isomorphic for all geometric points $\bar{x}$. These results have many applications. For example, if $\pi : X\to\mathrm{Spec}(R)$ is a smooth and proper morphism, where R is a discrete valuation ring with fraction field K and residue field k, and n is prime to $\mathrm{char}(k)$ and $\mathrm{char}(K)$, then the restriction of the representation of $\mathrm{Gal}(K_{\mathrm{s}}/K)$ in $\mathrm{H}^n(X^{\mathrm{s}},\mathbb{Z}/n)$ to the inertia group is trivial.

2.2.2 Étale and Galois cohomology

Let k be a field. Consider the small étale site on $\mathrm{Spec}(k)$. The underlying category consists of finite dimensional étale k-algebras, i.e. finite direct products of finite separable field extensions of k. Choose a separable closure k_{s} of k, and let $\Gamma=\mathrm{Gal}(k_{\mathrm{s}}/k)$ be the absolute Galois group of k.

Let $\mathcal{P}$ be a presheaf on $\mathrm{Spec}(k)$. For a finite, separable field extension k'/k we write $\mathcal{P}(k')$ for $\mathcal{P}(\mathrm{Spec}(k'))$. When k'/k is Galois, the Galois group $\mathrm{Gal}(k'/k)$ acts on $\mathcal{P}(k')$. If $\mathcal{P}$ sends disjoint unions of schemes to direct products of abelian groups, then it is a sheaf if and only if $\mathcal{P}(k')=\mathcal{P}(k'')^{\mathrm{Gal}(k''/k')}$ for every finite separable extension k'/k and every finite Galois extension k''/k', cf. [Mil80, Prop. II.1.5].

A continuous discrete Γ-module M defines a presheaf $\mathcal{F}_M$ on $\mathrm{Spec}(k)$ by the formula

$$\mathcal{F}_M(\prod_{i=1}^{n} k_i) = \prod_{i=1}^{n} M^{\mathrm{Gal}(k_{\mathrm{s}}/k_i)},$$

where the fields k_i are such that $k \subset k_i \subset k_{\mathrm{s}}$. One checks that $\mathcal{F}_M$ is a sheaf [Mil80, Lemma II.1.8], and that $M \mapsto \mathcal{F}_M$ defines an equivalence of the category of discrete Γ-modules with the category of étale sheaves on $\mathrm{Spec}(k)$, see [Mil80, Thm. II.1.9].

The inverse correspondence associates to a presheaf $\mathcal{P}$ on $\mathrm{Spec}(k)$ the discrete Galois module

$$M_{\mathcal{P}} = \varinjlim \mathcal{P}(k'),$$

where k'/k is a finite separable extension. Indeed, we can assume that k' is Galois over k, so that Γ acts on each $\mathcal{P}(k')$, and thus on $M_{\mathcal{P}}$. This module is discrete because $M_{\mathcal{P}}$ is the union of the invariants with respect to all open subgroups of Γ.

For a discrete Γ-module M the *Galois cohomology group* $\mathrm{H}^n(\Gamma, M)$ for $n \geq 0$ is defined as the inductive limit of $\mathrm{H}^n(\Gamma/U, M^U)$, where U ranges over all open normal subgroups of Γ, see [SerCG, Ch. I, §2]. Recall from the previous chapter that the resulting group is well defined up to unique isomorphism. It does not depend on the choice of k_{s} and is denoted by $\mathrm{H}^n(k, M)$, see [SerCG, II, §1].

The étale cohomology groups $\mathrm{H}^n_{\text{ét}}(\mathrm{Spec}(k), \mathcal{F}_M)$ are canonically isomorphic to the Galois cohomology groups $\mathrm{H}^n(k, M)$, since these are the right derived functors of $M \mapsto M^{\Gamma}$. Similarly, the Ext group $\mathrm{Ext}^n_{\mathrm{Spec}(k)}(\mathcal{F}_M, \mathcal{F}_{M'})$ in the category of étale sheaves on $\mathrm{Spec}(k)$ is the same as the Ext group $\mathrm{Ext}^n_k(M, M')$ in the category of discrete Γ-modules.

Now assume that $\pi : X \to \mathrm{Spec}(k)$ is a scheme over a field k equipped with the étale topology. Define $X^{\mathrm{s}} = X \times_k k_{\mathrm{s}}$, and let $\tilde{\mathcal{F}}$ be the inverse image of $\mathcal{F}$ with respect to the morphism $X^{\mathrm{s}} \to X$. The sheaf $\pi_*(\mathcal{F})$ on $\mathrm{Spec}(k)$ corresponds to the Γ-module

$$(\pi_*\mathcal{F})_{k_{\mathrm{s}}} = \varinjlim \mathcal{F}(X \times_k k') = \tilde{\mathcal{F}}(X^{\mathrm{s}}),$$

where k'/k ranges over finite subextensions of k_{s}/k. In the same way, the sheaf $(R^n\pi_*)(\mathcal{F})$ corresponds to the Γ-module

$$(R^n\pi_*)(\mathcal{F})_{k_{\mathrm{s}}} = \varinjlim \mathrm{H}^n(X \times_k k', \mathcal{F}) = \mathrm{H}^n(X^{\mathrm{s}}, \tilde{\mathcal{F}}).$$

2.2.3 Standard spectral sequences

Recall that when we have three abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, and left exact functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ such that F sends injective objects in $\mathcal{A}$ to G-acyclic objects in $\mathcal{B}$, then there is a convergent first quadrant *Grothendieck spectral sequence*

$$E_2^{p,q} = (R^pG)(R^qF)A \Rightarrow R^{p+q}(GF)A, \tag{2.2}$$

where $A \in \mathrm{Ob}(\mathcal{A})$, see [Wei94, Thm. 5.8.3], [Mil80, Appendix B]. Let $\mathcal{D}^+(\mathcal{A})$ denote the derived category of bounded below complexes in the abelian category $\mathcal{A}$ (for which we refer to [Wei94, Ch. X]). The above spectral sequence can be viewed as the spectral sequence of composed functors between derived categories $\mathbf{R}F : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ and $\mathbf{R}G : \mathcal{D}^+(\mathcal{B}) \to \mathcal{D}^+(\mathcal{C})$. In this interpretation (2.2) comes from the fact that $\mathbf{R}(GF)$ is the composition $\mathbf{R}G \circ \mathbf{R}F$, see [Wei94, Thm. 10.8.3].

Suppose that we have continuous morphisms of sites

$$X'' \xrightarrow{\pi'} X' \xrightarrow{\pi} X,$$

and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are the categories of sheaves on X'', X', X, respectively. Since π_* has a left adjoint functor π^* which is exact, π_* sends injectives to injectives, and hence for any sheaf $\mathcal{F}$ on X'' we obtain the Leray spectral sequence [Mil80, Thm. III.1.18(b)]

$$E_2^{p,q} = (R^p\pi_*)(R^q\pi'_*)\mathcal{F} \Rightarrow R^{p+q}(\pi\pi')_*\mathcal{F}. \tag{2.3}$$

Similarly, for a continuous morphism $\pi : X' \to X$ we obtain the spectral sequence [Mil80, Thm. III.1.18(a)]

$$E_2^{p,q} = \mathrm{H}^p(X, (R^q\pi_*)(\mathcal{F})) \Rightarrow \mathrm{H}^{p+q}(X', \mathcal{F}), \tag{2.4}$$

where $\mathcal{F}$ is a sheaf on X'.

Applications of these spectral sequences are many.

(1) Assume that X is a scheme over a field k equipped with the étale topology. Let $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$. Let us apply (2.4) to the structure morphism $\pi : X \to \mathrm{Spec}(k)$ and a sheaf $\mathcal{F}$ on X. To simplify notation we denote the inverse image of $\mathcal{F}$ on $X^{\mathrm{s}} = X \times_k k_{\mathrm{s}}$ also by $\mathcal{F}$. As we have seen in Section 2.2.2, the sheaf $(R^n\pi_*)(\mathcal{F})$ on $\mathrm{Spec}(k)$ corresponds to the Γ-module $\mathrm{H}^n(X^{\mathrm{s}}, \mathcal{F})$. Therefore, we obtain the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q(X^{\mathrm{s}}, \mathcal{F})) \Rightarrow \mathrm{H}^{p+q}(X, \mathcal{F}). \tag{2.5}$$

(2) Let $\pi : X_{\text{Ét}} \to X_{\text{ét}}$ be the continuous morphism induced by the identity on X. For a sheaf $\mathcal{F}$ on $X_{\text{ét}}$ there is a canonical isomorphism $\mathrm{H}^n_{\text{ét}}(X, \mathcal{F}) = \mathrm{H}^n_{\text{Ét}}(X, \pi^*\mathcal{F})$, see [Tam94, Thm. II. 3.3.1] or [Mil80, Prop. III.3.1]. Since π is induced by the identity on X, the functor π_* is clearly exact. Thus for any sheaf $\mathcal{G}$ on $X_{\text{Ét}}$ the spectral sequence (2.4) gives a canonical isomorphism

$$\mathrm{H}^n_{\text{ét}}(X, \pi_*\mathcal{G}) \xrightarrow{\sim} \mathrm{H}^n_{\text{Ét}}(X, \mathcal{G}).$$

In particular, if $\mathcal{G} = G_X$ is the sheaf on $X_{\text{Ét}}$ represented by a commutative group scheme G over X, then $\pi_*\mathcal{G}$ is the sheaf on $X_{\text{ét}}$ obtained by restricting $\mathcal{G}$ from the category of all X-schemes to the category of étale X-schemes, so $\pi_*\mathcal{G}$ is the sheaf on $X_{\text{ét}}$ represented by G. Thus we obtain a canonical isomorphism

$$\mathrm{H}^n_{\text{ét}}(X, G_X) \xrightarrow{\sim} \mathrm{H}^n_{\text{Ét}}(X, G_X). \tag{2.6}$$

For any commutative group scheme G over Y this allows one to define a natural map

$$f^* : \mathrm{H}^i_{\text{ét}}(Y, G_Y) \longrightarrow \mathrm{H}^i_{\text{ét}}(X, G_X) \tag{2.7}$$

for any morphism $f : X \to Y$, where G_X is the sheaf defined by the group X-scheme $G \times_Y X$. Indeed, in view of the canonical isomorphism (2.6) we can replace the small étale site by the big étale site. Then $f : X \to Y$ is in the underlying category of Y, so $f^* G_Y = G_X$, see Section 2.1.4. The adjunction morphism $G_Y \to f_* f^* G_Y = f_* G_X$ gives rise to the first arrow in

$$\mathrm{H}^i_{\text{Ét}}(Y, G_Y) \longrightarrow \mathrm{H}^i_{\text{Ét}}(Y, f_* G_X) \longrightarrow \mathrm{H}^i_{\text{Ét}}(X, G_X),$$

where the second arrow comes from the spectral sequence (2.4) attached to $f : X \to Y$. The map in (2.7) is defined as the composition of these two maps.

(3) Now let $\pi : X_{\text{fppf}} \to X_{\text{ét}}$ be the continuous morphism induced by the identity on X. We refer to [Mil80, Thm. III.3.9] for the following fact. If G is a smooth quasi-projective commutative group scheme over X, then $(R^i \pi_*)(G) = 0$ for $i > 0$. The Leray spectral sequence then gives isomorphisms

$$\mathrm{H}^n_{\text{ét}}(X, G) \xrightarrow{\sim} \mathrm{H}^n_{\text{fppf}}(X, G). \tag{2.8}$$

In fact, the assumption that G is quasi-projective can be dropped, see [Gro68, III, Thm. 11.7] and [Mil80, Rem. 3.11 (b)].

2.2.4 Passing to the limit

Suppose that we have a filtering projective system of quasi-compact and quasi-separated schemes X_i indexed by a set I, with affine morphisms $X_i \to X_j$ for all $i, j \in I$ such that $i \geq j$. Then there is a scheme $X = \varprojlim X_i$. Now assume that G_0 is a group scheme over X_0 for some $0 \in I$. For each $i \in I$ such that $i \geq 0$ define $G_i = G_0 \times_{X_0} X_i$. Let $G = G_0 \times_{X_0} X$. Then for any integer $n \geq 0$ the natural homomorphism

$$\varinjlim \mathrm{H}^n_{\text{ét}}(X_i, G_i) \xrightarrow{\sim} \mathrm{H}^n_{\text{ét}}(X, G)$$

is an isomorphism [SGA4, VII, Cor. 5.9], see also [Mil80, Ch. III, Lemma 1.16, Remark 1.17 (a)]. In particular, we have natural isomorphisms

$$\varinjlim \mathrm{H}^n_{\text{ét}}(X_i, \mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_m).$$

2.3 Cohomological purity

2.3.1 Absolute purity with torsion coefficients

Let X be a scheme. We write $\mathcal{D}^+(X_{\text{ét}})$ for the derived category of bounded below complexes of étale sheaves of abelian groups on X. Similarly, we write $\mathcal{D}^+(X_{\text{ét}}, \mathbb{Z}/\ell^m)$ for the derived category of bounded below complexes of étale

$\mathbb{Z}/\ell^m$-sheaves on X (for the comparison of the corresponding derived functors see [Mil80, Ch. III, Exercise 2.25]). A standard reference for derived categories is [Wei94].

Let $\mathcal{F}$ be a sheaf of abelian groups on X. Suppose that we have a closed immersion $i : Z \to X$. Let $U \subset X$ be the complement to Z. To an étale morphism $V \to X$ one associates the abelian group

$$\mathrm{Ker}[\mathcal{F}(V) \longrightarrow \mathcal{F}(V_U)].$$

The associated sheaf vanishes on U. It is the image under i_* of a sheaf on Z which is denoted by $i^!\mathcal{F}$. The functor from X-sheaves to abelian groups that sends $\mathcal{F}$ to $(i^!\mathcal{F})(Z)$ is left exact. Its derived functors are denoted by $\mathrm{H}^n_Z(X, \mathcal{F})$ and called the cohomology groups of $\mathcal{F}$ with support in Z.

At the level of sheaves we get the functor $\mathbf{R}i^! : \mathcal{D}^+(X_{\text{ét}}) \to \mathcal{D}^+(Z_{\text{ét}})$. The cohomology sheaves $(R^n i^!)(\mathcal{F})$ of $(\mathbf{R}i^!)\mathcal{F}$ are denoted by $\mathcal{H}^n_Z(X, \mathcal{F})$. By definition, these are the derived functors of the functor from X-sheaves to Z-sheaves sending $\mathcal{F}$ to $i^!\mathcal{F}$. There is a Grothendieck spectral sequence of composed functors involving $\mathbf{R}i^!$ and the derived functor of the sections functor $\Gamma(Z, \cdot)$ (see [Mil80, p. 241]):

$$E_2^{pq} = \mathrm{H}^p(Z, \mathcal{H}^q_Z(X, \mathcal{F})) \Rightarrow \mathrm{H}^{p+q}_Z(X, \mathcal{F}). \tag{2.9}$$

Assume n is invertible on X. For $c > 0$ one defines the sheaf $\mathbb{Z}/n(c)_X := \mu_{n,X}^{\otimes c}$. For $c = 0$, one writes $\mathbb{Z}/n(0)_X = \mathbb{Z}/n_X$. For $c < 0$, one defines $\mathbb{Z}/n(c)_X$ as the sheaf which associates to an étale $Y \to X$ the group $\mathrm{Hom}_Y(\mathbb{Z}/n(-c)_Y, \mathbb{Z}/n_Y)$. For a sheaf $\mathcal{F}$ of $\mathbb{Z}/n$-modules, one defines $\mathcal{F}(c) := \mathcal{F} \otimes \mathbb{Z}/n(c)$. See [Mil80, Ch. II, §3, p. 78/79] for a general definition of Hom sheaves and tensor product sheaves.

Theorem 2.3.1 (Gabber) *Let X be a regular scheme, let $i : Z \hookrightarrow X$ be a closed regular subscheme of codimension c everywhere, let ℓ be a prime different from the residual characteristics of X and let m be a positive integer. In $\mathcal{D}^+(Z_{\text{ét}})$ we have an isomorphism $\mathbb{Z}/\ell^m \xrightarrow{\sim} (\mathbf{R}i^!)(\mathbb{Z}/\ell^m)(c)[2c]$. In particular,*

$$\mathcal{H}^n_Z(X, \mathbb{Z}/\ell^m) = 0 \text{ for } n \neq 2c, \quad (\mathbb{Z}/\ell^m)(-c)_Z \xrightarrow{\sim} \mathcal{H}^{2c}_Z(X, \mathbb{Z}/\ell^m).$$

See [Rio14, Cor. 3.1.1, p. 324]. For schemes locally of finite type over a perfect field, the theorem was proved in [SGA4, XVI, Cor. 3.9], see also [Mil80, Thm. VI.5.1].

We record a useful corollary of Theorem 2.3.1. By a *strict normal crossing divisor* we understand an effective divisor $D = D_1 + \ldots + D_r$ in a regular scheme X such that each irreducible component D_i is regular and all intersections of these components are transversal. Transversality means that at each point $x \in D$ the local equations of the components D_i containing x form a part of a regular system of parameters for the local ring $\mathcal{O}_{X,x}$. The following corollary of Gabber's absolute purity theorem is proved in [Rio14, Cor. 3.1.4, p. 324].

Corollary 2.3.2 (Gabber) *Let X be a regular scheme and let $j : U \to X$ be an open immersion such that $X \smallsetminus U$ is a strict normal crossing divisor with the irreducible components $D_1, \dots, D_r$. Let ℓ be a prime different from the residual characteristics of X. For $n \geq 1$ we have canonical isomorphisms of X-sheaves*

$$(R^n j_*)(\mathbb{Z}/\ell^m) = \bigwedge^n (R^1 j_*)(\mathbb{Z}/\ell^m) = \bigwedge^n \Big(\bigoplus_{i=1}^r (\mathbb{Z}/\ell^m)(-1)_{D_i} \Big).$$

2.3.2 Gysin exact sequence

We return to the situation where X is a regular scheme, $i : Z \hookrightarrow X$ is a closed regular subscheme of codimension c everywhere, and ℓ is a prime invertible on X. Let $U = X \smallsetminus Z$ and let $j : U \to X$ be the natural open immersion. The functor j^* has a left adjoint functor $j_!$ which is exact [Mil80, p. 78]. This implies that we have an exact sequence of étale sheaves on X:

$$0 \longrightarrow \mathbb{Z}_U \longrightarrow \mathbb{Z}_X \longrightarrow \mathbb{Z}_Z \longrightarrow 0, \tag{2.10}$$

where $\mathbb{Z}_U = j_! j^* \mathbb{Z}$ and $\mathbb{Z}_Z = i_* i^* \mathbb{Z}$, see [Mil80, p. 92]. Applying the functor $\mathcal{E}xt(\cdot, \mathcal{F})$, defined as the derived functor of the internal $\mathcal{H}om$, to (2.10) gives a long exact sequence which breaks down into isomorphisms

$$(R^{n-1} j_*)(j^* \mathcal{F}) \tilde{\longrightarrow} \mathcal{H}^n_Z(X, \mathcal{F}), \quad n \geq 2, \tag{2.11}$$

see [Mil80, p. 242]. Thus the stalk of the sheaf $\mathcal{H}^n_Z(X, \mathcal{F})$ at a geometric point $\bar{z} \in Z$ is

$$\mathcal{H}^n_Z(X, \mathcal{F})_{\bar{z}} = \mathrm{H}^{n-1}(\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,\bar{z}}) \smallsetminus \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{Z,\bar{z}}), \mathcal{F}). \tag{2.12}$$

From (2.11) we obtain canonical isomorphisms

$$j_*(\mathbb{Z}/\ell^m) = \mathbb{Z}/\ell^m, \quad (R^{2c-1} j_*)(\mathbb{Z}/\ell^m) = (\mathbb{Z}/\ell^m)(-c)_Z \tag{2.13}$$

and

$$(R^n j_*)(\mathbb{Z}/\ell^m) = 0 \text{ for } n \neq 0,\ 2c-1.$$

In view of these isomorphisms the spectral sequence

$$E_2^{pq} = \mathrm{H}^p(X, (R^q j_*)(\mathbb{Z}/\ell^m)) \Rightarrow \mathrm{H}^{p+q}(U, \mathbb{Z}/\ell^m) \tag{2.14}$$

gives rise to the *Gysin exact sequence*

$$\ldots \to \mathrm{H}^{n-2c}(Z, \mathbb{Z}/\ell^m(-c)) \to \mathrm{H}^n(X, \mathbb{Z}/\ell^m) \to \mathrm{H}^n(U, \mathbb{Z}/\ell^m) \to \mathrm{H}^{n-2c+1}(Z, \mathbb{Z}/\ell^m(-c)) \to \ldots \tag{2.15}$$

Here we used the canonical isomorphism

$$\mathrm{H}^n(X, (\mathbb{Z}/\ell^m)_Z) = \mathrm{H}^n(X, i_*(\mathbb{Z}/\ell^m)) = \mathrm{H}^n(Z, \mathbb{Z}/\ell^m)$$

coming from the spectral sequence $\mathrm{H}^p(X, R^q i_*(\mathbb{Z}/\ell^m)) \Rightarrow \mathrm{H}^{p+q}(Z, \mathbb{Z}/\ell^m)$. Indeed, i_* is an exact functor, because the closed immersion $i : Z \to X$ is a finite morphism.

Alternatively, the Gysin sequence can be obtained as follows. Consider the long exact sequence

$$\ldots\to \mathrm{H}^n_Z(X,\mathbb{Z}/\ell^m)\to \mathrm{H}^n(X,\mathbb{Z}/\ell^m)\to \mathrm{H}^n(U,\mathbb{Z}/\ell^m)\to \mathrm{H}^{n+1}_Z(X,\mathbb{Z}/\ell^m)\to \ldots,$$

obtained by applying $\mathrm{Ext}_X(\cdot,\mathbb{Z}/\ell^m)$ to (2.10). Then one identifies $\mathrm{H}^n_Z(X,\mathbb{Z}/\ell^m)$ with $\mathrm{H}^{n-2c}(Z,\mathbb{Z}/\ell^m(-c))$ using the spectral sequence (2.9) and Theorem 2.3.1.

2.3.3 Cohomology of a henselian discrete valuation ring

Let A be a henselian discrete valuation ring with fraction field K and residue field k. If we set

$$X=\mathrm{Spec}(A),\quad Z=\mathrm{Spec}(k),\quad U=\mathrm{Spec}(K),$$

then $i:\mathrm{Spec}(k)\to\mathrm{Spec}(A)$ is a closed immersion of regular schemes of codimension $c=1$, so this is a particular case of the situation considered in the previous section. By Section 2.2.2 the étale cohomology groups of $\mathrm{Spec}(k)$ and $\mathrm{Spec}(K)$ coincide with Galois cohomology groups of k and K, respectively. We now describe how to interpret the étale cohomology of $\mathrm{Spec}(A)$ in terms of Galois cohomology.

As before, let $G=\mathrm{Gal}(K_{\mathrm{s}}/K)$, $I=\mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{nr}})$, $\Gamma=\mathrm{Gal}(K_{\mathrm{nr}}/K)=G/I$, where $K_{\mathrm{nr}}\subset K_{\mathrm{s}}$ is the maximal unramified extension of K, so K_{nr} is the field of fractions of the strict henselisation A^{sh}. The category of étale sheaves on $\mathrm{Spec}(A)$ is equivalent to the category of triples (M,N,φ), where M is a Γ-module, N is a G-module, and $\varphi: M\to N^I$ is a homomorphism of Γ-modules [Mil80, Example II.3.12]. A morphism of triples

$$(M,N,\varphi)\to(M',N',\varphi')$$

is a pair consisting of a map of Γ-modules $M\to M'$ and a map of G-modules $N\to N'$ such that the obvious resulting diagram is commutative. To a sheaf $\mathcal{F}$ on $\mathrm{Spec}(A)$ one associates the triple $(i^*\mathcal{F},j^*\mathcal{F},\varphi)$, where φ is the natural morphism $i^*\mathcal{F}\to i^*j_*j^*\mathcal{F}$. This agrees with the definition of triples, because the stalk of the $\mathrm{Spec}(A)$-sheaf j_*N at $\mathrm{Spec}(k_{\mathrm{s}})$ is computed at the strict henselisation, see (2.1), thus the $\mathrm{Spec}(k)$-sheaf i^*j_*N corresponds to the Γ-module N^I. In particular, the $\mathrm{Spec}(A)$-sheaf j_*M, where M is a G-module, corresponds to the triple (M^I,M,id).

Let $\mathcal{F}(M,N,\varphi)$ be the sheaf on $\mathrm{Spec}(A)$ corresponding to the triple (M,N,φ). It can be constructed as the fibred product of i_*M and j_*N over $i_*i^*j_*N$, see [Mil80, Thm. II.3.10]. The constant $\mathrm{Spec}(A)$-sheaf $\mathbb{Z}$ corresponds to the triple $(\mathbb{Z},\mathbb{Z},\mathrm{id})$, thus the group of sections of $\mathcal{F}(M,N,\varphi)$ is M^Γ. It follows that

$$\mathrm{H}^i(\mathrm{Spec}(A),\mathcal{F}(M,N,\varphi))=\mathrm{H}^i(k,M). \tag{2.16}$$

2.3.4 Gysin sequence: residues and functoriality

We continue the discussion of the previous section keeping the same notation. Let ℓ be a prime not equal to $\mathrm{char}(k)$. Then μ_{ℓ^m}, where m is a positive integer, is an étale sheaf on $\mathrm{Spec}(A)$. By (2.16) we have

$$\mathrm{H}^n(\mathrm{Spec}(A), \mu_{\ell^m}) = \mathrm{H}^n(k, \mu_{\ell^m})$$

for any $n \geq 1$. Thus, after twisting, the Gysin sequence (2.15) becomes the exact sequence

$$\ldots \to \mathrm{H}^n(k, \mu_{\ell^m}) \to \mathrm{H}^n(K, \mu_{\ell^m}) \to \mathrm{H}^{n-1}(k, \mathbb{Z}/\ell^m) \to \mathrm{H}^{n+1}(k, \mu_{\ell^m}) \to \ldots \quad (2.17)$$

This looks very similar to the exact sequence (1.9) with $C = \mu_{\ell^m}$:

$$0 \longrightarrow \mathrm{H}^n(k, \mu_{\ell^m}) \longrightarrow \mathrm{H}^n(K, \mu_{\ell^m}) \xrightarrow{r} \mathrm{H}^{n-1}(k, \mathbb{Z}/\ell^m) \longrightarrow 0. \quad (2.18)$$

These two sequences are indeed the same, at least up to inverting the sign of the residue map r.

Lemma 2.3.3 *The long exact sequences* (2.17) *and* (2.18) *coincide, after replacing r with $-r$.*

Proof. We need to check that these sequences come from identical spectral sequences. In our case the spectral sequence (2.14) has the form

$$\mathrm{H}^p(\mathrm{Spec}(A), (R^q j_*)(\mu_{\ell^m})) \Rightarrow \mathrm{H}^{p+q}(K, \mu_{\ell^m}). \quad (2.19)$$

whereas the Hochschild–Serre spectral sequence is

$$\mathrm{H}^p(\Gamma, \mathrm{H}^q(I, \mu_{\ell^m})) \Rightarrow \mathrm{H}^{p+q}(G, \mu_{\ell^m}).$$

On the one hand, the Hochschild–Serre spectral sequence is the spectral sequence of composed functors: the functor $M \mapsto M^I$ from continuous G-modules to continuous Γ-modules, followed by the functor of Γ-invariants. On the other hand, the spectral sequence (2.19) is the spectral sequence of composed functors j_* from $\mathrm{Spec}(K)$-sheaves to $\mathrm{Spec}(A)$-sheaves, followed by the functor of sections from $\mathrm{Spec}(A)$-sheaves to abelian groups. As we have seen in the previous section, the dictionary between étale $\mathrm{Spec}(A)$-sheaves and triples interprets the first of these as the functor sending a G-module M to the triple (M^I, M, id). The functor of sections sends this to M^G, which shows that the spectral sequences are indeed identical.

Finally, to compare the residue map r in (2.18) with the corresponding arrow in (2.17) we need to make sure that the identification of $\mathrm{Hom}(I, \mu_{\ell^m})$ with $\mathbb{Z}/\ell^m$ on the Hochschild–Serre side is compatible with the identification of $(R^1 j_*)(\mu_{\ell^m})$ with $(\mathbb{Z}/\ell^m)_{\mathrm{Spec}(k)}$ in (2.13). Since ℓ is coprime to the characteristic of k, the field K_{nr} contains roots of unity of degree ℓ^m and we have $\mathrm{Hom}(I, \mu_{\ell^m}) = \mathrm{Hom}(\mathrm{Gal}(K_{\mathrm{t}}/K_{\mathrm{nr}}), \mu_{\ell^m})$, where $K_{\mathrm{t}} \subset K_{\mathrm{s}}$ is the maximal tamely ramified extension of K. If π is a uniformiser of K, then the action of I on the K_{s}-points of the torsor $t^{\ell^m} = \pi$ for μ_{ℓ^m} factors as

$$I = \mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{nr}}) \longrightarrow \mathrm{Gal}(K_{\mathrm{t}}/K_{\mathrm{nr}}) \longrightarrow \mathrm{Gal}(K_{\mathrm{nr}}(\pi^{1/\ell^m})/K_{\mathrm{nr}}) = \mu_{\ell^m}.$$

This homomorphism corresponds to $1 \in \mathbb{Z}/\ell^m$ on the Hochschild–Serre side. As explained in [Rio14, p. 324], the isomorphism $(R^1 j_*)(\mu_{\ell^m}) = (\mathbb{Z}/\ell^m)_{\mathrm{Spec}(k)}$ in Gabber's absolute purity theorem is induced by the section of $(R^1 j_*)(\mu_{\ell^m})$ which is the *opposite* of the class of the torsor $t^{\ell^m} = \pi$. This finishes the proof of the lemma. $\square$

We now make some observations regarding the functoriality of the Gysin sequence.

Let $f : X' \to X$ be a morphism of integral regular schemes. Let $Z \subset X$ and $Z' \subset X'$ be regular integral closed subschemes of codimension 1 such that $f(Z') \subset Z$. Let $U = X \smallsetminus Z$ and $U' = X' \smallsetminus Z'$. Assume that $f(U') \subset U$, so that there is a commutative diagram

$$\begin{array}{ccccc} U' & \xrightarrow{j'} & X' & \xleftarrow{i'} & Z' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X & \xleftarrow{i} & Z \end{array}$$

Since $f(X')$ is not contained in Z, we have a well defined divisor $f^{-1}(Z) \subset X'$ supported on Z'. Thus we can write $f^{-1}(Z) = rZ'$, where r is a positive integer. Explicitly, since X is regular, any point of Z has an open affine neighbourhood $V \subset X$ such that $Z \cap V$ is the zero set of a regular function on V. If π is a local equation of $Z \subset X$ in such an open set V, where $V \cap f(X') \neq \emptyset$, then π gives rise to a non-zero rational function on X'; moreover, $v_{Z'}(\pi) = r$, where $v_{Z'}$ is the valuation of the discrete valuation ring $\mathcal{O}_{X',Z'}$.

Lemma 2.3.4 *Let ℓ be a prime invertible on X. There is a commutative diagram*

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathrm{H}^n(X', \mu_{\ell^m}) & \longrightarrow & \mathrm{H}^n(U', \mu_{\ell^m}) & \longrightarrow & \mathrm{H}^{n-1}(Z', \mathbb{Z}/\ell^m) & \longrightarrow & \dots \\ & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle [r]f^*} & & \\ \dots & \longrightarrow & \mathrm{H}^n(X, \mu_{\ell^m}) & \longrightarrow & \mathrm{H}^n(U, \mu_{\ell^m}) & \longrightarrow & \mathrm{H}^{n-1}(Z, \mathbb{Z}/\ell^m) & \longrightarrow & \dots \end{array}$$

Proof. By the construction of the Gysin sequence, the bottom row comes from the spectral sequence of composed functors $\mathbf{R}j_* : \mathcal{D}^+(U_{\text{ét}}) \to \mathcal{D}^+(X_{\text{ét}})$ and the sections functor $\mathbf{R}\Gamma : \mathcal{D}^+(X_{\text{ét}}) \to \mathcal{D}^+(\mathrm{Ab})$, where Ab is the category of abelian groups (and similarly for the top row). From the functoriality of the spectral sequence and the purity theorem we obtain the commutative diagram as above, where we only need to identify the map linking $\mathrm{H}^{n-1}(Z, \mathbb{Z}/\ell^m)$ and $\mathrm{H}^{n-1}(Z', \mathbb{Z}/\ell^m)$.

A canonical adjunction morphism $\mu_{\ell^m} \to (\mathbf{R}f_*)\mu_{\ell^m}$ in $\mathcal{D}^+(U_{\text{ét}})$ induces a morphism in $\mathcal{D}^+(X_{\text{ét}})$:

$$(\mathbf{R}j_*)\mu_{\ell^m} \longrightarrow (\mathbf{R}j_*)(\mathbf{R}f_*)\mu_{\ell^m} = (\mathbf{R}(jf)_*)\mu_{\ell^m} = (\mathbf{R}f_*)(\mathbf{R}j'_*)\mu_{\ell^m}.$$

Since f^* and f_* are adjoint functors, we obtain a canonical morphism in $\mathcal{D}^+(X'_{\text{ét}})$:

$$f^*(\mathbf{R}j_*)\mu_{\ell^m} \longrightarrow (\mathbf{R}j'_*)\mu_{\ell^m}.$$

We need to compute the induced map $f^*(R^1j_*)\mu_{\ell^m}\to(R^1j'_*)\mu_{\ell^m}$. Recall from the previous section that $(R^1j_*)\mu_{\ell^m}$ is identified with $(\mathbb{Z}/\ell^m)_Z$ in such a way that $1 \in \Gamma(Z, \mathbb{Z}/\ell^m)$ corresponds to the negative of the class of the torsor given by $t^{\ell^m} = \pi$. Since $v_{Z'}(\pi) = r$, the map $\mathrm{H}^{n-1}(Z, \mathbb{Z}/\ell^m)\to\mathrm{H}^{n-1}(Z', \mathbb{Z}/\ell^m)$ in the diagram is $[r]f^*$. □

2.4 H^1 with coefficients $\mathbb{Z}$ and $\mathbb{G}_m$

Lemma 2.4.1 *Let X be a scheme. Let L be a field and let $f : \mathrm{Spec}(L)\to X$ be a morphism. We have the following properties:*

(i) $\mathrm{H}^1_{\text{ét}}(X, f_*\mathbb{Z}_L) = 0$*;*
(ii) $\mathrm{H}^1_{\text{ét}}(X, f_*\mathbb{G}_{m,L}) = 0$*;*
(iii) $R^1f_*\mathbb{Z}_L = 0$*;*
(iv) $R^1f_*\mathbb{G}_{m,L} = 0$.

If $\mathcal{F}$ is a sheaf on $\mathrm{Spec}(L)$*, then for any $i \geq 1$*

(v) *the sheaf $R^if_*\mathcal{F}$ is a torsion sheaf;*
(vi) *the group $\mathrm{H}^i_{\text{ét}}(X, f_*\mathcal{F})$ is a torsion group.*

Proof. The spectral sequence (2.4) gives an injection

$$\mathrm{H}^1_{\text{ét}}(X, f_*(\mathcal{F})) \hookrightarrow \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(L), \mathcal{F}).$$

Statements (i) and (ii) then follow since $\mathrm{H}^1(L, \mathbb{Z}_L) = 0$ and $\mathrm{H}^1(L, \mathbb{G}_{m,L}) = 0$ (Hilbert's theorem 90).

The sheaf $R^1f_*\mathbb{Z}_L$ is associated to the presheaf sending an étale open set $U\to X$ to $\mathrm{H}^1_{\text{ét}}(U \times_X \mathrm{Spec}(L), \mathbb{Z}_L)$. But this group is zero, because $U \times_X \mathrm{Spec}(L)$ is either empty or the spectrum of a finite product of fields, and $\mathrm{H}^1_{\text{ét}}(E, \mathbb{Z}_E) = 0$ when E is a field. This proves (iii).

A similar argument, which uses Hilbert's theorem 90, proves (iv).

The sheaf $R^if_*\mathcal{F}$ is associated to the presheaf which sends an open set $U\to X$ to $\mathrm{H}^i_{\text{ét}}(U \times_X \mathrm{Spec}(L), \mathcal{F})$. This group is a direct sum of Galois cohomology groups, which are torsion groups for $i \geq 1$. This proves (v).

In our case the spectral sequence (2.4) takes the form

$$E_2^{pq} = \mathrm{H}^p_{\text{ét}}(X, R^qf_*\mathcal{F}) \Rightarrow \mathrm{H}^n_{\text{ét}}(\mathrm{Spec}(L), \mathcal{F}).$$

By part (v) the terms E_2^{pq} are torsion groups when $q \geq 1$. It follows that the kernel of the natural map

$$\mathrm{H}^i_{\text{ét}}(X, f_*\mathcal{F}) \longrightarrow \mathrm{H}^i_{\text{ét}}(\mathrm{Spec}(L), \mathcal{F})$$

is a torsion group. But $\mathrm{H}^i(\mathrm{Spec}(L), \mathcal{F})$ is also a torsion group for $i \geq 1$, so statement (vi) follows. □

Proposition 2.4.2 *Let X be a normal scheme. Then* $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}_X) = 0$.

Proof. Here $\mathbb{Z}_X$ is the sheaf associated to the constant presheaf $\mathbb{Z}$. We may assume that X is irreducible. Let $i : \eta \to X$ be the generic point of X. We have the natural map $\mathbb{Z}_X \to i_* \mathbb{Z}_\eta$. We claim it is an isomorphism. Indeed, let $U \to X$ be an étale morphism. Then U is normal. If it is connected, then it is integral. This shows that the map $\mathbb{Z}_X \to i_* \mathbb{Z}_\eta$ is an isomorphim. Then Lemma 2.4.1 (i) gives $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}_X) = 0$. □

The Proposition holds more generally under the assumption that X is geometrically unibranch.

2.5 The Picard group and the Picard scheme

Definition 2.5.1 *The* **Picard group** $\mathrm{Pic}(X)$ *of a scheme X is the group of invertible coherent sheaves of $\mathcal{O}_X$-modules, considered up to isomorphism.*

By this definition we have

$$\mathrm{Pic}(X) = \mathrm{H}^1_{\text{zar}}(X, \mathcal{O}_X^*) = \mathrm{H}^1_{\text{zar}}(X, \mathbb{G}_{m,X}).$$

Let $\pi : X_{\text{ét}} \to X_{\text{zar}}$ be the continuous morphism induced by the identity on X. We have $(R^1\pi_*)(\mathbb{G}_m) = 0$; this is Grothendieck's version of Hilbert's theorem 90, see [Mil80, Prop. III.4.9]. The Leray spectral sequence then entails a canonical isomorphism

$$\mathrm{Pic}(X) = \mathrm{H}^1_{\text{zar}}(X, \mathbb{G}_{m,X}) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_{m,X}). \tag{2.20}$$

The same is true for $\mathrm{H}^1_{\text{fppf}}(X, \mathbb{G}_{m,X})$. Alternatively, to an invertible sheaf $\mathcal{L}$ one directly associates a torsor T for $\mathbb{G}_{m,X}$ defined by $T(U) = \mathrm{Isom}_U(\mathcal{O}_U, f^*\mathcal{L})$, where $f : U \to X$ is étale. This gives an equivalence of the category of invertible sheaves of $\mathcal{O}_X$-modules and the category of étale X-torsors for $\mathbb{G}_{m,X}$, see [SGA4$\frac{1}{2}$, Arcata, Prop. II.2.3].

The rest of this section is based on Kleiman's excellent survey [Kle05], see also [BLR90, Ch. 8]. Fix a noetherian base scheme S and let $f : X \to S$ be a separated morphism of finite type. For an S-scheme T we write $X_T = X \times_S T$ and write $f_T : X_T \to T$ for the projection to T.

The *relative Picard functor* $\mathrm{Pic}_{X/S}$ is defined as follows:

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X_T)/\mathrm{Pic}(T).$$

Let $\mathrm{Pic}_{(X/S)\,\mathrm{Zar}}$, $\mathrm{Pic}_{(X/S)\,\text{Ét}}$, $\mathrm{Pic}_{(X/S)\,\mathrm{fppf}}$ be the associated sheaves in the big Zariski, big étale, and fppf topologies.

Proposition 2.5.2 *Assume that for any S-scheme T the canonical adjunction morphism $\mathcal{O}_T \to f_{T*} f_T^* \mathcal{O}_S = f_{T*}\mathcal{O}_{X_T}$ is an isomorphism. Then the following natural maps of presheaves on the category of schemes locally of finite type over S are injective:*

$$\mathrm{Pic}_{X/S} \hookrightarrow \mathrm{Pic}_{(X/S)\,\mathrm{Zar}} \hookrightarrow \mathrm{Pic}_{(X/S)\,\text{Ét}} \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,\mathrm{fppf}}, \tag{2.21}$$

and the last map is an isomorphism. The first two maps in (2.21) are isomorphisms if f has a section. The second map is an isomorphism if f has a section locally in the Zariski topology.

Proof. This is [Kle05, Thm. 2.5]. We sketch the proof given in [Kle05, Remark 2.11] because it is a good illustration of the use of the spectral sequence (2.4).

Take an S-scheme T. The Zariski sheaf on T, which is associated to the presheaf sending Z/T to $\mathrm{H}^1_{\mathrm{Zar}}(X_Z, \mathbb{G}_{m,X_Z})$, is $R^1 f_{T*}\mathbb{G}_{m,X_T}$. Hence

$$\mathrm{Pic}_{(X/S)\,\mathrm{Zar}}(T) = \mathrm{H}^0_{\mathrm{Zar}}(T, R^1 f_{T*}\mathbb{G}_{m,X_T}).$$

The morphism $f_T : X_T \to T$ gives rise to the spectral sequence (2.4):

$$\mathrm{H}^p_{\mathrm{Zar}}(T, R^q f_{T*}\mathbb{G}_{m,X_T}) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{Zar}}(X_T, \mathbb{G}_{m,X_T}).$$

The assumption $\mathcal{O}_T \xrightarrow{\sim} f_{T*}\mathcal{O}_{X_T}$ implies $\mathbb{G}_{m,T} \xrightarrow{\sim} f_*\mathbb{G}_{m,X_T}$. Hence the beginning of the exact sequence of low degree terms of the spectral sequence is

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{(X/S)\,\mathrm{Zar}}(T) \to \mathrm{H}^2_{\mathrm{Zar}}(T, \mathbb{G}_{m,T}) \to \mathrm{H}^2_{\mathrm{Zar}}(X_T, \mathbb{G}_{m,X_T}),$$

proving the injectivity of $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{(X/S)\,\mathrm{Zar}}$. A section of f induces a retraction of each canonical map

$$\mathrm{H}^n_{\mathrm{Zar}}(T, \mathbb{G}_{m,T}) \xrightarrow{\sim} \mathrm{H}^n_{\mathrm{Zar}}(X_T, \mathbb{G}_{m,X_T}),$$

which is therefore injective. This implies that the first map in (2.21) is an isomorphism.

Using (2.20), the same arguments apply to the étale and fppf topologies. Hence we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
\mathrm{Pic}(T) & \to & \mathrm{Pic}(X_T) & \to & \mathrm{Pic}_{(X/S)\,\mathrm{Zar}}(T) & \to & \mathrm{H}^2_{\mathrm{Zar}}(T, \mathbb{G}_m) & \to & \mathrm{H}^2_{\mathrm{Zar}}(X_T, \mathbb{G}_m) \\
\| & & \| & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Pic}(T) & \to & \mathrm{Pic}(X_T) & \to & \mathrm{Pic}_{(X/S)\,\text{Ét}}(T) & \to & \mathrm{H}^2_{\text{Ét}}(T, \mathbb{G}_m) & \to & \mathrm{H}^2_{\text{Ét}}(X_T, \mathbb{G}_m) \\
\| & & \| & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Pic}(T) & \to & \mathrm{Pic}(X_T) & \to & \mathrm{Pic}_{(X/S)\,\mathrm{fppf}}(T) & \to & \mathrm{H}^2_{\mathrm{fppf}}(T, \mathbb{G}_m) & \to & \mathrm{H}^2_{\mathrm{fppf}}(X_T, \mathbb{G}_m)
\end{array}$$

The injectivity of $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{(X/S)\,\text{Ét}}$ formally implies the injectivity of

$$\mathrm{Pic}_{(X/S)\,\mathrm{Zar}} \longrightarrow \mathrm{Pic}_{(X/S)\,\text{Ét}},$$

since the latter map is obtained from the former by passing from presheaves to associated Zariski sheaves, and this operation preserves injectivity by the exactness of the functor a from presheaves to sheaves [Mil80, Thm. 2.15 (a)].

In view of (2.8), the Five Lemma applied to the two lower rows of the diagram gives an isomorphism $\mathrm{Pic}_{(X/S)\,\text{Ét}} \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,\mathrm{fppf}}$. □

Remark 2.5.3 If $f : X \to S$ is flat and proper with geometrically integral fibres, then for any morphism $T \to S$, the map $\mathcal{O}_T \to f_{T*}\mathcal{O}_{X_T}$ is an isomorphism. (See [Kle05, Exercise 9.3.11].) This applies for instance when $S = \mathrm{Spec}(k)$ is the spectrum of a field and X is a proper, geometrically integral variety over k.

The following proposition shows that the condition that $\mathcal{O}_T \to f_{T*}\mathcal{O}_{X_T}$ is an isomorphism holds for any *flat* S-scheme T if it holds for $T = S$.

Proposition 2.5.4 *Let $f : X \to S$ be a separated morphism of noetherian schemes such that $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism. Then for any flat scheme $T \to S$ the map $\mathcal{O}_T \to f_{T*}\mathcal{O}_{X_T}$ is an isomorphism.*

Proof. The statement is local on S and T. We may thus assume $S = \mathrm{Spec}(A)$ and $T = \mathrm{Spec}(B)$ with B flat over A. Since X is separated, we can write X as a finite union $X = \cup_i X_i$ of affine open sets $X_i = \mathrm{Spec}(A_i)$ with affine intersections $X_{ij} = \mathrm{Spec}(A_{ij})$. We have the obvious exact sequence of A-modules

$$0 \longrightarrow \mathrm{H}^0(X, \mathcal{O}_X) \longrightarrow \prod_i A_i \longrightarrow \prod_{ij} A_{ij}.$$

The hypothesis that $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism then gives the exactness of the sequence of A-modules

$$0 \longrightarrow A \longrightarrow \prod_i A_i \longrightarrow \prod_{ij} A_{ij}.$$

Since B is flat over A, we have an exact sequence of B-modules

$$0 \longrightarrow B \longrightarrow \prod_i A_i \otimes_A B \longrightarrow \prod_{ij} A_{ij} \otimes_A B.$$

The scheme $X_T = X \times_S T$ is covered by open subsets $X_i \times_S T = \mathrm{Spec}(A_i \otimes_A B)$ with intersections $X_{ij} \times_S T = \mathrm{Spec}(A_{ij} \otimes_A B)$, hence we have an exact sequence

$$0 \longrightarrow \mathrm{H}^0(X_T, \mathcal{O}_{X_T}) \longrightarrow \prod_i A_i \otimes_A B \longrightarrow \prod_{ij} A_{ij} \otimes_A B.$$

Comparing the last two exact sequences, we find $\mathrm{H}^0(T, \mathcal{O}_T) = B = \mathrm{H}^0(X_T, \mathcal{O}_{X_T})$, thus $\mathcal{O}_T(T) \to f_{T*}\mathcal{O}_{X_T}(T)$ is an isomorphism. The same argument holds for any Zariski open subset of T. We thus obtain an isomorphism $\mathcal{O}_T \xrightarrow{\sim} f_{T*}\mathcal{O}_{X_T}$. □

Remark 2.5.5 This result is a particular case of the following general statement. Let $f : X \to S$ be a quasi-compact and quasi-separated morphism and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then the formation of the direct image sheaves $R^i f_*\mathcal{F}$, where $i \geq 0$, commutes with flat base change over S. See (EGA III, Prop. 1.4.15) and [Stacks, Lemma 02KH].

If any of the functors $\mathrm{Pic}_{X/S}$, $\mathrm{Pic}_{(X/S)\,\mathrm{Zar}}$, $\mathrm{Pic}_{(X/S)\,\mathrm{\acute{E}t}}$, $\mathrm{Pic}_{(X/S)\,\mathrm{fppf}}$ is representable, then the representing scheme (which is uniquely determined) is called the *Picard scheme* and is denoted by $\mathbf{Pic}_{X/S}$.

The main existence theorem for $\mathbf{Pic}_{X/S}$ is the following result of Grothendieck, see [Kle05, Thm. 4.8] for a slightly stronger statement.

Theorem 2.5.6 *Assume $f : X \to S$ is projective and flat with integral geometric fibres. Then the scheme $\mathbf{Pic}_{X/S}$ representing $\mathrm{Pic}_{(X/S)\,\text{Ét}}$ exists, is separated and locally of finite type over S.*

For $f : X \to S$ is projective and flat, Mumford showed that $\mathbf{Pic}_{X/S}$ exists if the condition that the geometric fibres are integral is weakened to the condition that the geometric fibres are reduced and connected, provided that the irreducible components of all fibres are geometrically irreducible, see [Kle05, Thm. 4.18.1]. Another important result of Grothendieck is the following theorem [Kle05, Thm. 4.18.2, Cor. 4.18.3].

Theorem 2.5.7 *Assume that S is integral and $X \to S$ is proper. Then there is a non-empty open subset $V \subset S$ such that $\mathbf{Pic}_{X_V/V}$ exists, represents $\mathrm{Pic}_{(X_V/V)\,\mathrm{fppf}}$, and is a disjoint union of open quasi-projective schemes. In particular, this holds for $S = \mathrm{Spec}(k)$, where k is a field.*

Corollary 2.5.8 *Let X be a proper and geometrically integral variety over a field k. Then for any k-scheme T there is an exact sequence of abelian groups*

$$0 \longrightarrow \mathrm{Pic}_{X/k}(T) \longrightarrow \mathbf{Pic}_{X/k}(T) \longrightarrow \mathrm{Br}(T) \longrightarrow \mathrm{Br}(X_T). \qquad (2.22)$$

If $X(k) \neq \emptyset$, then $\mathbf{Pic}_{X/k}(T) = \mathrm{Pic}_{X/k}(T)$ for any k-scheme T, so that the Picard group scheme $\mathbf{Pic}_{X/k}$ represents the relative Picard functor $\mathrm{Pic}_{X/k}$.

Proof. By the representability of $\mathrm{Pic}_{(X/k)\,\text{Ét}}$ we obtain (2.22) from the middle row of the commutative diagram in the proof of Proposition 2.5.2. If $X(k) \neq \emptyset$, then the morphism $X_T \to T$ has a section, so that the map $\mathrm{Br}(T) \longrightarrow \mathrm{Br}(X_T)$ is injective. □

Corollary 2.5.9 *Let X be a proper and geometrically integral variety over a field k. Then there is an exact sequence of abelian groups*

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathbf{Pic}_{X/k}(k) \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X). \qquad (2.23)$$

If K is a finite Galois extension of k with Galois group $G = \mathrm{Gal}(K/k)$ such that $X(K) \neq \emptyset$, then we have a canonical isomorphism

$$\mathbf{Pic}_{X/k}(k) \cong \mathrm{Pic}(X_K)^G.$$

Proof. The exact sequence (2.23) is obtained from (2.22) by taking $T = \mathrm{Spec}(k)$. Taking $T = \mathrm{Spec}(K)$ in (2.22), we obtain a compatible exact sequence

$$0 \longrightarrow \mathrm{Pic}(X_K) \longrightarrow \mathbf{Pic}_{X/k}(K) \longrightarrow \mathrm{Br}(K) \longrightarrow \mathrm{Br}(X_K).$$

This is also a sequence of G-modules. Since $X(K) \neq \emptyset$, Corollary 2.5.8 gives an isomorphism $\mathrm{Pic}(X_K) \xrightarrow{\sim} \mathbf{Pic}_{X/k}(K)$. For the group k-scheme $\mathbf{Pic}_{X/k}$, we have $\mathbf{Pic}_{X/k}(k) = \mathbf{Pic}_{X/k}(K)^G$. □

2.6 Appendix. The language of stacks

This section will only be used in our sketch of de Jong's proof of Gabber's theorem in Section 3.3. Our goal here is to give a very short list of key concepts with some examples. This is not a replacement for a detailed introduction to stacks, algebraic spaces and gerbes, for which we refer the reader to a very helpful book by Olsson [Ols16], see also [Gir71] and [Vis05].

2.6.1 Fibred categories

We start with the definition of a fibred category [Ols16, §3.1].

Let C be a category. (We shall be mostly interested in the case when C is the category of schemes over a base scheme S.) A *category over* C is a category F together with a functor $p : F \to C$. For an object U of C define the *fibre* $F(U)$ over U as the category whose objects are the objects u of F over U, i.e. such that $p(u) = U$, and whose morphisms are morphisms in F that lift $\mathrm{id} : U \to U$.

A *fibred category over* C is a category F equipped with a functor $p : F \to C$ such that for every morphism $f : U \to V$ in C and for every $v \in F(V)$ there exist $u \in F(U)$ and a lifting $\phi : u \to v$ of f such that the following property holds. If $\psi : w \to v$ is a morphism in F such that $p(\psi) = fh$ is the precomposition of $f = p(\phi)$ with a morphism $h : p(w) \to p(u) = U$, then there exists a unique lifting $\lambda : w \to u$ of h such that $\psi = \phi\lambda$. In this case the morphism ϕ is called *cartesian* and u is called a *pullback of* v *along* f and is written $u = f^*v$.

A morphism of fibred categories $p : F \to C$ to $q : G \to C$ is a functor $g : F \to G$ sending cartesian morphisms to cartesian morphisms such that there is an equality of functors $p = q \circ g$.

Examples 1. Let X be an object of a category C. Write C/X for the localisation of C at the object X. This is the category whose objects are the pairs (Y, f) with Y an object of C and f is a morphism $Y \to X$, and the morphisms are the morphisms in C making the obvious triangles commutative. The forgetful functor $C/X \to C$ is a fibred category.

2. Let $F : C^{\mathrm{op}} \to (Sets)$ be a contravariant functor from a category C to the category of sets. Let $\mathcal{F}$ be the category of pairs (U, x), where U is an object of C and $x \in F(U)$. A morphism $(U', x') \to (U, x)$ is a morphism $g : U' \to U$ such that $F(g)x = x'$. It is easy to check that the functor $\mathcal{F} \to C$ sending (U, x) to U is a fibred category. This allows one to view presheaves as categories fibred in sets, see [Ols16, Prop. 3.2.8]. We shall return to this example in the particular case when C is the category of schemes over a base scheme S.

Categories fibred in groupoids

The reference is [Ols16, §3.4].

A fibred category $p : F \to C$ is a *category fibred in groupoids* if the fibre $F(U)$ is a groupoid for every U in C, i.e., every morphism in $F(U)$ is an isomorphism. Equivalently, every morphism in F is cartesian [Ols16, Exercise 3.D, p. 85].

Let $p : F \to C$ be a category fibred in groupoids. For X in C and for objects x_1 and x_2 in $F(X)$ define the functor

$$\underline{\mathrm{Isom}}(x_1, x_2) : (C/X)^{\mathrm{op}} \longrightarrow (Sets)$$

that associates to $f : Y \to X$ the set $\mathrm{Isom}_{F(Y)}(f^*x_1, f^*x_2)$, for some chosen pullbacks f^*x_1 and f^*x_2 along f. The definition of a category fibred in groupoids then implies that a morphism $g : Z \to Y$ gives rise to a canonical map

$$\underline{\mathrm{Isom}}(x_1, x_2)(f : Y \to X) \longrightarrow \underline{\mathrm{Isom}}(x_1, x_2)(fg : Z \to X),$$

so this is indeed a functor. Up to canonical isomorphism it does not depend on the choice of pullbacks.

As a particular case, for an object x of $F(X)$ we get a functor

$$\underline{\mathrm{Aut}}_x = \underline{\mathrm{Isom}}(x, x) : (C/X)^{\mathrm{op}} \longrightarrow (Groups).$$

2.6.2 Stacks

The references for this section are [Ols16, §4.2, §4.6].

Let $p : F \to C$ be a category fibred in groupoids, where C has finite fibred products. For a set of morphisms $\{X_i \to X\}$, $i \in I$, one defines $F(\{X_i \to X\})$ to be the category of *descent data*, consisting of objects E_i of $F(X_i)$, for $i \in I$, and isomorphisms $\sigma_{ij} : \mathrm{pr}_1^*(E_i) \to \mathrm{pr}_2^*(E_j)$ in $F(X_i \times_X X_j)$, for each $i, j \in I$, satisfying the standard compatibility condition on triple intersections. If the natural functor $F(X) \to F(\{X_i \to X\})$ is an equivalence of categories, then one says that the set of morphisms $\{X_i \to X\}$, $i \in I$, is of *effective descent* for F.

Now let C be a site, i.e. a category with a Grothendieck topology on it, for example, the category Sch/S of schemes with the étale topology over a base scheme S. A category fibred in groupoids $p : F \to C$ is a *stack* if for every object X and any covering family $\{X_i \to X\}$, $i \in I$, the functor $F(X) \to F(\{X_i \to X\})$ is an equivalence of categories.

Equivalently [Ols16, Prop. 4.6.2], for any covering of any X in C any descent datum with respect to this covering is effective, and $\underline{\mathrm{Isom}}(x_1, x_2)$ is a sheaf, for any x_1 and x_2 in $F(X)$. In particular, $\underline{\mathrm{Aut}}_x$ is also a sheaf.

Example 1 *The stack associated to a sheaf on a site.* A set is canonically turned into a groupoid by defining morphisms to be the identity maps on the elements of this set. In Example 2 of Section 2.6.1 we have seen that a functor $f : C^{\mathrm{op}} \to (Sets)$ naturally gives rise to a category fibred in sets over C, whose fibre over X is the set $f(X)$. This category is a stack if and only if f is a sheaf [Vis05, Prop. 4.9].

Yoneda's lemma

For an S-scheme X we have the functor of points $h_X : (Sch/S)^{\mathrm{op}} \to (Sets)$ defined by $h_X(Y) = \mathrm{Hom}_S(Y, X)$. Yoneda's lemma says that the functor of points is a fully faithful functor $Sch/S \to \mathrm{Hom}((Sch/S)^{\mathrm{op}}, (Sets))$, hence it gives an embedding of Sch/S into the category of contravariant functors from Sch/S to $(Sets)$. Moreover, for any functor $F : (Sch/S)^{\rm op}\to(Sets)$ we have a bijection

$$\mathrm{Hom}(h_X, F)\tilde{\longrightarrow}F(X)$$

given by evaluating on the object $\mathrm{id} : X\to X$ of $h_X(X)$. This allows one to replace an S-scheme X by its functor of points h_X, which is an object of a larger category.

This operation can be refined as follows. As we have seen, for an S-scheme X the category Sch/X of X-schemes is a fibred category over Sch/S, via the functor that forgets X. This is a replacement for h_X. The 2-Yoneda lemma [Ols16, §3.2] says that if $p : F\to Sch/S$ is another fibred category, then the functor

$$\xi : \mathrm{HOM}_{Sch/S}(Sch/X, F) \longrightarrow F(X)$$

that sends a morphism of fibred categories to the value of this morphism on the object $\mathrm{id} : X\to X$ of Sch/X, is an equivalence of categories.

Example 2 *The stack associated to an S-scheme.* This allows one to replace an S-scheme X by the fibred category $Sch/X\to Sch/S$. One immediately checks that this is a category fibred in groupoids, more precisely, in sets with the identity maps. Moreover, it is a stack since, by a theorem of Grothendieck, h_X is a sheaf in fpqc, hence also in fppf and big étale topologies [Vis05, Thm. 2.55] (this is also trivially true for the big Zariski topology).

2.6.3 Algebraic spaces and algebraic stacks

The definition of algebraic stacks [Ols16, §8.1] uses algebraic spaces, so we need to recall their definition too, see [Ols16, Ch. 5].

Since morphisms of schemes can be obtained by glueing morphisms on Zariski open coverings, any S-scheme X gives rise to the big Zariski sheaf h_X. Assume that S is an affine scheme and let Aff_S be the category of affine schemes over S. Let $F : \mathrm{Aff}_S^{\rm op}\to(Sets)$ be a functor which is a big Zariski sheaf. Then F is representable by a separated S-scheme if and only if

(1) the diagonal morphism $F\to F\times F$ is an affine closed embedding, and

(2) there is a family of affine S-schemes X_i and affine open embeddings $h_{X_i}\to F$ such that the map of Zariski sheaves $\coprod_i h_{X_i}\to F$ is surjective.

See [Ols16, Prop. 1.4.11]. (A map of Zariski sheaves $A\to B$ is surjective if for any affine S-scheme U and for any section in $B(U)$ there is a Zariski open covering $\{U_i\}$ of U such that the restriction of this section to each U_i is in the image of $A(U_i)\to B(U_i)$.)

Here we use the terminology that if F and G are functors $\mathrm{Aff}_S^{\rm op}\to(Sets)$, then a morphism of functors $F\to G$ has a property like "affine closed embedding" if it is representable, i.e. for every $Z\in\mathrm{Aff}_S$ and any morphism $h_Z\to G$ the fibred product functor $h_Z\times_G F$ is isomorphic to h_Y for some $Y\in\mathrm{Aff}_S$, and the resulting morphism $Y\to X$ is an affine closed embedding.

Let S be a scheme and let F be a sheaf on (Sch/S) with the étale topology. An important observation is that if the diagonal morphism $F \to F \times F$ is representable (by schemes), then any morphism $h_T \to F$, where T is an S-scheme, is representable too. This follows from the isomorpism $T \times_F Z \cong (T \times_S Z) \times_{F \times F} F$, for any S-scheme Z and any morphism $h_Z \to F$.

If in the above characterisation of S-schemes as big Zariski sheaves with certain additional properties we replace the Zariski topology with the big étale topology, we obtain the definition of an algebraic space. Namely [Ols16, Def. 5.1.10], an *algebraic space* over S is a functor X from $(Sch/S)^{\mathrm{op}}$ to $(Sets)$ which is a big étale sheaf such that

(1) the diagonal $X \to X \times_S X$ is representable by schemes, and

(2) there is a surjective étale morphism $U \to X$, where U is an S-scheme.

Condition (2) makes sense in view of the observation we made above.

Alternatively, one can define algebraic spaces as quotients of schemes by étale equivalence relations [Ols16, §5.2]. (In particular, this leads to examples of algebraic spaces which are quotients of schemes by free group actions, which may not be schemes.)

Like schemes, algebraic spaces with quasi-compact diagonal are sheaves for the fpqc and hence for fppf topology [Ols16, Thm. 5.5.2].

Consider stacks over Sch/S with étale topology. Since an algebraic space is a big étale sheaf, it gives rise to a stack (see Example 1 in Section 2.6.2). A morphism of stacks $\mathcal{X} \to \mathcal{Y}$ is called *representable* if for every algebraic space V and every morphism $V \to \mathcal{Y}$ the fibred product $\mathcal{X} \times_{\mathcal{Y}} V$ is an algebraic space.

A stack $\mathcal{X}$ is called *algebraic* (or an Artin stack) if

(1) the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, and

(2) there exists a smooth surjective morphism from an S-scheme to $\mathcal{X}$.

Property (1) is equivalent to the following property: for every S-scheme U and any two objets u_1 and u_2 in $\mathcal{X}(U)$ the sheaf $\underline{\mathrm{Isom}}(u_1, u_2)$ is an algebraic space [Ols16, Lemma 8.1.8].

An algebraic stack is a Deligne–Mumford stack if there is a surjective étale morphism from an S-scheme to it.

Important examples of algebraic stacks over the category C of S-schemes are quotient stacks [Ols16, Example 8.1.12]. If G is a smooth group S-scheme that acts on an algebraic space X over S, then $[X/G]$ is defined as the stack whose objects are triples $(T, \mathcal{P}, \pi)$, where T is an S-scheme, $\mathcal{P}$ is a sheaf of torsors for $G \times_S T$ on the big étale site of T, and $\pi : \mathcal{P} \to X \times_S T$ is a $G \times_S T$-equivariant morphism of sheaves. In the particular case when G acts trivially on S, the quotient stack $[S/G]$ is called the *classifying stack* of G and is denoted by BG.

We summarise the logical links between the concepts we discussed above in the following diagram:

$$\begin{array}{c}
(S\text{-schemes}) \\
\big\downarrow \\
(S\text{-algebraic spaces}) \\
\big\downarrow \\
(\text{algebraic stacks over } S) \\
\big\downarrow \\
(\text{stacks over } S) \\
\big\downarrow \\
(\text{categories fibred in groupoids over } S)
\end{array}$$

2.6.4 Gerbes and twisted sheaves

The references for this section are [Ols16, §12.2], [deJ], [Lie08].

Let G be a sheaf of *abelian* groups on the big étale site of Sch/S. For an S-scheme X, by an abuse of notation, we write G for the sheaf of abelian groups on Sch/X induced by G.

Gerbes

A G-*gerbe* over Sch/S is a stack $p : F \to C$ together with an isomorphism of sheaves of groups $\iota_x : G \xrightarrow{\sim} \underline{\mathrm{Aut}}_x$ for every object x in F such that the following conditions hold.

(G1) Objects exist locally: every S-scheme Y has a covering $\{f_i : Y_i \to Y\}$ such that all $F(Y_i)$ are non-empty.

(G2) Any two objects are locally isomorphic: for any objects y and y' in $F(Y)$ there exists a covering $\{f_i : Y_i \to Y\}$ such that $f_i^* y$ and $f_i^* y'$ are isomorphic in $F(Y_i)$ for all i.

(G3) For every S-scheme Y if $\sigma : y \to y'$ is an isomorphism in $F(Y)$, then the induced isomorphism $\sigma : \underline{\mathrm{Aut}}_y \to \underline{\mathrm{Aut}}_{y'}$ is compatible with the isomorphisms ι_x, that is, $\iota_{y'} = \sigma \iota_y$.

By (G1) and (G2) the sheaf $\underline{\mathrm{Isom}}(x_1, x_2)$ is a G-torsor on Sch/X, for every S-scheme X and every x_1 and x_2 in $F(X)$, see [Ols16, Remark 12.2.3].

A *morphism of gerbes* is defined as a morphism of stacks $f : F' \to F$ such that for every object x of F' the composition $G \xrightarrow{\iota_x} \underline{\mathrm{Aut}}_x \xrightarrow{f_*} \underline{\mathrm{Aut}}_{f(x)}$ is equal to $G \xrightarrow{\iota_{f(x)}} \underline{\mathrm{Aut}}_{f(x)}$. Any morphism of G-gerbes is in fact an isomorphism [Ols16, Lemma 12.2.4].

If G is a smooth group S-scheme, for example $G = \mathbb{G}_m$, then any G-gerbe on the big étale site of S is an algebraic stack [Ols16, Exercise 12.E].

Gerbe of liftings of a torsor Let us give an example of a gerbe. Consider an exact sequence of sheaves of groups (where G is abelian but not necessarily H and K) on the big étale site of a given scheme S:

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1. \tag{2.24}$$

A K-torsor P over S gives rise to the G-gerbe over Sch/S whose objects are liftings of P to an H-torsor. More precisely, consider the fibred category $\mathcal{G}_P$ over Sch/S whose objects are triples (X, R, ϵ), where X is an S-scheme, R is an H-torsor over Sch/X, and ϵ is an isomorphism of the push-forward of R along $H \to K$ (the quotient of R by G) with P. It is clear that $\mathcal{G}_P$ is a category fibred in groupoids over Sch/S, via the forgetful functor sending (X, R, ϵ) to X, and for any object x of $\mathcal{G}_P$ the sheaf $\underline{\mathrm{Aut}}_x$ is canonically isomorphic to G over Sch/X. Using the effectivity of descent for sheaves and for morphisms of sheaves one shows that $\mathcal{G}_P$ is a G-gerbe [Ols16, Prop. 12.2.6].

Gerbe associated to a cohomology class Using the previous construction, one associates a G-gerbe to any cohomology class $\alpha \in \mathrm{H}^2(S, G)$. Namely, consider an exact sequence of sheaves of *abelian* groups (2.24) such that H is *injective*. The boundary map induces an isomorphism

$$\mathrm{H}^1(S, K) \tilde{\longrightarrow} \mathrm{H}^2(S, G),$$

so α gives rise to a K-torsor over Sch/S, to which we associate the gerbe of its liftings to an H-torsor as above. The fact that morphisms of G-gerbes are isomorphisms implies that another injective resolution gives rise to an isomorphic gerbe. A theorem from Giraud's book [Gir71, Thm. IV.3.4.2 (i)] says that this gives an isomorphism between $\mathrm{H}^2(S, G)$ and the group of isomorphism classes of G-gerbes over Sch/S.

In particular, the Brauer group $\mathrm{Br}(S) = \mathrm{H}^2(S, \mathbb{G}_m)$ is identified with the isomorphism classes of $\mathbb{G}_m$-gerbes over Sch/S. For each $\alpha \in \mathrm{Br}(S)$ we denote by $\mathcal{S}_\alpha$ a $\mathbb{G}_m$-gerbe over Sch/S whose isomorphism class is defined by α.

Suppose that

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 1 \tag{2.25}$$

is an exact sequence of sheaves of abelian groups. Consider the map that sends a section of D to its inverse image under $C \to D$. This inverse image is a B/A-torsor; so we obtain a map $\mathrm{H}^0(S, D) \to \mathrm{H}^1(S, B/A)$. (According to [Gir71, III.3.5.5.1] this map is the opposite of the map defined using injective resolutions.) Next, associating to a B/A-torsor the gerbe of its liftings to a B-torsor defines a map $\mathrm{H}^1(S, B/A) \to \mathrm{H}^2(S, A)$, which is in fact a homomorphism, see [Gir71, IV.3.4.1.1]. By [Gir71, Thm. IV.3.4.2 (ii)] the above identification of $\mathrm{H}^2(S, G)$ with the isomorphism classes of G-gerbes over Sch/S is such that the composition

$$\mathrm{H}^0(S, D) \longrightarrow \mathrm{H}^1(S, B/A) \longrightarrow \mathrm{H}^2(S, A) \tag{2.26}$$

is the opposite of the map defined using injective resolutions.

If B and C in (2.25) are injective, the first map in (2.26) is surjective with kernel the image of $\mathrm{H}^0(S,C)$, and the second map is an isomorphism. Lift a cohomology class in $\mathrm{H}^2(S,A)$ to a section in $\mathrm{H}^0(S,D)$. The gerbe attached to a cohomology class in $\mathrm{H}^2(S,A)$ is isomorphic to the gerbe of liftings of this section to a section of C.

Twisted sheaves

We refer to [Ols16, Ch. 9] for the theory of quasi-coherent sheaves on algebraic stacks over a given scheme S. This uses the lisse-étale site of a stack.

Let $\pi : \mathcal{S}\to S$ be a $\mathbb{G}_m$-gerbe over Sch/S given by $\alpha \in \mathrm{H}^2(S,\mathbb{G}_m)$. Let n be an integer. A quasi-coherent sheaf of $\mathcal{O}_{\mathcal{S}}$-modules $\mathcal{E}$ is called an *n-twisted sheaf* if for any field k and any morphism $x : \mathrm{Spec}(k)\to\mathcal{S}$ the natural action of $\underline{\mathrm{Aut}}_x \cong \mathbb{G}_{m,k}$ on the k-vector space $x^*\mathcal{E}$ is via the character $t \mapsto t^n$.

It is easy to see ([Ols16, Lemma 12.3.3], [Lie08, Lemma 3.1.1.7]) that the tensor product of an n-twisted sheaf and an m-twisted sheaf is an $(n+m)$-twisted sheaf; the Hom sheaf of an n-twisted sheaf with values in an m-twisted sheaf is an $(m-n)$-twisted sheaf. The functor π^* sends quasi-coherent $\mathcal{O}_S$-modules to 0-twisted sheaves on $\mathcal{S}$, and induces an equivalence of these categories. In particular, if $\mathcal{E}$ is an n-twisted sheaf on the gerbe $\mathcal{S}$, then the sheaf $\mathcal{E}nd(\mathcal{E})$ is a 0-twisted sheaf and hence isomorphic to $\pi^*\mathcal{A}$ for a unique quasi-coherent sheaf of $\mathcal{O}_S$-algebras.

There is a closely related notion of α-twisted sheaf. By a theorem of Artin, if S is quasi-projective over an affine scheme, for any class $\alpha \in \mathrm{H}^2(S,\mathcal{F})$ where $\mathcal{F}$ is a sheaf on the small étale site of S, there is an étale covering $\{U_i\to S\}$ of S such that α is represented by a Čech cocycle $\alpha_{ijk} \in \Gamma(U_{ijk},\mathcal{F})$, see [Mil80, Thm. III.2.17]. Here we use the standard notation $U_{ijk} = U_i \times_X U_j \times_X U_k$. Now let $\mathcal{F} = \mathbb{G}_m$. For $\alpha \in \mathrm{H}^2(S,\mathbb{G}_m)$ an *α-twisted sheaf* (with respect to this covering) is given by quasi-coherent sheaves of $\mathcal{O}_{U_i}$-modules $\mathcal{M}_i$ together with isomorphisms $\varphi_{ij} : \mathcal{M}_i|_{U_{ij}} \tilde{\longrightarrow} \mathcal{M}_j|_{U_{ij}}$ such that restricting to U_{ijk} we have

$$\varphi_{jk}\varphi_{ij} = \alpha_{ijk}\varphi_{ik}.$$

Note that in general an α-twisted sheaf is not a sheaf on a scheme in the usual sense. If $\beta_{ijk} \in \Gamma(U_{ijk},\mathbb{G}_m)$ is another Čech cocycle, defining a class $\beta \in H^2(X,\mathbb{G}_m)$, then the naturally defined tensor product is an $(\alpha+\beta)$-sheaf.

Lemma 2.6.1 *Let $\alpha \in \mathrm{H}^2(S,\mathbb{G}_m)$. The category of α-twisted sheaves on the scheme S is equivalent to the category of 1-twisted sheaves on the $\mathbb{G}_m$-gerbe $\mathcal{S}$ defined by α.*

Sketch of proof. See [deJ, Lemma 2.10]. To construct an α-twisted sheaf from a 1-twisted sheaf one chooses an exact sequence (2.25) with $A = \mathbb{G}_m$ and B, C injective. Choose a section in $\mathrm{H}^0(S,D)$ that lifts $\alpha \in \mathrm{H}^2(S,\mathbb{G}_m)$. Choose an étale covering $\{U_i\to S\}$ that trivialises α. Since α restricts to 0 on each U_i, this section lifts to a section of C over U_i. Hence the morphism $U_i\to S$ lifts to a morphism $U_i \to \mathcal{S}$. Then the pullback of our 1-twisted sheaf to U_i is a quasi-coherent sheaf of $\mathcal{O}_{U_i}$-modules. The differences of sections of C on U_{ij} lift to a section of A which we use to define φ_{ij}. One then checks the formula $\varphi_{jk}\varphi_{ij} = \alpha_{ijk}\varphi_{ik}$. □

Chapter 3

Brauer groups of schemes

There are two ways to generalise the Brauer groups of fields to schemes. The definition of the Brauer group of a field k in terms of central simple algebras over k readily extends to schemes as the group of equivalence classes of Azumaya algebras. We call it the Brauer–Azumaya group. The Brauer–Azumaya group $\mathrm{Br}_{\mathrm{Az}}(X)$ of a quasi-compact scheme X is a torsion group. The cohomological description $\mathrm{Br}(k) = \mathrm{H}^2(k, \bar{k}^*)$ also extends and gives rise to the Brauer–Grothendieck group $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{m,X})$. There is a natural inclusion of $\mathrm{Br}_{\mathrm{Az}}(X)$ in $\mathrm{Br}(X)$. In Section 3.3 we reproduce de Jong's proof of Gabber's theorem which says that this defines an isomorphism of $\mathrm{Br}_{\mathrm{Az}}(X)$ with the torsion subgroup of $\mathrm{Br}(X)$ when X is a quasi-projective scheme. Note that there exist integral normal noetherian schemes such that $\mathrm{Br}(X)$ is not a torsion group, for example, already some normal complex surfaces are like this, see Chapter 7. In Section 3.5 we prove a theorem of Grothendieck that the Brauer–Grothendieck group $\mathrm{Br}(X)$ of a regular integral scheme X is naturally a subgroup of the Brauer group of its field of functions F. In particular, $\mathrm{Br}(X)$ is then a torsion group.

The purity theorem for the Brauer group of a regular integral scheme X is discussed in Section 3.6 in the special case of schemes of dimension 1, and in Section 3.7 in the general case. For torsion of order invertible on X the purity theorem can be stated and proved in terms of residues at the generic points of the irreducible divisors on X. We state the absolute purity theorem for the Brauer group of a regular scheme, whose proof has been recently completed. This leads to a description of the Brauer group of a regular integral scheme in terms of discrete valuations of its function field.

3.1 The Brauer–Azumaya group of a scheme

The following theorem is due to Azumaya, Auslander and Goldman, and Grothendieck, see [Gro68, I, Thm. 5.1] and [Mil80, Ch. IV, §2].

Theorem 3.1.1 *Let X be a scheme and let A be an $\mathcal{O}_X$-algebra which is a locally free $\mathcal{O}_X$-module. The following conditions are equivalent:*

(i) *For each $x \in X$ the fibre $A \otimes k(x)$ is a central simple algebra over the residue field $k(x)$.*

(ii) *The natural map $A \otimes_{\mathcal{O}_X} A^{\mathrm{op}} \to \mathcal{E}nd_{\mathcal{O}_X\text{-mod}}(A)$ is an isomorphism.*

(iii) *For each $x \in X$ there exist a positive integer r, a Zariski open set $U \subset X$ with $x \in U$ and a finite, surjective, étale morphism $U' \to U$ such that $A_{U'} \cong M_r(\mathcal{O}_{U'})$.*

(iv) *For each $x \in X$ there exist a positive integer r, a Zariski open set $U \subset X$ with $x \in U$ and a surjective étale morphism $U' \to U$ such that $A_{U'} \cong M_r(\mathcal{O}_{U'})$.*

An algebra A satisfying these equivalent conditions is called an *Azumaya algebra*. If X is connected, then the integer r in (iii) is constant on X. It is called the *degree* of the algebra.

A generalisation of the Skolem–Noether theorem leads to a proof that the set of isomorphism classes of Azumaya algebras of degree r on X is in a natural bijection with the étale Čech cohomology pointed set $\check{\mathrm{H}}^1_{\text{ét}}(X, \mathrm{PGL}_{r,X})$, see [Mil80, p. 122]. This pointed set classifies PGL_r-torsors on X [Mil80, Cor. III.4.7].

Two Azumaya algebras A and B on X are called *equivalent* if there exist locally free $\mathcal{O}_X$-modules P and Q locally of finite rank and an isomorphim of $\mathcal{O}_X$-algebras

$$A \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X\text{-mod}}(P) \cong B \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X\text{-mod}}(Q).$$

The set of equivalence classes is called the *Brauer–Azumaya group* $\mathrm{Br}_{\mathrm{Az}}(X)$. Tensor product makes it into a commutative monoid such that the class of $\mathcal{O}_X$ is the identity element. It is actually an abelian group.

The group $\mathrm{Br}_{\mathrm{Az}}(X)$ is a torsion group when X has finitely many connected components, which is the case when X is quasi-compact [Mil80, Prop. IV.2.7].

The equivalence of (iii) and (iv) in the above theorem is due to the following remarkable fact: if A is a local ring, then any $\mathrm{PGL}_{n,A}$-torsor is split by a finite étale extension of A. More generally, we have the following theorem.

Theorem 3.1.2 *Let A be a semilocal ring and let G be a semisimple group scheme over A. Then any G-torsor over A is split by a finite étale extension of A. The same holds if G is a reductive group scheme over a normal noetherian ring A.*

Proof. See [SGA3, XXIV, Thm. 4.1.5, Cor. 4.1.6]. □

3.2 The Brauer–Grothendieck group of a scheme

Grothendieck's definition of the (cohomological) Brauer group formally resembles his formula for the Picard group (2.20).

Definition 3.2.1 *The Brauer–Grothendieck group of a scheme X is*

$$\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{m,X}).$$

For an affine scheme $X = \mathrm{Spec}(A)$, where A is a commutative ring, one often writes $\mathrm{Br}(A) := \mathrm{Br}(X)$. In the particular case $X = \mathrm{Spec}(k)$, where k is field, we obtain the classical description of the Brauer group of a field in terms of continuous 2-cocycles of its absolute Galois group $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$, where k_{s} is a separable closure of k:

$$\mathrm{Br}(k) = \mathrm{H}^2(k, k_{\mathrm{s}}^*) = \mathrm{H}^2(\Gamma, k_{\mathrm{s}}^*).$$

One may also consider the Zariski cohomological Brauer group of a scheme X. Let us denote it by $\mathrm{H}^2_{\mathrm{zar}}(X, \mathbb{G}_m)$. Write $\pi : X_{\text{ét}} \to X_{\mathrm{zar}}$ for the morphism of sites. Then we have $\mathbb{G}_{m,\mathrm{zar}} = \pi_*\mathbb{G}_m$ and $R^1\pi_*(\mathbb{G}_m) = 0$. From the spectral sequence (2.4) we get an injection

$$\mathrm{H}^2_{\mathrm{zar}}(X, \mathbb{G}_m) \hookrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m).$$

Note, however, that this injection need not be an isomorphism. Indeed, if X is integral and locally factorial, then $\mathrm{H}^2_{\mathrm{zar}}(X, \mathbb{G}_m) = 0$, see Remark 3.5.1.

A morphism of schemes $f : X \to Y$ which is locally of finite type gives rise to a morphism (2.7). In the case of $G = \mathbb{G}_m$ we obtain

$$f^* : \mathrm{H}^n_{\text{ét}}(Y, \mathbb{G}_{m,Y}) \longrightarrow \mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_{m,X}). \tag{3.1}$$

For $n = 2$ this gives a natural map of Brauer groups $f^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$, which is sometimes referred to as the *restriction* map. If K is a field and $M : \mathrm{Spec}(K) \to X$ is a K-point of X, then one writes $A(M) = M^*(A) \in \mathrm{Br}(K)$ and refers to $A(M)$ as the *value*, or specialisation, of A at M.

The Brauer group and cohomology with finite coefficients

The link of the Brauer group to étale cohomology with finite coefficients is provided by the Kummer exact sequence

$$1 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_{m,X} \xrightarrow{x \mapsto x^{\ell^n}} \mathbb{G}_{m,X} \longrightarrow 1.$$

Here ℓ is a prime invertible on X and n is a positive integer. The associated long exact sequence of cohomology gives an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X)/\ell^n \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \longrightarrow \mathrm{Br}(X)[\ell^n] \longrightarrow 0. \tag{3.2}$$

At the level of H^1 the Kummer sequence gives an exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \mathbb{G}_m)/\mathrm{H}^0(X, \mathbb{G}_m)^{\ell^n} \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mu_{\ell^n}) \longrightarrow \mathrm{Pic}(X)[\ell^n] \longrightarrow 0,$$

where $\mathrm{H}^0(X, \mathbb{G}_m)^{\ell^n}$ stands for the group of ℓ^n-powers of invertible regular functions on X. At the level of H^3 we have another useful exact sequence

$$0 \longrightarrow \mathrm{Br}(X)/\ell^n \longrightarrow \mathrm{H}^3_{\text{ét}}(X, \mu_{\ell^n}) \longrightarrow \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)[\ell^n] \longrightarrow 0. \tag{3.3}$$

The Mayer–Vietoris sequence

Theorem 3.2.2 *Let X be a scheme and let $X = U \cup V$ be an open Zariski covering. Write $W = U \cap V$. Then there is an infinite exact sequence*

$$\begin{aligned} 0 &\longrightarrow \Gamma(X, \mathcal{O}_X^*) \longrightarrow \Gamma(U, \mathcal{O}_U^*) \oplus \Gamma(V, \mathcal{O}_V^*) \longrightarrow \Gamma(W, \mathcal{O}_W^*) \\ &\longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(U) \oplus \mathrm{Pic}(V) \longrightarrow \mathrm{Pic}(W) \\ &\longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(U) \oplus \mathrm{Br}(V) \longrightarrow \mathrm{Br}(W) \longrightarrow \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m) \longrightarrow \cdots \end{aligned}$$

Here the arrows like $\mathrm{Pic}(X) \to \mathrm{Pic}(U) \oplus \mathrm{Pic}(V)$ are restriction maps, and the arrows like $\mathrm{Pic}(U) \oplus \mathrm{Pic}(V) \to \mathrm{Pic}(W)$ are differences of restriction maps. This is a particular case of the Mayer–Vietoris sequence for an étale sheaf $\mathbb{G}_{m,X}$ on X [Mil80, Ch. III, §2, Exercise 2.24].

As a consequence of Theorem 3.2.2, if the open set U is locally factorial, for instance if U is regular, then one has a short exact sequence

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(U) \oplus \mathrm{Br}(V) \longrightarrow \mathrm{Br}(W).$$

This can be compared with Theorem 3.5.5 below.

Passing to the reduced subscheme

Proposition 3.2.3 *Let X be a noetherian scheme. Let $X_{\text{red}} \subset X$ be the reduced subscheme.*

(i) *If X is affine, then the natural map $\mathrm{Br}(X) \to \mathrm{Br}(X_{\text{red}})$ is an isomorphism.*

(ii) *If $\dim(X) \leq 1$, then $\mathrm{Br}(X) \to \mathrm{Br}(X_{\text{red}})$ is an isomorphism.*

(iii) *If $\dim(X) \leq 2$, then the natural map $\mathrm{Br}(X) \to \mathrm{Br}(X_{\text{red}})$ is surjective.*

Proof. Cf. [De75], [CTOP02, Lemma 1.6]. There are closed immersions

$$X_{\text{red}} = X_0 \subset X_1 \subset \ldots \subset X_n = X$$

and ideals $\mathcal{I}_j \subset \mathcal{O}_{X_j}$, for $j = 1, \ldots, n$, such that $\mathcal{O}_{X_{j-1}} = \mathcal{O}_{X_j}/\mathcal{I}_j$ and $\mathcal{I}_j^2 = 0$. On each X_j we have an exact sequence of sheaves for the étale topology

$$0 \longrightarrow \mathcal{I}_j \longrightarrow \mathbb{G}_{m,X_j} \longrightarrow r_*\mathbb{G}_{m,X_{j-1}} \longrightarrow 1,$$

where $r : X_{j-1} \to X_j$ is the given closed immersion, the coherent ideal $\mathcal{I}_j$ is viewed as a sheaf for the étale topology, and the map $\mathcal{I}_j \to \mathbb{G}_{m,X_j}$ is given by $x \mapsto 1 + x$. For any i we have $\mathrm{H}^i_{\text{ét}}(X_j, \mathcal{I}_j) = \mathrm{H}^i_{\text{zar}}(X_j, \mathcal{I}_j)$. If X is affine, then all these groups vanish for $i \geq 1$. If $\dim(X) \leq 1$, then these groups vanish for $i \geq 2$. If $\dim(X) \leq 2$, these groups vanish for $i \geq 3$. Thus

$$\mathrm{H}^2_{\text{ét}}(X_j, \mathbb{G}_m) \longrightarrow \mathrm{H}^2_{\text{ét}}(X_j, r_*\mathbb{G}_{m,X_{j-1}})$$

is an isomorphism if X is affine or if $\dim(X) \leq 1$. If $\dim(X) \leq 2$, then this map is surjective. Since r is a closed immersion, we have $R^i r_*(F) = 0$ for $i \geq 1$ and any sheaf F. The Leray spectral sequence for the immersion $X_{j-1}\to X_j$ and the sheaf $\mathbb{G}_m$ gives

$$\mathrm{H}^2_{\text{ét}}(X_j, r_*\mathbb{G}_{m,X_{j-1}})\tilde{\longrightarrow}\mathrm{H}^2_{\text{ét}}(X_{j-1},\mathbb{G}_m).$$

Thus the natural map $\mathrm{H}^2_{\text{ét}}(X_j,\mathbb{G}_m)\to\mathrm{H}^2_{\text{ét}}(X_{j-1},\mathbb{G}_m)$ is an isomorphism if X is affine or $\dim(X)\leq 1$; it is surjective if $\dim(X)\leq 2$. □

As we shall see in Section 7.1, as soon as $\dim(X)\geq 2$, the map $\mathrm{Br}(X)\to\mathrm{Br}(X_{\text{red}})$ need not be injective.

Proposition 3.2.4 *Let X be a noetherian scheme. Let n be a positive integer invertible on X. Then we have the following statements.*

(a) *The natural map* $\mathrm{Br}(X)/n\to\mathrm{Br}(X_{\text{red}})/n$ *is injective.*

(b) *The natural map* $\mathrm{Br}(X)[n]\to\mathrm{Br}(X_{\text{red}})[n]$ *is surjective.*

(c) *If X is a scheme over a field of characteristic* 0*, then the natural map* $\mathrm{Br}(X)_{\text{tors}}\to\mathrm{Br}(X_{\text{red}})_{\text{tors}}$ *is surjective.*

Proof. If F is a coherent sheaf on X, then multiplication by n on $\mathrm{H}^i_{\text{zar}}(X,F)=\mathrm{H}^i_{\text{ét}}(X,F)$ is an isomorphism for any $i\geq 0$. The arguments from the proof of Proposition 3.2.3 then give an exact sequence

$$A\longrightarrow\mathrm{Br}(X)\longrightarrow\mathrm{Br}(X_{\text{red}})\longrightarrow B$$

with A and B uniquely n-divisible. The three statements then follow from a diagram chase. The second statement may also be established by using the Kummer sequence and invariance of étale cohomology with coefficients μ_n for $X_{\text{red}}\to X$ when n is invertible on X [SGA4, VII, §1]. □

3.3 Comparison of the two Brauer groups

Let us fix an integer $n>1$. There is a natural exact sequence of group schemes over X

$$1\longrightarrow\mathbb{G}_{m,X}\longrightarrow\mathrm{GL}_{n,X}\longrightarrow\mathrm{PGL}_{n,X}\longrightarrow 1,\qquad(3.4)$$

where $\mathbb{G}_{m,X}\to\mathrm{GL}_{n,X}$ is the central subgroup of scalar matrices. It gives rise to a boundary map of pointed cohomology sets

$$\delta_n:\check{\mathrm{H}}^1_{\text{ét}}(X,\mathrm{PGL}_{n,X})\longrightarrow\check{\mathrm{H}}^2_{\text{ét}}(X,\mathbb{G}_m)\hookrightarrow\mathrm{H}^2_{\text{ét}}(X,\mathbb{G}_m)=\mathrm{Br}(X).$$

Theorem 3.3.1 *Let X be a scheme. Then we have the following statements.*

(i) *The set* $\check{\mathrm{H}}^1_{\text{ét}}(X,\mathrm{PGL}_{n,X})$ *can be identified with the set of isomorphism classes of Azumaya algebras of degree n on X.*

(ii) *The boundary maps δ_n for $n>1$ are compatible and induce a homomorphism of abelian groups* $\mathrm{Br}_{\text{Az}}(X)\to\mathrm{Br}(X)$.

(iii) *This homomorphism* $\mathrm{Br}_{\text{Az}}(X)\to\mathrm{Br}(X)$ *is injective.*

(iv) $\delta_n(\check{\mathrm{H}}^1_{\text{ét}}(X,\mathrm{PGL}_{n,X}))\subset\mathrm{Br}(X)[n]$.

Proof. See [Mil80, Thm. IV.2.5]. Milne also gives a proof of (iii) via gerbes, which does not use the exact sequence (3.4). (See Proposition 3.3.3 below.) □

The fundamental result linking the Brauer–Azumaya group to the Brauer–Grothendieck group is the following theorem of Gabber. Previous results in this direction were obtained by Gabber in his thesis and also by Hoobler. A proof in the affine case is given in [Lie08, Cor. 3.1.4.2].

Theorem 3.3.2 (Gabber) *Let X be a quasi-compact separated scheme with an ample invertible sheaf, for example, a quasi-projective scheme over an affine scheme. Then the map*

$$\mathrm{Br}_{\mathrm{Az}}(X) \longrightarrow \mathrm{Br}(X)_{\mathrm{tors}}$$

is an isomorphism.

By definition (see [Stacks, Def. 01PS]) X has an ample invertible sheaf means that there exists an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules such that for any $x \in X$ there is an $s \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})$ for some $n \geq 1$ such that $s(x) \neq 0$ and the open subset $s \neq 0$ is affine. This holds for any quasi-projective scheme over an affine scheme.

The separateness assumption is necessary. Indeed, there exist non-separated, normal varieties X over $\mathbb{C}$ with torsion elements in $\mathrm{Br}(X)$ that are not in the image of $\mathrm{Br}_{\mathrm{Az}}(X)$, see [EHKV01] and [Ber05].

The remaining part of this section is a sketch of de Jong's proof of Theorem 3.3.2, see [deJ].

We begin by interpreting the map $\mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}(X)$ as a map that associates to an Azumaya algebra a certain $\mathbb{G}_m$-gerbe.

To an Azumaya algebra A on X one attaches the category $\mathcal{X}(A)$ whose objects are triples $(T, \mathcal{M}, j)$, where T is an X-scheme, $\mathcal{M}$ is a locally free $\mathcal{O}_T$-module, and j is an isomorphism $j : \mathcal{E}nd(\mathcal{M}) \tilde{\longrightarrow} A_T$. A morphism of triples $(T, \mathcal{M}, j) \to (T', \mathcal{M}', j')$ is a pair (f, i) consisting of a morphism of X-schemes $f : T \to T'$ and an isomorphism $i : f^*\mathcal{M}' \tilde{\longrightarrow} \mathcal{M}$ compatible with j and j'. Note that there is a natural map $\mathbb{G}_m(T) \to \mathrm{Aut}(T, \mathcal{M}, j)$ sending u to (id_T, u).

Proposition 3.3.3 *The forgetful functor $\pi : \mathcal{X}(A) \to Sch/X$ is a $\mathbb{G}_m$-gerbe.*

Proof. [Ols16, Prop. 12.3.6] One checks that $\mathcal{X}(A)$ is a stack. The verification that $\mathcal{X}(A)$ is a $\mathbb{G}_m$-gerbe can be done locally, so one can assume that $A = \mathcal{E}nd(\mathcal{O}_X^n)$. Furthermore, we can assume that $\mathcal{M} = \mathcal{O}_T^n$. After localising again, we can assume that j comes from the conjugation by an element of $\mathrm{Aut}(\mathcal{O}_T^n)$. Thus any object in $\mathcal{X}(A)$ is locally isomorphic to $(T, \mathcal{O}_T^n, \mathrm{id})$, so any two objects are locally isomorphic. Now the automorphism sheaf of the object $(T, \mathcal{O}_T^n, \mathrm{id})$ is $\mathbb{G}_m$ acting by scalar multiplication on $\mathcal{O}_T^n$. □

Since the isomorphism classes of $\mathbb{G}_m$-gerbes over X are classified by the elements of $\mathrm{H}^2(X, \mathbb{G}_m)$, this gives a map $\mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}(X)$. The class in $\mathrm{Br}(X)$ associated to A can be described as follows. Assume that A is an Azumaya algebra over X of dimension n^2. Consider (3.4) as an exact sequence of sheaves of groups for the étale topology. Let P be the functor on Sch/X sending $Y\to X$ to $\mathrm{Isom}_{\mathcal{O}_Y}(M_n(\mathcal{O}_Y), A_Y)$. Using essentially the Noether–Skolem theorem one shows that this functor is a PGL_n-torsor over Sch/X. Then the class associated to A is the image of the class of this torsor under the map $\mathrm{H}^1(X, \mathrm{PGL}_n)\to\mathrm{H}^2(X, \mathbb{G}_m)$ which sends a PGL_n-torsor to the gerbe of its liftings to a GL_n-torsor as defined in Section 2.6.4, see [Ols16, Lemma 12.3.9].

To any cohomology class $\alpha \in \mathrm{Br}(X)$ one associates a $\mathbb{G}_m$-gerbe $\mathcal{X}_\alpha$ (well defined up to isomorphism) using the construction of a gerbe associated to a cohomology class in Section 2.6.4. Namely, one takes (2.24) to be the extension (3.4). One wants to show that $\mathcal{X}_\alpha$ is isomorphic to $\mathcal{X}(A)$ for some A.

A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called *finite locally free* if every point $x \in X$ has a Zariski open neighbourhood $U \subset X$ such that $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus n}$ for some n. We refer to [Ols16, Ch. 9] for the theory of quasi-coherent sheaves on algebraic stacks; then one also has the notion of finite locally free sheaves in this context.

The gerbe $\mathcal{X}(A)$ has a tautological finite locally free 1-twisted sheaf $\mathcal{M}$ together with an isomorphism $\mathcal{E}nd(\mathcal{M}) \cong \pi^*A$ of algebras over $\mathcal{X}(A)$. Then $A = \pi_*\mathcal{E}nd(\mathcal{M})$.

Proposition 3.3.4 *A $\mathbb{G}_m$-gerbe $\mathcal{X}$ over Sch/X is isomorphic to the gerbe $\mathcal{X}(A)$ for some Azumaya algebra A on X if and only if $\mathcal{X}$ has a finite locally free 1-twisted $\mathcal{O}_\mathcal{X}$-module $\mathcal{M}$ of positive rank. In this case $A = \pi_*\mathcal{E}nd(\mathcal{M})$ is an Azumaya algebra on X, the adjunction map $\pi^*A\to\mathcal{E}nd(\mathcal{M})$ is an isomorphism, and $\mathcal{X} \cong \mathcal{X}(A)$.*

Proof. See [Ols16, Prop. 12.3.11] □

The goal is thus to show that this is the case for the $\mathbb{G}_m$-gerbe $\mathcal{X} = \mathcal{X}_\alpha$, for any $\alpha \in \mathrm{Br}(X)_{\mathrm{tors}}$, when X has an ample invertible sheaf. Recall that by Lemma 2.6.1 the categories of 1-twisted sheaves on $\mathcal{X}$ and of α-twisted sheaves on X are equivalent. So our task is to construct a finite locally free α-twisted sheaf on X.

Let $\alpha \in \mathrm{Br}(X)_{\mathrm{tors}}$, say $n\alpha = 0$ for some $n \geq 1$. Let $\mathcal{L}$ be an ample invertible sheaf on X. One can represent $(X, \mathcal{L})$ as a filtering projective limit of pairs $(X_i, \mathcal{L}_i)$, where X_i is of finite type over $\mathbb{Z}$ and $\mathcal{L}_i$ is an ample invertible sheaf on X_i, with affine transition morphisms $X_i\to X_j$. By Section 2.2.4 the group $\mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_m)$ is naturally isomorphic to the direct limit of the groups $\mathrm{H}^n_{\text{ét}}(X_i, \mathbb{G}_m)$. Hence $\mathrm{Br}(X)_{\mathrm{tors}}$ is the direct limit of the groups $\mathrm{Br}(X_i)_{\mathrm{tors}}$. Thus without loss of generality we can assume that X is a quasi-projective scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$, so X is noetherian.

In the course of the proof X will be repeatedly replaced by X_R for some ring R which is finite and flat over $\mathbb{Z}$. This is justified by a lemma of Hoobler [Hoo82, Prop. 3] that says that if $\varphi : Y\to X$ is a finite locally free morphism (which is the same as finite and flat as X is noetherian, see [Stacks, Lemma 02KB]), then $\alpha \in \mathrm{H}^2(X, \mathbb{G}_m)$ comes from an Azumaya algebra on X if and only if $\alpha_Y \in \mathrm{H}^2(Y, \mathbb{G}_m)$ comes from an Azumaya algebra on Y. To prove this lemma, de Jong argues as follows. If α_Y comes from an Azumaya algebra on Y, then there is a finite locally free α_Y-twisted sheaf $\mathcal{F}$ on Y. Then the naturally defined direct image $\varphi_*\mathcal{F}$ is a finite locally free α-twisted sheaf on X.

The proof starts with an application of a theorem of Gabber which solves the problem in the particular case when X is affine. We do not reproduce the proof of this result here; it can be found in [Ga81, Ch. 2, Thm. 1], see also [KO81, Thm. 3.1] and [Lie08, Cor. 3.1.4.2]. There is a section $s \in \mathrm{H}^0(X, \mathcal{L}^{\otimes m})$, for some $m \geq 1$, such that the open set X_s is affine. By this result, the restriction of α to X_s is represented by an Azumaya algebra A. Hence there is a finite locally free α-twisted sheaf $\mathcal{F}_s$ on X_s. By taking direct sum on the connected components we can assume that $\mathcal{F}_s$ has constant rank. Let us write $j : \mathcal{X}_s \to \mathcal{X}$ for the open immersion defined by $X_s \hookrightarrow X$. Then $j_*\mathcal{F}_s$ is a quasi-coherent α-twisted sheaf on X. Representing it as a direct limit of coherent sheaves allows one to find a coherent α-twisted subsheaf $\mathcal{F} \subset j_*\mathcal{F}_s$ such that $j^*\mathcal{F} = \mathcal{F}_s$.

We can ensure that X_s contains any given finite set of closed points (see [EGA II, Cor. 4.5.4]), so our coherent α-twisted sheaf $\mathcal{F}$ is finite locally free at each of these points.

A quasi-coherent sheaf of $\mathcal{O}_X$-modules is finite and locally free if and only if it is flat and of finite type [Stacks, Lemma 05P2]. Thus the task is to ensure that our α-twisted sheaf $\mathcal{F}$ is flat. Let $\mathrm{Sing}(\mathcal{F})$ be the set of points of X at which $\mathcal{F}$ is not flat. What we have obtained now is the case $c = 1$ of the following statement.

(H_c) *For any finite set T of closed points of X, after a finite flat ring extension of R, there exists an α-twisted sheaf $\mathcal{F}$ which is finite and locally free at T, of constant positive rank outside of* $\mathrm{Sing}(\mathcal{F})$*, and such that* $\mathrm{codim}_X \mathrm{Sing}(\mathcal{F}) \geq c$.

The strategy of the proof is to use ring extensions to increase c; in view of Hoobler's lemma, the theorem will be proved if one can make $c = \dim(X) + 1$.

Step 1

Assume that (H_c) holds for a finite set of closed points $T \subset X$. The claim of this step is that, after replacing R by a finite flat extension ring, there exist $n+1$ coherent α-twisted sheaves $\mathcal{F}_0, \ldots, \mathcal{F}_n$ (recall that $n\alpha = 0$) and finite sets of closed points $S_0, \ldots, S_n$ in X with the following properties:

(1) T is disjoint from $\bigcup_{i=0}^n \mathrm{Sing}(\mathcal{F}_i)$;

(2) each $\mathcal{F}_i$ has constant positive rank on $X \smallsetminus \mathrm{Sing}(\mathcal{F}_i)$;

(3) each irreducible component of $\mathrm{Sing}(\mathcal{F}_i)$ of codimension c contains a point of S_i;

(4) for any $i \neq j$ the sheaf $\mathcal{F}_j$ is finite locally free at all the points of S_i.

Indeed, (H_c) ensures the existence of $\mathcal{F}_0$ which is locally free at T. Choose a closed point in each irreducible component of $\mathrm{Sing}(\mathcal{F}_0)$ of codimension c; let $S_0 \subset X$ be the set of these points. Define $T_0 = T \cup S_0$. Now (H_c) ensures the existence of $\mathcal{F}_1$ which is locally free at T_0. If a codimension c irreducible component of $\mathrm{Sing}(\mathcal{F}_1)$ is contained in $\mathrm{Sing}(\mathcal{F}_0)$, then it is a codimension c irreducible component of $\mathrm{Sing}(\mathcal{F}_0)$, but this is not possible because $\mathcal{F}_1$ is locally free at some closed point of this component. Thus we can choose a closed point in each codimension c irreducible component of $\mathrm{Sing}(\mathcal{F}_1)$ which is not in $\mathrm{Sing}(\mathcal{F}_0)$. Let $S_1 \subset X$ be the set of these points, and let $T_1 = T_0 \cup S_1$. The pairs $(\mathcal{F}_0, S_0)$ and $(\mathcal{F}_1, S_1)$ satisfy properties (1) to (4) with $n = 2$. Next, one constructs $\mathcal{F}_2$ and so on. If $\mathcal{F}_0, \ldots, \mathcal{F}_{j-1}$ are already constructed so that properties (1) to (4) are satisfied, one constructs $\mathcal{F}_j$ which is locally free at all the points of $T \cup S_0 \cup \ldots \cup S_{j-1}$ and chooses S_j in $\mathrm{Sing}(\mathcal{F}_j)$ outside of the union of $\mathrm{Sing}(\mathcal{F}_i)$ for $i = 0, \ldots, j-1$.

Step 2

Replacing each $\mathcal{F}_i$ by $\mathcal{F}_i^{\oplus m_i}$ for appropriate positive integers m_i we ensure that there is a positive integer r such that the rank of $\mathcal{F}_i$ on $X \smallsetminus \mathrm{Sing}(\mathcal{F}_i)$ is r. Later on we shall assume that r is large. Define

$$\mathcal{G}_1 = (\mathcal{F}_0 \oplus \ldots \oplus \mathcal{F}_n)^{\oplus r^n}, \qquad \mathcal{G}_2 = \mathcal{F}_0 \otimes \ldots \otimes \mathcal{F}_n.$$

It is clear that $\mathcal{G}_1$ is an α-twisted sheaf; in fact, $\mathcal{G}_2$ is also an α-twisted sheaf since $n\alpha = 0$. It follows that

$$\mathcal{H} = \mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)$$

is a 0-twisted sheaf on X, so is a coherent $\mathcal{O}_X$-module. Recall that $\mathcal{L}$ is an ample invertible sheaf on X. Replacing X by X_R preserves the ampleness of $\mathcal{L}$.

Let ψ be a section of $\mathcal{H} \otimes \mathcal{L}^{\otimes N}$ over X for some positive integer N, and let $\mathcal{F}$ be the kernel of the map $\psi : \mathcal{G}_1 \to \mathcal{G}_2 \otimes \mathcal{L}^{\otimes N}$.

Let U be the complement to $\bigcup_{i=0}^{n} \mathrm{Sing}(\mathcal{F}_i)$ in X. The aim of Step 2 is to give conditions for $\mathcal{F}$ to be finite locally free of positive rank on a larger open set than U. More precisely, one gives conditions ensuring that $\mathrm{codim}_X \mathrm{Sing}(\mathcal{F}) \geq c+1$, in terms of pullbacks at closed points of X.

To define the pullback at a geometric point $\bar{x} = \mathrm{Spec}(\kappa(\bar{x})) \in X$ one chooses a lifting of the morphism $\bar{x} \to X$ to a morphism $\bar{x} \to \mathcal{X}$, which is possible as $\alpha \in \mathrm{Br}(X)$ is annihilated by the restriction to the algebraically closed residue field $\kappa(\bar{x})$. The same works for a closed point with finite residue field, by the triviality of the Brauer group of a finite field.

Claim. *Let $\mathcal{F} = \mathrm{Ker}[\psi : \mathcal{G}_1 \to \mathcal{G}_2 \otimes \mathcal{L}^{\otimes N}]$, where ψ is a section of $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2 \otimes \mathcal{L}^{\otimes N})$ over X, for some positive integer N. Assume that the following conditions are satisfied.*

(a) *For every geometric point $\bar{x} = \mathrm{Spec}(\kappa(\bar{x})) \in U$ the pullback to $\bar{x}$ gives a surjective map of $\kappa(\bar{x})$-vector spaces*

$$\psi_{\bar{x}} : \mathcal{G}_1 \otimes \kappa(\bar{x}) \longrightarrow \mathcal{G}_2 \otimes \mathcal{L}^{\otimes N} \otimes \kappa(\bar{x}).$$

(b) *For any $i = 0, \ldots, n$ and any $s = \mathrm{Spec}(\kappa(s)) \in S_i$ the composition*

$$\mathcal{F}_i^{\oplus r^n} \otimes \kappa(s) \hookrightarrow \mathcal{G}_1 \otimes \kappa(s) \longrightarrow \mathcal{G}_2 \otimes \mathcal{L}^{\otimes N} \otimes \kappa(s)$$

is an isomorphism, whereas the following composition is zero:

$$(\oplus_{j \neq i} \mathcal{F}_j)^{\oplus r^n} \otimes \kappa(s) \hookrightarrow \mathcal{G}_1 \otimes \kappa(s) \longrightarrow \mathcal{G}_2 \otimes \mathcal{L}^{\otimes N} \otimes \kappa(s).$$

Then $\mathcal{F}$ is an α-twisted sheaf on X such that $\mathrm{Sing}(\mathcal{F}) \subset \bigcup_{i=0}^n \mathrm{Sing}(\mathcal{F}_i)$ *and* $\mathrm{Sing}(\mathcal{F})$ *is disjoint from* $S = \cup_{i=0}^n S_i$. *In particular,* $\mathrm{codim}_X \mathrm{Sing}(\mathcal{F}) \geq c+1$.

This shows that if ψ satisfying (a) and (b) exists, then (H_c) implies (H_{c+1}).

Proof of Claim. It is clear that $\mathcal{F} = \mathrm{Ker}(\psi)$ is an α-twisted sheaf on X. The last sentence of the statement is a consequence of the fact that each codimension c irreducible component of $\mathrm{Sing}(\mathcal{F}_i)$ contains a point of S_i.

Condition (a) implies that the restriction of $\mathcal{H}$ to the open subscheme $U \subset X$ is the kernel of a surjective map of finite locally free sheaves. Locally such a map has a section, so its kernel is finite locally free.

Let us prove that condition (b) implies that $\mathcal{F}$ is finite locally free at each $x \in S$. Let $\mathcal{O}_{X,x}$ be the local ring at x and let $\mathcal{O}_{X,x}^{\mathrm{h}}$ be the henselisation of $\mathcal{O}_{X,x}$. The Brauer group $\mathrm{Br}(\mathcal{O}_{X,x}^{\mathrm{h}})$ is canonically isomorphic to the Brauer group of the residue field $\mathrm{Br}(\kappa(x))$, see Theorem 3.4.2 (i). Since $\kappa(x)$ is finite, we have $\mathrm{Br}(\kappa(x)) = 0$. It follows that there is a finite étale extension of local rings $\mathcal{O}_{X,x} \subset B$ with trivial residue field extension such that the image of α in $\mathrm{Br}(B)$ is zero. Thus there is a lifting $\mathrm{Spec}(B) \to \mathcal{X}$ of $\mathrm{Spec}(B) \to X$ so that each $\mathcal{F}_i$ pulls back to the quasi-coherent sheaf on $\mathrm{Spec}(B)$ associated to a finitely generated B-module M_i.

We have $x \in S_i$ for some i. Then for $j \neq i$ the B-module M_j is free of rank r. Let us write $M = M_i$. If $\mathcal{H}_x$ is the stalk of $\mathcal{H}$ at x, then

$$\mathcal{H}_x \otimes B = \mathrm{Hom}_B(M^{\oplus r^n}, M^{\oplus r^n}) \oplus \mathrm{Hom}_B(B^{\oplus nr^{n+1}}, M^{\oplus r^n}).$$

Write $\psi \otimes B = \psi_1 \oplus \psi_2$. Since the residue field of B is $\kappa(x)$, condition (b) gives that ψ_1 is an isomorphism and $\psi_2 = 0$. Hence $\mathcal{F} = \mathrm{Ker}(\psi)$ is the direct summand $B^{\oplus nr^{n+1}} \subset \mathcal{G}_{1,B}$, so $\mathcal{F}$ is finite locally free at each point of S. This proves the claim.

As a preparation for the last step of the proof we point out that the fibre of $\mathcal{H}$ at a geometric point $\bar{x} \in U$ is the $\kappa(\bar{x})$-vector space of matrices of size $(n+1)r^{n+1} \times r^{n+1}$. Condition (a) at $\bar{x}$ is satisfied if $\psi_{\bar{x}}$ avoids the subset of matrices of rank less than r^{n+1}. This is a closed homogeneous subset of codimension

$$(n+1)r^{n+1} - r^{n+1} + 1 > nr^{n+1}.$$

We can make r arbitrarily large and thus ensure that this codimension is greater than $\dim(X)+1$.

Step 3

It remains to show that if N is sufficiently large, then there exists a section ψ satisfying conditions (a) and (b) above. This is a purely algebraic-geometric statement, so this part of the proof has nothing to do with either Brauer elements or gerbes.

Let R be a ring which is finite and flat over $\mathbb{Z}$, and let X be a quasi-projective scheme over R with an invertible sheaf $\mathcal{L}$. We write L for the line bundle on X whose sheaf of sections is $\mathcal{L}$. Let $\mathcal{H}$ be a coherent $\mathcal{O}_X$-module which is finite locally free over an open subscheme $U \subset X$. We write H for the vector bundle on U whose sheaf of sections is the restriction of $\mathcal{H}$ to U. For any point x in X fix an isomorphism $\mathcal{L} \otimes \kappa(x) \cong L_x \cong \kappa(x)$.

Suppose that for every $u \in U$ we are given a closed homogeneous subset $C_u \subset H_u$ of codimension greater than $\dim(X)+1$. Suppose also that for a finite set of closed points $S \subset X \smallsetminus U$ we are given $\psi_s \in \mathcal{H} \otimes \kappa(s)$, for each $s \in S$. Then there exists a positive integer N, a finite flat extension of rings $R \subset R'$ and a section $\psi \in \Gamma(X_{R'}, \mathcal{H} \otimes \mathcal{L}^{\otimes N})$ such that $\psi_u \notin C_u \otimes L_u^{\otimes N}$ for $u \in U_{R'}$, and for each closed point s' of $X_{R'}$ over a point $s \in S$ the value of ψ at s' is a non-zero multiple of ψ_s.

This isomorphism $L_u \cong \kappa(u)$ identifies $C_u \subset H_u$ with $C_u \otimes L_u^{\otimes N} \subset H_u \otimes L_u^{\otimes N}$.

For the proof we may assume $R = \mathbb{Z}$.

Let $\mathcal{I}_S \subset \mathcal{O}_X$ be the sheaf of ideals defined by S. For all N sufficiently large one can find sections $\Psi_i \in \Gamma(X, \mathcal{I}_S \otimes \mathcal{H} \otimes \mathcal{L}^{\otimes N})$, for i in a finite set I, such that the map of sheaves $\mathcal{O}_X^I \to \mathcal{I}_S \otimes \mathcal{H} \otimes \mathcal{L}^{\otimes N}$ sending 1_i to Ψ_i is surjective. In particular, the sections Ψ_i generate the sheaf $\mathcal{H} \otimes \mathcal{L}^{\otimes N}$ over U.

Next, by increasing N further, for each $s \in S$ one finds a section $\Psi_s \in \Gamma(X, \mathcal{I}_{S \smallsetminus \{s\}} \otimes \mathcal{H} \otimes \mathcal{L}^{\otimes N})$ whose value at s is ψ_s.

Let $\mathbb{A} = \operatorname{Spec}(\mathbb{Z}[x_i, y_s; i \in I, s \in S])$ be the affine space over $\mathbb{Z}$ of relative dimension $|I| + |S|$. Write $\mathbb{A} \times X$ for $\mathbb{A} \times_{\mathbb{Z}} X$ and consider the universal section

$$\Psi = \sum_{i \in I} x_i \Psi_i + \sum_{s \in S} y_s \Psi_s$$

of the pullback of $\mathcal{H} \otimes \mathcal{L}^{\otimes N}$ to $\mathbb{A} \times X$. The value $\Psi_{a,u}$ of Ψ at $(a, u) \in \mathbb{A} \times U$ is an element of $H_u \otimes L^{\otimes N}$ which we identified with H_u. Let $Z \subset \mathbb{A} \times X$ be the closed subset defined by the condition $\Psi_{a,u} \in C_u$. The values of the sections Ψ_i, for $i \in I$, generate the $\kappa(u)$-vector space H_u, hence the dimension of each fibre of the natural projection $Z \to U$ is at most $|I| + |S| - \operatorname{codim}_{H_u}(C_u)$. By assumption we have $\operatorname{codim}_{H_u}(C_u) > \dim(X) + 1$, hence $\dim(Z) \leq |I| + |S| - 1 = \dim(\mathbb{A}) - 2$. Thus the Zariski closure Z' of the projection of Z to $\mathbb{A}$ has codimension at least 2.

Let $\pi : \mathbb{A} \to \mathbb{Z}$ be the structure morphism. For each $s \in S$ define $Z_s \subset \mathbb{A}$ to be the closed subscheme defined by the ideal $(\pi(s), y_s)$. To finish the proof we need to find a point in $\mathbb{A}(R)$ outside of the codimension 2 closed subset $Z' \cup \bigcup_{s \in S} Z_s$, for some finite flat extension $\mathbb{Z} \subset R$. Note that π induces a surjective morphism $\mathbb{A} \smallsetminus Z \to \operatorname{Spec}(\mathbb{Z})$. The result then follows from Rumely's local-to-global principle [Rum86] in the form of [Mor89, Thm. 1.7]: an irreducible scheme V which is separated and of finite type over a ring of integers $\mathcal{O}_K$ of a number field K has a point in the ring of *all* algebraic integers if the structure morphism $V \to \operatorname{Spec}(\mathcal{O}_K)$ is surjective with geometrically irreducible generic fibre V_K. It is clear that such a point is defined over a finite extension of $\mathbb{Z}$. □

3.4 Localising elements of the Brauer group

Lemma 3.4.1 *Let X be a scheme. For any element $\alpha \in \mathrm{Br}(X)$ there exists an étale cover $f : U \to X$ such that $f^*\alpha = 0 \in \mathrm{Br}(U)$.*

Proof. This is a special case of a general statement: for any étale sheaf F on a scheme X, any $i > 0$ and any cohomology class $\alpha \in \mathrm{H}^i(X, F)$ there exists an étale cover $\{U_j \to X\}_{j \in J}$ such that the restriction of α to each $\mathrm{H}^i(U_j, F)$ is zero [Mil80, Prop. III.2.9, Remark III.2.11 (a)]. Take $U = \coprod_{j\in J} U_j$. □

Theorem 3.4.2 (Azumaya) *Let R be a henselian local ring with residue field k.*

(i) *The embedding of the closed point* $\mathrm{Spec}(k) \to \mathrm{Spec}(R)$ *induces an isomorphism* $\mathrm{Br}(R) \tilde{\longrightarrow} \mathrm{Br}(k)$.

(ii) *If R is a strictly henselian local ring, i.e. if k is separably closed, then* $\mathrm{Br}(R) = 0$.

Proof. For any smooth quasi-projective commutative group R-scheme G we have an isomorphism $\mathrm{H}^i(\mathrm{Spec}(R), G) \tilde{\longrightarrow} \mathrm{H}^i(k, G \times_R k)$ when $i \geq 1$, see [Mil80, Remark III.3.11 (a)]. For $G = \mathbb{G}_m$ we get the desired statement $\mathrm{Br}(R) \tilde{\longrightarrow} \mathrm{Br}(k)$.

(ii) follows from (i). Alternatively, by [Mil80, Thm. I.4.2 (d)] an étale morphism $U \to \mathrm{Spec}(R)$ has a section provided U contains a k-point which goes to the closed point of $\mathrm{Spec}(R)$. Thus (ii) is a consequence of Lemma 3.4.1. □

The original theorem of Azumaya concerns the Brauer–Azumaya group. We briefly outline the proof given in [Mil80, Cor. IV.2.13]. Let $\alpha \in \mathrm{Br}(R)$. Since R is local henselian, Lemma 3.4.1 implies that there exists a *finite* étale cover R'/R of henselian local rings such that α goes to 0 under the natural map $\mathrm{Br}(R) \to \mathrm{Br}(R')$. This implies that α belongs to $\mathrm{Br}_{\mathrm{Az}}(R)$. Therefore, $\mathrm{Br}_{\mathrm{Az}}(R) = \mathrm{Br}(R)$. Then one applies Hensel's lemma to suitable auxiliary smooth schemes over R to show that $\mathrm{Br}_{\mathrm{Az}}(R) = \mathrm{Br}(k)$.

Corollary 3.4.3 *Let R be a noetherian henselian local ring with maximal ideal $\mathfrak{m}$. Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Then the natural map* $\mathrm{Br}(R) \to \mathrm{Br}(\widehat{R})$ *is an isomorphism.*

Proof. Since R is noetherian, $\widehat{R}$ is a complete local ring with residue field $R/\mathfrak{m}$. In particular, $\widehat{R}$ is a henselian local ring. Now Azumaya's theorem says that the natural map $\mathrm{Br}(R) \to \mathrm{Br}(\widehat{R})$ is the identity map on $\mathrm{Br}(R/\mathfrak{m})$. □

Corollary 3.4.4 *Let k be a field, let X be a k-scheme and let $P \in X(k)$ be a k-point. For any $\alpha \in \mathrm{Br}(X)$ with $\alpha(P) = 0 \in \mathrm{Br}(k)$ there exist an étale morphism $f : U \to X$ and a k-point $M \in U(k)$ such that $f(M) = P$ and $f^*\alpha = 0 \in \mathrm{Br}(U)$.*

Proof. Let R be the henselisation of the local ring of X at P. By Theorem 3.4.2 (i) the image of α under the natural map $\mathrm{Br}(X) \to \mathrm{Br}(R)$ is zero. The ring R is a filtering direct limit of rings R_i, each of them equipped with an étale map $f_i : \mathrm{Spec}(R_i) \to X$ and a k-point M_i such that $f_i(M_i) = P$. The group $\mathrm{Br}(R)$ is the direct limit of the groups $\mathrm{Br}(R_i)$, see Section 2.2.4. Thus α goes to zero in $\mathrm{Br}(R_i)$ for some i, so we can take $U = \mathrm{Spec}(R_i)$. □.

Lemma 3.4.5 *Let k be a field and let X be a variety over k. Let $A \in \mathrm{Br}(X)$. There exists an integer $n > 0$ such that nA vanishes in each residue field of X.*

Proof. Suppose this has been proved for all varieties of dimension at most d. Let X be a variety of dimension $d+1$. To prove the result for X we may assume that it is reduced and irreducible. Let $k(X)$ be its function field. By Section 2.2.4, the torsion group $\mathrm{Br}(k(X))$ is the direct limit of the groups $\mathrm{Br}(U)$, where U is non-empty open in X. Thus there exists a non-empty open set $U \subset X$ such that the restriction of A to U is an element of $\mathrm{Br}(U)$ annihilated by some positive integer n. Let $Z = X \smallsetminus U$. By the induction hypothesis there exists an integer $m > 0$ such that the restriction of mA to residue fields of Z vanishes. Thus the restriction of nmA to residue fields of X vanishes. □

3.5 Going over to the generic point

A noetherian scheme X is called *geometrically locally factorial* if for any étale $U \to X$ each local ring of U is a unique factorisation domain. In particular, X is normal.

The notion is local on X for the Zariski topology. A regular local ring is geometrically locally factorial. More generally, a noetherian local ring which is a complete intersection in a regular local ring and which is regular in codimension ≤ 3 is geometrically locally factorial. For this result of Auslander and Buchsbaum, see [SGA2, Thm. XI.3.14].

Let X be a normal integral noetherian scheme and let $j : \mathrm{Spec}(F) \hookrightarrow X$ be its generic point. There is a natural exact sequence of sheaves in étale topology, which describes the embedding of the group of invertible regular functions into the group of non-zero rational functions as the kernel of the divisor map:

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow j_*\mathbb{G}_{m,F} \longrightarrow \bigoplus_{D \in X^{(1)}} i_{D*}\mathbb{Z}_{k(D)}, \tag{3.5}$$

see [Mil80, Example II.3.9]. Here $i_D : \mathrm{Spec}(k(D)) \hookrightarrow X$ is the embedding of the generic point of an irreducible divisor $D \subset X$; the direct sum ranges over all such divisors. The map $j_*\mathbb{G}_{m,F} \to i_{D*}\mathbb{Z}_{k(D)}$ can be described on an étale open set $U \to X$ as follows. Let D' be an irreducible divisor on U contained in $D \times_X U$. Since X is normal, U is also normal, hence the local ring $\mathcal{O}_{U,D'}$ is a discrete valuation ring with valuation $v_{D'} : \mathcal{O}_{U,D'} \smallsetminus \{0\} \to \mathbb{N}$. The group of sections $\mathrm{H}^0(U, j_*\mathbb{G}_{m,F})$ is the group of invertible elements in the ring of *rational* functions on U. The map $\mathrm{H}^0(U, j_*\mathbb{G}_{m,F}) \to \mathrm{H}^0(U, i_{D*}\mathbb{Z}_{k(D)})$ sends a function f to the integer $v_{D'}(f)$.

Now assume, in addition, that X is geometrically locally factorial. Then Weil divisors are the same as Cartier divisors, i.e. any divisor locally at each point is given by one equation. Thus (3.5) extends to an exact sequence

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow j_*\mathbb{G}_{m,F} \longrightarrow \bigoplus_{D \in X^{(1)}} i_{D*}\mathbb{Z}_{k(D)} \longrightarrow 0. \tag{3.6}$$

Remark 3.5.1 The exact sequence (3.6) restricted to the Zariski site of X is a flasque resolution of the Zariski sheaf $\mathbb{G}_{m,X}$. Recall that a Zariski sheaf $\mathcal{F}$ on X is *flasque* if for any Zariski open set $U \subset X$ the restriction map $\mathrm{H}^0(X,\mathcal{F})\to\mathrm{H}^0(U,\mathcal{F})$ is surjective. As remarked by Grothendieck in [Gro57], this implies $\mathrm{H}^i_{\mathrm{zar}}(X,\mathbb{G}_{m,X}) = 0$ for $i \geq 2$. This argument can be applied to any X which is locally factorial (in the usual sense, i.e. for the Zariski topology) and not necessarily regular.

Lemma 3.5.2 *Let X be a geometrically locally factorial integral scheme, for example, a regular integral noetherian scheme. Then the groups $\mathrm{H}^n_{\text{ét}}(X,\mathbb{G}_{m,X})$ are torsion groups for $n \geq 2$. In particular, the Brauer group $\mathrm{Br}(X)$ is a torsion group.*

Proof. This follows from Lemma 2.4.1 and the long exact sequence of cohomology attached to (3.6). □

Lemma 3.5.3 *Let X be a geometrically locally factorial (for example, regular) integral scheme with generic point $j : \mathrm{Spec}(F) \hookrightarrow X$. If $D \subset X$ is an irreducible divisor, we denote its generic point by $\mathrm{Spec}(k(D))$. There is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{H}^2(X, j_*\mathbb{G}_{m,F}) \longrightarrow \bigoplus_{D\in X^{(1)}} \mathrm{H}^1(k(D),\mathbb{Q}/\mathbb{Z}). \tag{3.7}$$

Proof. By Lemma 2.4.1 the long exact sequence of cohomology groups attached to (3.6) gives

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{H}^2(X, j_*\mathbb{G}_{m,F}) \longrightarrow \bigoplus_{D\in X^{(1)}} \mathrm{H}^2(X, i_{D*}\mathbb{Z}_{k(D)}).$$

By the same Lemma 2.4.1 the spectral sequence

$$\mathrm{H}^p(X,(R^q i_{D*})(\mathbb{Z}_{k(D)})) \Rightarrow \mathrm{H}^{p+q}(k(D),\mathbb{Z})$$

gives an injective map $\mathrm{H}^2(X, i_{D*}\mathbb{Z}_{k(D)})\to\mathrm{H}^2(k(D),\mathbb{Z})$. Multiplication by any non-zero integer is an automorphism of the abelian group $\mathbb{Q}$; however, any Galois cohomology group of positive degree is a torsion group [SerCG, Cor. 2.2.3], so $\mathrm{H}^n(k(D),\mathbb{Q}) = 0$ for $n > 0$. Thus the long exact sequence associated to the exact sequence of trivial Galois modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \tag{3.8}$$

gives an isomorphism $\mathrm{H}^1(k(D),\mathbb{Q}/\mathbb{Z})\xrightarrow{\sim}\mathrm{H}^2(k(D),\mathbb{Z}))$. This gives (3.7). □

Theorem 3.5.4 *Let X be a geometrically locally factorial (for example, regular) integral scheme with generic point $\mathrm{Spec}(F)$. The natural map $\mathrm{Br}(X)\to\mathrm{Br}(F)$ is injective. For any non-empty open subset $U \subset X$ this map factors through the natural map $\mathrm{Br}(X)\to\mathrm{Br}(U)$, which is therefore also injective.*

Proof. By Lemma 2.4.1 the spectral sequence

$$\mathrm{H}^p(X,(R^qj_*)(\mathbb{G}_{m,F}))\Rightarrow \mathrm{H}^{p+q}(F,\mathbb{G}_{m,F}) \tag{3.9}$$

implies that $\mathrm{H}^2(X,j_*\mathbb{G}_{m,F})$ is a subgroup of $\mathrm{H}^2(F,\mathbb{G}_{m,F})=\mathrm{Br}(F)$. Now (3.7) shows that $\mathrm{Br}(X)$ is naturally a subgroup of $\mathrm{Br}(F)$. □

Theorem 3.5.5 [Ber05] *Let X be a separated noetherian scheme and let $U\subset X$ be an open subscheme. Assume that U contains every generic point and every singular point of X. Then the restriction homomorphism $\mathrm{Br}(X)\to\mathrm{Br}(U)$ is injective.*

Proof. Let V be the open set of regular points. Then $X=U\cup V$. Let $W=U\cap V$. Since V is regular, the restriction map $\mathrm{Pic}(V)\to\mathrm{Pic}(W)$ is surjective. By the Mayer–Vietoris sequence (Theorem 3.2.2) the diagonal restriction map $\mathrm{Br}(X)\to\mathrm{Br}(U)\oplus\mathrm{Br}(V)$ is injective. If $\alpha\in\mathrm{Br}(X)$ has a trivial image in $\mathrm{Br}(U)$, then it has a trivial image at each generic point of U, hence it has a trivial image in $\mathrm{Br}(V)$. Indeed, as V is regular, the restriction map to the generic points is injective (Theorem 3.5.4). Thus $\alpha=0\in\mathrm{Br}(X)$. □.

Remark 3.5.6 In Section 7.7 we give counter-examples to the injectivity of the restriction map $\mathrm{Br}(R)\to\mathrm{Br}(K)$, where R is an integral local ring which is a local complete intersection and K is the field of fractions of R. In the second counter-example R is normal of dimension 2, in the third counter-example R is regular in codimension 2, but not in codimension 3. The ring R is not geometrically locally factorial.

3.6 Regular 1-dimensional schemes

This section follows [Gro68, III, §2] and [Mil80, III, Example 2.22]. Proposition 1.4.3, whose proof uses the Krull–Akizuki Theorem, enables one to recover all results stated in [Gro68, III, §2], without the excellence assumption added in [Mil80, III, Example 2.22].

Proposition 3.6.1 *Let X be an integral regular scheme of dimension 1 with generic point $\mathrm{Spec}(F)$.*

(i) *For any prime ℓ invertible on X there is an infinite exact sequence*

$$0\to\mathrm{H}^2(X,\mathbb{G}_m)\{\ell\}\to\mathrm{H}^2(F,\mathbb{G}_m)\{\ell\}\to\bigoplus_{x\in X^{(1)}}\mathrm{H}^1(k(x),\mathbb{Q}_\ell/\mathbb{Z}_\ell)\to\mathrm{H}^3(X,\mathbb{G}_m)\{\ell\}\to\ldots$$

$$\ldots\to\mathrm{H}^i(X,\mathbb{G}_m)\{\ell\}\to\mathrm{H}^i(F,\mathbb{G}_m)\{\ell\}\to\bigoplus_{x\in X^{(1)}}\mathrm{H}^{i-1}(k(x),\mathbb{Q}_\ell/\mathbb{Z}_\ell)\to\ldots$$

where $k(x)$ is the residue field of the point $x\in X$.

(ii) *If for each closed point $x \in X$ the residue field $k(x)$ is perfect, then there is an infinite exact sequence*

$$0\to\mathrm{Br}(X)\to\mathrm{Br}(F)\to \bigoplus_{x\in X^{(1)}} \mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^3(X,\mathbb{G}_m)\to\mathrm{H}^3(F,\mathbb{G}_m)\to\ldots$$
$$\ldots\to\mathrm{H}^i(X,\mathbb{G}_m)\to\mathrm{H}^i(F,\mathbb{G}_m)\to \bigoplus_{x\in X^{(1)}} \mathrm{H}^{i-1}(k(x),\mathbb{Q}/\mathbb{Z})\to\ldots$$

For each $x \in X^{(1)}$ the map $\mathrm{Br}(F)\to\mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z})$ is the composition of the restriction $\mathrm{Br}(F)\to\mathrm{Br}(F_x^{\mathrm{h}})$, where F_x^{h} is the field of fractions of the henselisation of the local ring $\mathcal{O}_{X,x}$, and the Witt residue $r_W : \mathrm{Br}(F_x^{\mathrm{h}})\to\mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z})$.

Proof. The exact sequence of sheaves (3.6)

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow j_*\mathbb{G}_{m,F} \longrightarrow \bigoplus_{D\in X^{(1)}} i_{D*}\mathbb{Z}_{k(D)} \longrightarrow 0$$

gives rise to the long exact sequence of étale cohomology groups

$$\ldots\to\mathrm{H}^i(X,\mathbb{G}_m)\to\mathrm{H}^i(X,j_*\mathbb{G}_{m,F})\to\mathrm{H}^i(X, \bigoplus_{x\in X^{(1)}} i_{x*}\mathbb{Z})\to\mathrm{H}^{i+1}(X,\mathbb{G}_m)\to\ldots$$

Since $\dim(X) = 1$, each inclusion $i_x : x\to X$ is a closed immersion, hence a finite morphism. Thus for any sheaf $\mathcal{F}$ on x we have $R^q i_{x*}(\mathcal{F}) = 0$ for $i \geq 1$. Therefore, we can re-write the above sequence as follows:

$$\ldots\to\mathrm{H}^i(X,\mathbb{G}_m)\to\mathrm{H}^i(X,j_*\mathbb{G}_{m,F})\to \bigoplus_{x\in X^{(1)}} \mathrm{H}^i(k(x),\mathbb{Z})\to\mathrm{H}^{i+1}(X,\mathbb{G}_m)\to\ldots$$

In particular, we have a long exact sequence

$$0\to\mathrm{H}^2(X,\mathbb{G}_m)\to\mathrm{H}^2(X,j_*\mathbb{G}_{m,F})\to \bigoplus_{x\in X^{(1)}} \mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^3(X,\mathbb{G}_m)\to\ldots$$

By Lemma 2.4.1 we have $R^1 j_*\mathbb{G}_{m,F} = 0$. For $q \geq 2$ the stalk of $R^q j_*\mathbb{G}_{m,F}$ at the generic point of X is the Galois cohomology group $\mathrm{H}^q(F_{\mathrm{s}},\mathbb{G}_m)$, where F_{s} is a separable closure of F, hence this stalk is zero. The stalk at a geometric point $\bar{x}$ above a closed point $x \in X$ is $\mathrm{H}^q(F_{\bar{x}}^{\mathrm{sh}},\mathbb{G}_m)$. By Proposition 1.4.3 (ii) this group is p_x-primary, where p_x is the characteristic exponent of the residue field $k(x)$. If $k(x)$ is perfect, then $\mathrm{H}^q(F_{\bar{x}}^{\mathrm{sh}},\mathbb{G}_m) = 0$ for all $q \geq 1$, by Proposition 1.4.3 (iv). If this holds for all x, then $R^q j_*\mathbb{G}_{m,F} = 0$ all $q \geq 1$.

From the spectral sequence (3.9)

$$\mathrm{H}^p(X, R^q j_*\mathbb{G}_{m,F}) \Rightarrow \mathrm{H}^{p+q}(F,\mathbb{G}_{m,F})$$

we then deduce the following statements.

- For $q \geq 2$ the natural map $\mathrm{H}^q(X,j_*\mathbb{G}_{m,F})\to\mathrm{H}^q(F,\mathbb{G}_m)$ induces an isomorphism of the ℓ-primary subgroups, for each prime ℓ invertible on X.

- The natural map $\mathrm{H}^q(X, j_*\mathbb{G}_{m,F}) \to \mathrm{H}^q(F, \mathbb{G}_m)$ is an isomorphism for all $q \geq 2$ if for each closed point $x \in X$ the residue field $k(x)$ is perfect.

This gives the exact sequences in the proposition.

To identify the map $\mathrm{Br}(F) \to \mathrm{H}^1(k(x), \mathbb{Q}/\mathbb{Z})$ with the Witt residue we can assume that $X = \mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{X,x})$. Let $K = F^{\mathrm{h}}_x$ be the field of functions of $\mathcal{O}^{\mathrm{h}}_{X,x}$. We follow the arguments from the proof of Lemma 2.3.3 using similar notation. Let K_{s} be a separable closure of K. Then $F^{\mathrm{sh}}_{\bar{x}}$ coincides with the maximal unramified extension K_{nr} of K in K_{s}. Define

$$G = \mathrm{Gal}(K_{\mathrm{s}}/K),\ I = \mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{nr}}),\ \Gamma = \mathrm{Gal}(k(x)_{\mathrm{s}}/k(x)) = \mathrm{Gal}(K_{\mathrm{nr}}/K) = G/I.$$

As discussed in Section 2.3.3, the category of étale sheaves on $\mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{X,x})$ is equivalent to the category of triples (M, N, φ), where M is a Γ-module, N is a G-module, and $\varphi : M \to N^I$ is a homomorphism of Γ-modules. Under the correspondence of sheaves on $\mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{X,x})$ and triples, the sheaf $j_*\mathbb{G}_{m,K}$ corresponds to the triple $(K^*_{\mathrm{nr}}, K^*_{\mathrm{s}}, \mathrm{id})$, the sheaf $i_*\mathbb{Z}_{k(x)}$ corresponds to $(\mathbb{Z}, 0, 0)$, and the map $j_*\mathbb{G}_{m,K} \to i_*\mathbb{Z}_{k(x)}$ is given by the valuation $K^*_{\mathrm{nr}} \to \mathbb{Z}$, see [Mil80, Example II.3.15]. According to (2.16) there is a canonical isomorphism

$$\mathrm{H}^2(\mathcal{O}^{\mathrm{h}}_{X,x}, j_*\mathbb{G}_{m,K}) = \mathrm{H}^2(\Gamma, K^*_{\mathrm{nr}}).$$

Under this isomorphism, the map

$$\mathrm{H}^2(\mathcal{O}^{\mathrm{h}}_{X,x}, j_*\mathbb{G}_{m,K}) \longrightarrow \mathrm{H}^2(k(x), \mathbb{Z}) = \mathrm{H}^1(k(x), \mathbb{Q}/\mathbb{Z})$$

becomes the Witt residue $\mathrm{H}^2(\Gamma, K^*_{\mathrm{nr}}) \to \mathrm{H}^2(\Gamma, \mathbb{Z}) = \mathrm{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z})$. □

The following theorem gives a description of the Brauer group of a henselian discrete valuation field K in the case when the residue field k is perfect. It can be compared to a similar description (1.14), where n is coprime to the characteristic of k but k is not necessarily perfect.

Theorem 3.6.2 (Witt) [Wit37] *Let R be a henselian discrete valuation ring with fraction field K and perfect residue field k. Then there is a split exact sequence*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(K) \longrightarrow \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0. \tag{3.10}$$

Proof. By the functoriality of étale cohomology the embedding of the closed point $\mathrm{Spec}(k) \to \mathrm{Spec}(R)$ gives rise to the specialisation map $\mathrm{Br}(R) \to \mathrm{Br}(k)$. This map is an isomorphism by Theorem 3.4.2. Now (3.10) follows from Proposition 3.6.1 in view of the surjectivity of the Witt residue, see Section 1.4.3. □

Corollary 3.6.3 *Let R be a henselian discrete valuation ring with fraction field K and* finite *residue field k. Then $\mathrm{Br}(K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$.*

Proof. By Theorem 1.2.11 (Wedderburn) we have $\mathrm{Br}(k) = 0$. In this case the Galois group Γ is the profinite completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$ generated by the Frobenius automorphism. Hence $\mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$. □

In particular, when $K = F_v$ is the completion of a global field F at a non-archimedean place v we obtain an isomorphism

$$\mathrm{inv}_v : \mathrm{Br}(F_v) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z},$$

called the *local invariant*. For example, if F_v is the field of p-adic numbers $\mathbb{Q}_p$, $p \neq 2$, and $a \in \mathbb{Z}_p^*$, by formula (1.16), $\mathrm{inv}_p(a,p) = 0$ if and only if the Legendre symbol $\left(\frac{a}{p}\right) = 1$.

There are other cases when the exact sequence of Proposition 3.6.1 can be completed by 0 on the right.

Theorem 3.6.4 *Let A be a semi-local Dedekind domain with field of fractions K. Let ℓ be a prime invertible in A. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(A)\{\ell\} \longrightarrow \mathrm{Br}(K)\{\ell\} \longrightarrow \bigoplus_{\mathfrak{p}} \mathrm{H}^1(A/\mathfrak{p}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow 0,$$

where $\mathfrak{p}$ ranges over the maximal ideals of A.

Proof. By Proposition 3.6.1 it remains to prove the surjectivity of the third map in the sequence. Choose a maximal ideal $\mathfrak{p} \subset A$ and let $x \in \mathrm{H}^1(A/\mathfrak{p}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. The group $\mathrm{H}^1(A/\mathfrak{p}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is the union of subgroups $\mathrm{H}^1(A/\mathfrak{p}, \mathbb{Z}/\ell^m)$, so x is in $\mathrm{H}^1(A/\mathfrak{p}, \mathbb{Z}/n)$ for some $n = \ell^m$. It is enough to find an element $\alpha \in \mathrm{Br}(K)[n]$ such that $\partial_{\mathfrak{p}}(\alpha) = x$ and $\partial_{\mathfrak{p}'}(\alpha) = 0$ for all maximal ideals $\mathfrak{p}' \neq \mathfrak{p}$ of A.

Let $A_{\mathfrak{p}}$ be the localisation of A at $\mathfrak{p}$ and let $A_{\mathfrak{p}}^{\mathrm{h}}$ be the henselisation of the local ring $A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}^{\mathrm{h}}$ is a henselian local ring, the specialisation map

$$\mathrm{H}^1(A_{\mathfrak{p}}^{\mathrm{h}}, \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{H}^1(A/\mathfrak{p}, \mathbb{Z}/n)$$

is an isomorphism. Let $\tilde{x} \in \mathrm{H}^1(A_{\mathfrak{p}}^{\mathrm{h}}, \mathbb{Z}/n)$ be the inverse image of x under this isomorphism.

Consider a finite separable field extension $K \subset L$ with the following two properties: if B is the integral closure of $A_{\mathfrak{p}}$ in L, then the embedding of the closed point $\mathrm{Spec}(A/\mathfrak{p}) \to \mathrm{Spec}(A_{\mathfrak{p}})$ factors as

$$\mathrm{Spec}(A/\mathfrak{p}) \longrightarrow \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A_{\mathfrak{p}})$$

and the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A_{\mathfrak{p}})$ is étale at the image of $\mathrm{Spec}(A/\mathfrak{p})$ in $\mathrm{Spec}(B)$. Let $\mathfrak{q} \subset B$ be the prime ideal such that $\mathrm{Spec}(B/\mathfrak{q})$ is this image of $\mathrm{Spec}(A/\mathfrak{p})$, and let $B_{\mathfrak{q}}$ be the localisation of B at $\mathfrak{q}$. Then, as was recalled in Section 2.1.3,

$$A_{\mathfrak{p}}^{\mathrm{h}} = \varinjlim B_{\mathfrak{q}}.$$

We have an isomorphism of residue fields $A/\mathfrak{p} = A_{\mathfrak{p}}/\mathfrak{p} \cong B_{\mathfrak{q}}/\mathfrak{q} = B/\mathfrak{q}$. Since L is separable over K, the A-algebra B is a finitely generated A-module. By the Krull–Akizuki theorem, B is a semi-local Dedekind domain, so B has finitely many maximal ideals. (See [SerCL, Ch. I, §4].)

Since $\mathrm{H}^1(A_{\mathfrak{p}}^{\mathrm{h}}, \mathbb{Z}/n)$ is the inductive limit of $\mathrm{H}^1(B_{\mathfrak{q}}, \mathbb{Z}/n)$ (see Section 2.2.4), our element $\tilde{x} \in \mathrm{H}^1(A_{\mathfrak{p}}^{\mathrm{h}}, \mathbb{Z}/n)$ comes from an element $\rho \in \mathrm{H}^1(B_{\mathfrak{q}}, \mathbb{Z}/n)$ for some ring B as above. The injective map $\mathrm{H}^1(B_{\mathfrak{q}}, \mathbb{Z}/n) \to \mathrm{H}^1(L, \mathbb{Z}/n)$ allows us to consider ρ as an element of $\mathrm{H}^1(L, \mathbb{Z}/n)$.

By the independence of valuations we can choose $t \in B$ such that the valuation of t at $\mathfrak{q}$ is 1 and $t \equiv 1 \bmod \mathfrak{q}'$ for each maximal ideal $\mathfrak{q}' \subset B$, $\mathfrak{q}' \neq \mathfrak{q}$. Let $\beta \in \mathrm{H}^2(L, \mu_n) = \mathrm{Br}(L)[n]$ be the cup-product of the class of t in $L^*/L^{*n} = \mathrm{H}^1(L, \mu_n)$ and the class $\rho \in \mathrm{H}^1(L, \mathbb{Z}/n)$. By Proposition 1.4.6, corestriction gives rise to a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(L)[n] & \longrightarrow & \bigoplus_{J \subset B} \mathrm{H}^1(B/J, \mathbb{Z}/n) \\ \downarrow{\scriptstyle \mathrm{cores}_{L/K}} & & \downarrow{\scriptstyle \mathrm{cores}_{(B/J)/(A/I)}} \\ \mathrm{Br}(K)[n] & \longrightarrow & \bigoplus_{I \subset A} \mathrm{H}^1(A/I, \mathbb{Z}/n) \end{array}$$

where the horizontal maps are residues, I ranges over the maximal ideals of A, and J ranges over the maximal ideals of B. We have $\partial_{\mathfrak{q}}(\beta) = x$ and $\partial_{\mathfrak{q}'}(\beta) = 0$ when $\mathfrak{q}' \subset B$ is a maximal ideal $\mathfrak{q}' \neq \mathfrak{q}$. Now let $\alpha = \mathrm{cores}_{L/K}(\beta)$. From the diagram we obtain $\partial_{\mathfrak{p}}(\alpha) = x$ and $\partial_{\mathfrak{p}'}(\alpha) = 0$ when $\mathfrak{p}' \subset A$ is a maximal ideal $\mathfrak{p}' \neq \mathfrak{p}$. $\square$

Remark 3.6.5 The same proof gives exact sequences

$$0 \longrightarrow \mathrm{H}^i(A, \mu_n^{\otimes j}) \longrightarrow \mathrm{H}^i(K, \mu_n^{\otimes j}) \longrightarrow \bigoplus_{\mathfrak{p} \subset A} \mathrm{H}^{i-1}(A/\mathfrak{p}, \mu_n^{\otimes j-1}) \longrightarrow 0,$$

where n is invertible in A, for any $i, j \in \mathbb{Z}$, $i \geq 1$, see [CTKH97, Cor. B.3.3]. It works also for various other theories such as Milnor's K-theory with torsion coefficients (H. Gillet).

3.7 Purity for the Brauer group

The results in this section were proved by Grothendieck in the case of smooth varieties over a field for the torsion prime to the characteristic of the field. Thanks to Gabber's absolute purity (Theorem 2.3.1) we can state Grothendieck's purity theorem for the Brauer group in a more general form.

Theorem 3.7.1 *Let X be a regular integral scheme, let $U \subset X$ be a dense open subscheme and let ℓ be a prime different from the residual characteristics of X. Let $D_1, \ldots, D_m$ be the irreducible components of the regular locus* [1] *of $X \setminus U$ that have codimension 1 in X. Then we have an exact sequence*

$$0 \longrightarrow \mathrm{Br}(X)\{\ell\} \longrightarrow \mathrm{Br}(U)\{\ell\} \longrightarrow \bigoplus_{i=1}^{m} \mathrm{H}^1(D_i, \mathbb{Q}_\ell/\mathbb{Z}_\ell). \qquad (3.11)$$

[1] In [Gro68, Chap. III, §6 formula (6.4) and Thm. 6.1] this regularity condition should have been added.

We denote the image of $\alpha \in \mathrm{Br}(U)\{\ell\}$ in $\mathrm{H}^1(D_i, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \subset \mathrm{H}^1(k(D_i), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ by $\partial_{D_i}(\alpha)$.

This theorem immediately implies the following

Theorem 3.7.2 *Let X be a regular integral scheme, let $U \subset X$ be a dense open subscheme and let ℓ be a prime different from the residual characteristics of X. Then we have an exact sequence*

$$0 \longrightarrow \mathrm{Br}(X)\{\ell\} \longrightarrow \mathrm{Br}(U)\{\ell\} \longrightarrow \bigoplus_{D} \mathrm{H}^1(k(D), \mathbb{Q}_\ell/\mathbb{Z}_\ell), \tag{3.12}$$

where D ranges over the irreducible divisors of X with support in $X \smallsetminus U$ and $k(D)$ denotes the residue field at the generic point of D.

The *residue* of $\alpha \in \mathrm{Br}(U)\{\ell\}$ at the generic point of D_i is defined as the image

$$\partial_{D_i}(\alpha) \in \mathrm{H}^1(k(D_i), \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Passing to the inductive limit over U one deduces the following corollary.

Corollary 3.7.3 *Let X be a regular integral scheme with generic point* $\mathrm{Spec}(F)$ *and let ℓ be a prime different from the residual characteristics of X. Then we have an exact sequence*

$$0 \longrightarrow \mathrm{Br}(X)\{\ell\} \longrightarrow \mathrm{Br}(F)\{\ell\} \longrightarrow \bigoplus_{D \in X^{(1)}} \mathrm{H}^1(k(D), \mathbb{Q}_\ell/\mathbb{Z}_\ell), \tag{3.13}$$

where $k(D)$ denotes the residue field at the generic point of D.

Proof of Theorem 3.7.1. Let $Z = X \smallsetminus U$. Applying the functor $\mathrm{Ext}_X(\cdot, \mathbb{G}_m)$ to the exact sequence (2.10) we obtain a long exact sequence of cohomology with support:

$$\ldots \longrightarrow \mathrm{H}^n_Z(X, \mathbb{G}_m) \longrightarrow \mathrm{H}^n(X, \mathbb{G}_m) \longrightarrow \mathrm{H}^n(U, \mathbb{G}_m) \longrightarrow \mathrm{H}^{n+1}_Z(X, \mathbb{G}_m) \longrightarrow \ldots$$

Let us first consider the case when Z is regular of codimension c in X at each point of Z. By the Kummer sequence the sheaf $\mathcal{H}^n_Z(X, \mathbb{G}_m)[\ell^m]$ is a quotient of the sheaf $\mathcal{H}^n_Z(X, \mu_{\ell^m})$. The latter sheaf is 0 when $n \leq 2c - 1$ by Gabber's absolute purity (Theorem 2.3.1). Thus for $c \geq 2$ the spectral sequence (2.9) with $\mathcal{F} = \mathbb{G}_{m,X}$, namely,

$$\mathrm{H}^p(Z, \mathcal{H}^q_Z(X, \mathbb{G}_{m,X})) \Rightarrow \mathrm{H}^{p+q}_Z(X, \mathbb{G}_{m,X}) \tag{3.14}$$

gives $\mathrm{H}^2_Z(X, \mathbb{G}_m)\{\ell\} = \mathrm{H}^3_Z(X, \mathbb{G}_m)\{\ell\} = 0$. Hence in this case the above long exact sequence gives an isomorphism

$$\mathrm{Br}(X)\{\ell\} \xrightarrow{\sim} \mathrm{Br}(U)\{\ell\}, \tag{3.15}$$

which gives the desired statement.

Now let $c = 1$. The exact sequence (3.2) based on the Kummer sequence gives rise to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Pic}(X)/\ell^n & \to & \mathrm{H}^2_{\text{ét}}(X,\mu_{\ell^n}) & \to & \mathrm{Br}(X)[\ell^n] & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Pic}(U)/\ell^n & \to & \mathrm{H}^2_{\text{ét}}(U,\mu_{\ell^n}) & \to & \mathrm{Br}(U)[\ell^n] & \to & 0 \end{array}$$

Since X is regular, the left hand vertical map is surjective, and the right hand vertical map $\mathrm{Br}(X)[\ell^n]\to\mathrm{Br}(U)[\ell^n]$ is injective by Proposition 3.5.4. We have $c = 1$, so Z is a divisor in X. Since Z is regular, it is a disjoint union of its irreducible components $D_1, \ldots, D_m$. The snake lemma applied to the above commutative diagram combined with the Gysin exact sequence (2.15) gives the exact sequence

$$0\to\mathrm{Br}(X)[\ell^n]\to\mathrm{Br}(U)[\ell^n]\to\bigoplus_{i=1}^{m}\mathrm{H}^1(D_i,\mathbb{Z}/\ell^n)\to\mathrm{H}^3(X,\mu_{\ell^n})\to\mathrm{H}^3(U,\mu_{\ell^n}). \tag{3.16}$$

Taking the limit as $n\to\infty$ we obtain (3.11).

For an arbitrary proper closed reduced subscheme $Z \subset X$ we define a descending chain of closed subschemes

$$Z = Z_0 \supset Z_1 \supset Z_2 \supset \ldots$$

as follows. For $n \geq 1$ define Z_n as the union of the singular locus of Z_{n-1} and the union of irreducible components of Z_{n-1} which have codimension at least $n+1$ in X. Then Z is the disjoint union of locally closed regular subschemes $Z_{n-1} \smallsetminus Z_n$ for $n \geq 1$. We note that $Z_{n-1} \smallsetminus Z_n$ is either empty or of pure codimension n in $X \smallsetminus Z_n$.

Unless Z_0 is regular and of pure codimension 1, the last non-empty complement $Z_{n-1} \smallsetminus Z_n$, where $n \geq 2$, is a closed regular subscheme of X of constant codimension n, thus removing it from X does not affect the ℓ-primary torsion of the Brauer group, as we have seen in the beginning of the proof. Repeating the operation we end up with an isomorphism $\mathrm{Br}(X)\{\ell\} = \mathrm{Br}(X \smallsetminus Z_1)\{\ell\}$. If $Z = Z_1$, we are done. Otherwise, we can apply (3.16) to the regular subscheme $Z \smallsetminus Z_1$ of $X \smallsetminus Z_1$ to obtain (3.11). □

The embedding $i_D : \mathrm{Spec}(k(D))\to X$ of the generic point of D factors as

$$\mathrm{Spec}(k(D))\to\mathrm{Spec}(\widehat{\mathcal{O}_{X,D}})\to\mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{X,D})\to\mathrm{Spec}(\mathcal{O}_{X,D})\to X,$$

where $\widehat{\mathcal{O}_{X,D}}$ is the completion and $\mathcal{O}^{\mathrm{h}}_{X,D}$ is the henselisation of the discrete valuation ring $\mathcal{O}_{X,D}$ (the henselisation and the completion of a noetherian local ring do not affect the residue field). Each residue map $\mathrm{Br}(F)\{\ell\}\to\mathrm{H}^1(k(D),\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ can be computed at the level of the local ring $\mathcal{O}_{X,D}$ which is a discrete valuation ring with residue field $k(D)$ and field of fractions $k(X)$. By Lemma 2.3.3 it equals $-r$, where r is the residue map with finite coefficients μ_{ℓ^n} in the exact sequence (1.9), see Section 1.4.1.

By Proposition 3.6.1 the residue map of Section 3.6 is the Witt residue with coefficients in $\mathbb{G}_m$. By Theorem 1.4.10 it coincides with the residue map discussed in this section (when both maps are defined); indeed, each of these maps is equal to $-r$.

It is important to understand the functorial behaviour of residues.

Theorem 3.7.4 *Let X be a regular scheme and let $Y \subset X$ be a regular irreducible divisor. Let X' be a regular integral scheme and let $f : X' \to X$ be a morphism such that $f(X')$ is not contained in Y. The divisor $f^{-1}(Y) \subset X'$ can be written as a finite sum $\sum_{t\in T} r_t Z_t$, where $Z_t \subset X'$ is an irreducible divisor and r_t is a positive integer, for $t \in T$.*

Let ℓ be a prime invertible on X. For any $\alpha \in \mathrm{Br}(X \smallsetminus Y)\{\ell\}$ and any $t \in T$ the residue $\partial_{Z_t}(f^*(\alpha))$ is the image of $r_t \partial_Y(\alpha)$ under the composite map

$$\mathrm{H}^1(Y, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \mathrm{H}^1(Z_t, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \mathrm{H}^1(k(Z_t), \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Proof. Let $U = X \smallsetminus Y$ and let $U' = f^{-1}(U) = X' \smallsetminus f^{-1}(Y)$. Let Z' be a regular dense open subset of Z_t. By removing a closed subset from X' we can assume that $X' \smallsetminus U' = Z'$.

Let $m \geq 1$ be such that $\ell^m \alpha = 0$. Then α comes from some $\tilde{\alpha} \in \mathrm{H}^2(U, \mu_{\ell^m})$. We have $f^*\tilde{\alpha} \in \mathrm{H}^2(U', \mu_{\ell^m})$. As (X, Y) and (X', Z') are regular pairs of codimension 1, we have the associated Gysin sequences. The commutative diagram from Lemma 2.3.4 implies that $\partial_{Z'}(f^*\tilde{\alpha}) = r f^* \partial_Y(\tilde{\alpha})$. The proof is finished by taking the restriction to the generic point $\mathrm{Spec}(k(Z_t)) = \mathrm{Spec}(k(Z'))$. □

The following general result, many special cases of which had been earlier established, was recently proved by Česnavičius [Čes].

Theorem 3.7.5 *Let X be a regular integral scheme and let $U \subset X$ be an open set whose complement is of codimension at least 2. Then the restriction map*

$$\mathrm{Br}(X) \longrightarrow \mathrm{Br}(U)$$

is an isomorphism.

For the ℓ-primary subgroup of the Brauer group, where ℓ is a prime invertible on X, this is a special case of Theorem 3.7.2, itself a consequence of Gabber's purity theorem. Česnavičius' proof uses the result in dimension ≤ 2 (Auslander–Goldman, Grothendieck [Gro68, II, Thm. 2.1]), the result in dimension 3 (Gabber [Ga81, Thm. 2', p. 131]), Theorem 3.7.2 and other results by Gabber, as well as Scholze's recent theory of perfectoid spaces and tilting equivalence to handle p-torsion in the local unequal characteristic case.

As an easy consequence, we have

Theorem 3.7.6 *Let X be a noetherian, regular, integral scheme with function field F. Then $\mathrm{Br}(X) \subset \mathrm{Br}(F)$ is the subgroup*

$$\bigcap_{x\in X^{(1)}} \mathrm{Br}(\mathcal{O}_{X,x}).$$

Proof. The inclusion $\mathrm{Br}(X) \subset \bigcap_{x\in X^{(1)}} \mathrm{Br}(\mathcal{O}_{X,x}) \subset \mathrm{Br}(F)$ is clear. Let $\alpha \in \mathrm{Br}(F)$ be in $\bigcap_{x\in X^{(1)}} \mathrm{Br}(\mathcal{O}_{X,x})$. Using the fact that the Brauer group commutes with limits (Section 2.2.4), one finds a non-empty open set $U \subset X$ and an element $\beta \in \mathrm{Br}(U)$ such that β maps to $\alpha \in \mathrm{Br}(F)$. Let U be a maximal open subset of X with this property. Suppose that there exists a codimension 1 point $x \in X$ which is not in U. Since α is in the image of $\mathrm{Br}(\mathcal{O}_{X,x})$, there exists an open set $V \subset X$ containing x and an element $\gamma \in \mathrm{Br}(V)$ that maps to $\alpha \in \mathrm{Br}(F)$. Consider the Mayer–Vietoris exact sequence (Theorem 3.2.2)

$$\mathrm{Br}(U \cup V) \longrightarrow \mathrm{Br}(U) \oplus \mathrm{Br}(V) \longrightarrow \mathrm{Br}(U \cap V).$$

Since X is regular, by Theorem 3.5.4 the map $\mathrm{Br}(U \cap V)\to\mathrm{Br}(F)$ is injective. Thus there exists $\delta \in \mathrm{Br}(U \cup V)$ that goes to α. Since $x \notin U$, we have a contradiction. Thus the complement to U in X has codimension at least 2. By the purity theorem (Theorem 3.7.5) the inclusion $\mathrm{Br}(X) \subset \mathrm{Br}(U)$ is an equality. This completes the proof. □

This immediately implies

Proposition 3.7.7 *Let X be a regular integral scheme with function field F. Let $A_i \subset F$, for $i \in I$, be the discrete valuation rings $A \subset F$ with fraction field F which lie over X, that is, such that the map $\mathrm{Spec}(F)\to X$ factors through $\mathrm{Spec}(F)\to\mathrm{Spec}(A)$. Then $\mathrm{Br}(X) \subset \mathrm{Br}(F)$ is the subgroup $\bigcap_{i\in I} \mathrm{Br}(A_i) \subset \mathrm{Br}(F)$.* □

Proposition 3.7.8 *Let S be a scheme, let X be a regular integral scheme with function field F and let $X\to S$ be a proper morphism. Let $A_i \subset F$, $i \in I$, be the discrete valuation rings $A \subset F$ with fraction field F which lie over S, that is, such that the composition $\mathrm{Spec}(F)\to X\to S$ factors through $\mathrm{Spec}(F)\to\mathrm{Spec}(A)$. Then $\mathrm{Br}(X) \subset \mathrm{Br}(F)$ is the subgroup $\bigcap_{i\in I} \mathrm{Br}(A_i) \subset \mathrm{Br}(F)$.* □

Proof. The morphism $X\to S$ is proper, in particular, it is separated and of finite type. By the valuative criterion of properness [Stacks, Lemma 0BX5] there exists a unique morphism $\mathrm{Spec}(A)\to X$ such that the composition $\mathrm{Spec}(F)\to X\to S$ factors as

$$\mathrm{Spec}(F) \longrightarrow \mathrm{Spec}(A) \longrightarrow X \longrightarrow S.$$

It remains to apply Proposition 3.7.7. □

This proposition can be applied to a smooth, proper, integral variety X over a field k to deduce the birational invariance of $\mathrm{Br}(X)$, see Proposition 5.2.2.

Proposition 3.7.9 *Let S be a scheme. Let X and Y be integral, regular, proper S-schemes, with function fields F_X and F_Y, respectively. Suppose there exists an isomorphism $g : F_X \xrightarrow{\sim} F_Y$ such that $\mathrm{Spec}(F_Y)\xrightarrow{\sim}\mathrm{Spec}(F_X)$ is an isomorphism of S-schemes. Then the induced isomorphism $\mathrm{Br}(F_X)\xrightarrow{\sim}\mathrm{Br}(F_Y)$ restricted to the subgroup $\mathrm{Br}(X)$ is an isomorphism $\mathrm{Br}(X)\xrightarrow{\sim}\mathrm{Br}(Y)$ compatible with natural maps $\mathrm{Br}(S)\to\mathrm{Br}(X)$ and $\mathrm{Br}(S)\to\mathrm{Br}(Y)$.*

Proof. Note that in Proposition 3.7.8 the collection of A_i, $i \in I$, is defined solely in terms of the morphism $\mathrm{Spec}(F)\to S$. Therefore, an isomorphism of S-schemes $\mathrm{Spec}(F_Y) \cong \mathrm{Spec}(F_X)$ gives rise to the desired isomorphisms. □

3.8 The Brauer group and finite morphisms

Let X be a connected scheme. Let $f : Y \to X$ be a finite locally free morphism of schemes. This means that locally for the Zariski topology on X the morphism is of the form $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, where B a free A-module of finite rank. Since X is connected, the rank is constant; let us denote it by d. If X is locally noetherian, the hypothesis on f is equivalent to f being flat and finite.

The norm of $b \in B$ is the determinant of the matrix that gives the multiplication by b on B with respect to some basis of B. It does not depend on the basis. The norm is multiplicative; the norm of $a \in A$ is a^d. We obtain a map of coherent sheaves $f_*\mathcal{O}_Y \to \mathcal{O}_X$. The composition of the canonical map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ with $f_*\mathcal{O}_Y \to \mathcal{O}_X$ sends u to u^d, cf. [Mum66, Lecture 10]. The étale sheaf $\mathbb{G}_{m,X}$ is defined by setting $\mathbb{G}_{m,X}(U) = \Gamma(U, \mathcal{O}_U)^*$ for any étale morphism $U \to X$, and similarly for $\mathbb{G}_{m,Y}$. We thus obtain natural morphisms of sheaves

$$\mathbb{G}_{m,X} \longrightarrow f_*\mathbb{G}_{m,Y} \longrightarrow \mathbb{G}_{m,X},$$

whose composition sends u to u^d. By the finiteness of f, the functor f_* from the category of étale sheaves on Y to the category of étale sheaves on X is exact [Mil80, Cor. II.3.6]. Thus the Leray spectral sequence (2.4) gives an isomorphism $\mathrm{H}^n_{\text{ét}}(X, f_*\mathbb{G}_{m,Y}) \xrightarrow{\sim} \mathrm{H}^n_{\text{ét}}(Y, \mathbb{G}_{m,Y})$ which identifies the canonical map (3.1) with $\mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_{m,X}) \to \mathrm{H}^n_{\text{ét}}(X, f_*\mathbb{G}_{m,Y})$. We thus obtain the *restriction* and *corestriction* maps

$$\mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_{m,X}) \xrightarrow{\mathrm{res}_{Y/X}} \mathrm{H}^n_{\text{ét}}(Y, \mathbb{G}_{m,Y}) \xrightarrow{\mathrm{cores}_{Y/X}} \mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_{m,X})$$

whose composition is the multiplication by d. Here the restriction $\mathrm{res}_{Y/X}$ is the canonical map $f^* : \mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_{m,X}) \to \mathrm{H}^n_{\text{ét}}(Y, \mathbb{G}_{m,Y})$. For $n = 2$ we obtain the restriction and corestriction maps of Brauer groups

$$\mathrm{res}_{Y/X} : \mathrm{Br}(X) \longrightarrow \mathrm{Br}(Y), \quad \mathrm{cores}_{Y/X} : \mathrm{Br}(Y) \to \mathrm{Br}(X).$$

The following proposition, which will be used in Section 5.3, is a standard formalism that applies to various functors.

Proposition 3.8.1 *Let Y and X be schemes and let $f : Y \to X$ be a finite locally free morphism of constant rank. Let $i : V \to X$ be a morphism and let $W = V \times_X Y$. Let $j : W \to Y$ and $g : W \to V$ be the natural projections; here g is a finite locally free morphism of constant rank. The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Br}(Y) & \xrightarrow{j^*} & \mathrm{Br}(W) \\ \downarrow{\scriptstyle \mathrm{cores}_{Y/X}} & & \downarrow{\scriptstyle \mathrm{cores}_{W/V}} \\ \mathrm{Br}(X) & \xrightarrow{i^*} & \mathrm{Br}(V) \end{array}$$

Proof. We have $fj = ig$, hence $f_*j_*\mathbb{G}_{m,W} = i_*g_*\mathbb{G}_{m,W}$. There is a commutative diagram of étale sheaves on X

$$\begin{array}{ccc} f_*\mathbb{G}_{m,Y} & \longrightarrow & i_*g_*\mathbb{G}_{m,W} \\ \downarrow & & \downarrow \\ \mathbb{G}_{m,X} & \longrightarrow & i_*\mathbb{G}_{m,V} \end{array} \tag{3.17}$$

where the left vertical arrow is the norm map associated to f and the right vertical arrow is induced by the norm map $g_*\mathbb{G}_{m,W}\to\mathbb{G}_{m,V}$. Applying cohomology to (3.17), we see that the bottom left square of the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}^2(Y,\mathbb{G}_{m,Y}) & \longrightarrow & \mathrm{H}^2(Y,j_*\mathbb{G}_{m,W}) & \longrightarrow & \mathrm{H}^2(W,\mathbb{G}_{m,W}) \\ \cong\uparrow & & \cong\uparrow & & \cong\uparrow \\ \mathrm{H}^2(X,f_*\mathbb{G}_{m,Y}) & \longrightarrow & \mathrm{H}^2(X,i_*g_*\mathbb{G}_{m,W}) & \longrightarrow & \mathrm{H}^2(V,g_*\mathbb{G}_{m,W}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^2(X,\mathbb{G}_{m,X}) & \longrightarrow & \mathrm{H}^2(X,i_*\mathbb{G}_{m,V}) & \longrightarrow & \mathrm{H}^2(V,\mathbb{G}_{m,V}) \end{array}$$

The right hand horizontal and the top vertical arrows are natural maps $E_2^{2,0}\to E^2$ in the spectral sequence attached to a morphism. In the case of top vertical maps these are finite morphisms f and g, hence the functor f_* from the category of étale sheaves on Y to the category of étale sheaves on X is exact [Mil80, Cor. II.3.6], and the same applies to g_*. Thus the top vertical maps are isomorphisms. The bottom vertical maps are induced by the norm maps $f_*\mathbb{G}_{m,Y}\to\mathbb{G}_{m,X}$ and $g_*\mathbb{G}_{m,W}\to\mathbb{G}_{m,V}$. All this ensures that the whole diagram is commutative.

Retaining the four corners of the last diagram we obtain the commutative diagram of the proposition. □

The definitions of restriction and corestriction given above can be applied to the case when X is a scheme over a field k. A finite (not necessarily separable) extension $k \subset L$ gives rise a finite locally free morphism $X_L = X \times_k L\to X$ of rank $[L : k]$, so we obtain the restriction and corestriction maps

$$\mathrm{Br}(X) \xrightarrow{\mathrm{res}_{L/k}} \mathrm{Br}(X_L) \xrightarrow{\mathrm{cores}_{L/k}} \mathrm{Br}(X)$$

whose composition is multiplication by $[L : k]$. E.g., if $X = \mathrm{Spec}(k)$, we get the corestiction map $\mathrm{cores}_{L/k} : \mathrm{Br}(L)\to\mathrm{Br}(k)$.

The composition $\mathrm{cores}_{L/k} \circ \mathrm{res}_{L/k}$ is the multiplication by $[L : k]$ on $\mathrm{Br}(k)$.

One application is the following proposition.

Proposition 3.8.2 *Let K be a field of transcendence degree 1 over a separably closed field k of characteristic $p > 0$. Then $\mathrm{Br}(K)$ is a p-primary torsion group.*

Proof. There is a geometrically integral curve C over k such that $K = k(C)$. Let $\bar{k}$ be an algebraic closure of k. Take any $\alpha \in \mathrm{Br}(k(C))$. By Tsen's Theorem 1.2.12, the image of α in $\mathrm{Br}(\bar{k}(C))$ is zero. Using Theorem 1.3.5, one sees that there is a finite extension $k \subset E \subset \bar{k}$ such that $\mathrm{res}_{E(C)/k(C)}(\alpha) = 0$. The degree $[E : k] = [E(C) : k(C)]$ is a power of p. By the corestriction-restriction formula, we see that $\alpha \in \mathrm{Br}(k(C))$ is annihilated by a power of p. □

Proposition 3.8.3 *Let X and Y be regular integral schemes and let $f : Y \to X$ be a dominant, generically finite morphism of degree d. Then the kernel of the natural map $f^* : \mathrm{Br}(X) \to \mathrm{Br}(Y)$ is killed by d. In particular, for any integer $n > 1$ coprime to d the map $f^* : \mathrm{Br}(X)[n] \to \mathrm{Br}(Y)[n]$ is injective.*

Proof. By Proposition 3.5.4 the embedding of the generic point $\mathrm{Spec}(k(X))$ in X induces an injective map $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k(X))$, and similarly for Y. Since the composition of restriction and corestriction

$$\mathrm{cores}_{k(Y)/k(X)} \circ \mathrm{res}_{k(Y)/k(X)} : \mathrm{Br}(k(X)) \longrightarrow \mathrm{Br}(k(Y)) \longrightarrow \mathrm{Br}(k(X))$$

is the multiplication by d, the kernel of the natural map $f^* : \mathrm{Br}(X) \to \mathrm{Br}(Y)$ is killed by d, so our statement follows. □

Theorem 3.8.4 *Let X and Y be regular integral schemes and let $f : Y \to X$ be a finite flat morphism of degree d such that $k(Y)$ is a Galois extension of $k(X)$ with Galois group G. Then $d\,\mathrm{Br}(Y)^G \subset f^*\mathrm{Br}(X) \subset \mathrm{Br}(Y)$.*

In particular, for any integer $n > 1$ coprime to $d = |G|$ the natural map $f^* : \mathrm{Br}(X)[n] \to \mathrm{Br}(Y)[n]^G$ is an isomorphism.

Proof. For $\mathrm{Spec}(A) \subset X$ an affine open set in X, the inverse image in Y is an affine scheme $\mathrm{Spec}(B)$. The ring B is regular hence normal, is finite over A, and its fraction field is $k(Y)$. Hence B is the integral closure of A in $k(Y)$. Thus the action of G on L induces an action of G on B. Covering X by affine open sets, we get that the action of G on $k(Y)$ induces an action of G on Y. This induces an action of G on $\mathrm{Br}(Y)$.

We claim that the composition

$$\mathrm{res}_{Y/X} \circ \mathrm{cores}_{Y/X} : \mathrm{Br}(Y) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(Y)$$

is given by the formula

$$\alpha \mapsto \sum_{\sigma \in G} \sigma^*(\alpha).$$

Since X and Y are regular, the embedding of the generic point into X induces an injective map $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k(X))$, and there is a similar map for Y. The claim is thus reduced to a similar claim for a finite Galois extension of fields, which is well known, see [GS17, Ch. 3, Exercice 3].

Thus for $\alpha \in \mathrm{Br}(Y)^G$ we obtain

$$\mathrm{res}_{Y/X} \circ \mathrm{cores}_{Y/X}(\alpha) = \sum_{\sigma \in G} \sigma^*(\alpha) = d\alpha \in \mathrm{Br}(Y).$$

Thus $d\alpha = f^*(\mathrm{cores}_{Y/X}(\alpha))$ belongs to $f^*(\mathrm{Br}(X)) \subset \mathrm{Br}(Y)$.

For the last statement of the theorem, the surjectivity is clear since we have $\mathrm{Br}(Y)^G[n] \subset d\mathrm{Br}(Y)^G$. The injectivity follows from Proposition 3.8.3. □

The following lemma will be used in Section 5.3.

Lemma 3.8.5 *Let k be a field and let A be a finite-dimensional commutative k-algebra. Let $A = \prod_{i=1}^m A_i$, where each A_i is a local k-algebra. For $i = 1, \ldots, m$, let k_i be the residue field of A_i, and let $n_i = \dim_k(A_i)/[k_i : k]$. For $\alpha \in \mathrm{Br}(A)$ write $\alpha_i \in \mathrm{Br}(k_i)$ for the image of α under the evaluation map $\mathrm{Br}(A) \to \mathrm{Br}(k_i)$. Then we have*

$$\mathrm{cores}_{A/k}(\alpha) = \sum_{i=1}^m n_i(\mathrm{cores}_{k_i/k}(\alpha_i)) \ \in \ \mathrm{Br}(k).$$

Proof. For any $x \in A$, one has the formula [BouVIII, §12, no. 2, Prop. 6]

$$\mathrm{N}_{A/k}(x) = \prod_{i=1}^m \mathrm{N}_{k_i/k}(x_i)^{n_i},$$

where $x_i \in k_i$ is the image of x in k_i, for each $i = 1, \ldots, m$. One needs to prove an analogue of this formula for the Brauer group. It is clearly enough to consider the case when A is a local k-algebra. Here some work is needed when A is not a field. Details can be found in [ABBB, §3]. □

Chapter 4

Smooth varieties

In this chapter we describe a general technique for computing the Brauer group $\mathrm{Br}(X)$ of a smooth projective variety X over a field k. Let k_{s} be a separable closure of k and let $X^{\mathrm{s}} = X \times_k k_{\mathrm{s}}$. The Galois group $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$ acts on the *geometric Picard group* $\mathrm{Pic}(X^{\mathrm{s}})$ and on the *geometric Brauer group* $\mathrm{Br}(X^{\mathrm{s}})$. One would like to understand the kernel and the cokernel of the natural map $\mathrm{Br}(X) \to \mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$. This can be done (with some success) using a Leray spectral sequence which involves Galois cohomology groups with coefficients in $\mathrm{Pic}(X^{\mathrm{s}})$ and $\mathrm{Br}(X^{\mathrm{s}})$. The structure of $\mathrm{Pic}(X^{\mathrm{s}})$ is discussed in the first section, and the structure of $\mathrm{Br}(X^{\mathrm{s}})$ is the subject of the second section. The spectral sequence and its differentials, with applications to the computation of $\mathrm{Br}(X)$, are discussed in the third section. In Section 4.4, under general geometric hypotheses on X, one obtains more precise results about $\mathrm{Br}(X)$. In Section 4.5 we discuss the Brauer groups of curves. The last section of this chapter concerns the computation of the Picard and Brauer groups of a product of two smooth projective varieties.

4.1 The Picard group of a variety over a field

In this section we recall a number of important results on the Picard group. Basic references are the books [BLR90] by Bosch, Lütkebohmert and Raynaud, and Kleiman's contribution [Kle05] to [FGI+05].

Let k be a field and let X be a variety over k. Assume that X is *geometrically integral* and *proper*. Then for any k-scheme T the canonical map $\mathcal{O}_T \to f_{T*}\mathcal{O}_{X_T}$ is an isomorphism, thus Proposition 2.5.2 tells us that the natural map between the relative Picard functors

$$\mathrm{Pic}_{(X/S)\,\text{ét}} \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,\mathrm{fppf}}$$

is an isomorphism. By a fundamental result of Grothendieck (Theorem 2.5.7), this functor is representable by a commutative group scheme $\mathbf{Pic}_{X/k}$ which is a disjoint union of open quasi-projective schemes.

Let $\mathbf{Pic}^0_{X/k} \subset \mathbf{Pic}_{X/k}$ be the connected component of identity [SGA3, VI$_A$, 2]. This is the smallest connected open subgroup of $\mathbf{Pic}_{X/k}$. It is a k-group of finite type. For any field extension K/k we have $\mathbf{Pic}^0_{X/k} \times_k K \cong \mathbf{Pic}^0_{X_K/K}$. The Néron–Severi group is defined as the quotient $\mathrm{NS}_{X/k} = \mathbf{Pic}_{X/k}/\mathbf{Pic}^0_{X/k}$. It is étale over k [SGA3, VI$_A$, 5.5]. In particular, we have $\mathrm{NS}_{X/k}(k_{\mathrm{s}}) = \mathrm{NS}_{X/k}(\bar{k})$. This is a finitely generated abelian group (Néron–Severi, [SGA6, XIII]). If K is any field containing k_{s}, then the natural map $\mathrm{NS}_{X/k}(k_{\mathrm{s}}) \to \mathrm{NS}_{X/k}(K)$ is an isomorphism.

An invertible sheaf on $\overline{X}$ is algebraically equivalent to 0 if and only if the corresponding point in $\mathbf{Pic}_{X/k}(\bar{k})$ belongs to $\mathbf{Pic}^0_{X/k}$ [Kle05, Prop. 9.5.10].

The tangent space to $\mathbf{Pic}_{X/k}$ at 0 is the coherent cohomology group $\mathrm{H}^1(X, \mathcal{O}_X)$ [Kle05, Thm. 9.5.11]. It follows that $\dim \mathbf{Pic}_{X/k} \leq \dim \mathrm{H}^1(X, \mathcal{O}_X)$, and the equality holds if and only if $\mathbf{Pic}_{X/k}$ is smooth. If the characteristic of k is 0, then $\mathbf{Pic}_{X/k}$ is smooth by Cartier's theorem, so $\mathbf{Pic}_{X/k}$ has the same dimension $\dim \mathrm{H}^1(X, \mathcal{O}_X)$ at every point. As recalled in [Kle05, Rem. 9.5.15, Prop. 9.5.19], Mumford proved in [Mum66, Ch. 27] that for any field k the tangent space to $\mathbf{Pic}_{X/k,\mathrm{red}}$ at 0 is the intersection of kernels of the Bockstein homomorphisms $\mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{H}^2(X, \mathcal{O}_X)$. It follows that $\mathbf{Pic}_{X/k}$ is smooth if either $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ or $\mathrm{H}^2(X, \mathcal{O}_X) = 0$.

If X is *projective, geometrically integral* and *geometrically normal*, then $\mathbf{Pic}^0_{X/k}$ is projective [Kle05, Thm. 9.5.4]. Using properness of $\mathbf{Pic}^0_{X/k}$, Grothendieck proved that the reduced subscheme $\mathbf{Pic}^0_{X/k,\mathrm{red}}$ is an abelian variety [FGA6, Prop. 3.1, Cor. 3.2, p. 236]. It is called the *Picard variety* of X. If $\mathbf{Pic}_{X/k}$ is smooth, then $\mathbf{Pic}^0_{X/k}$ coincides with the Picard variety of X.

We summarise the basic properties of the Picard scheme of a normal projective variety over a field in the following theorem.

Theorem 4.1.1 *Let X be a projective, geometrically integral and geometrically normal variety over a field k.*

(i) *There is an exact sequence of Γ-modules*

$$0 \longrightarrow \mathbf{Pic}^0_{X/k}(k_{\mathrm{s}}) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}}) \longrightarrow \mathrm{NS}(X^{\mathrm{s}}) \longrightarrow 0,$$

where $\mathbf{Pic}^0_{X/k}$ is a projective connected algebraic group, whose tangent space at 0 is the coherent cohomology group $\mathrm{H}^1(X, \mathcal{O}_X)$.

(ii) *If $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ or if $\mathrm{H}^2(X, \mathcal{O}_X) = 0$, or if $\mathrm{char}(k) = 0$, then $\mathbf{Pic}^0_{X/k}$ is smooth, hence an abelian variety of dimension $\dim \mathrm{H}^1(X, \mathcal{O}_X)$.*

(iii) *We have $\mathrm{NS}(X^{\mathrm{s}}) = \mathrm{NS}(\overline{X})$ and this group is finitely generated.*

(iv) *For $\ell \neq \mathrm{char}(k)$, we have $\mathrm{NS}(\overline{X})\{\ell\} \cong \mathrm{H}^2(\overline{X}, \mathbb{Z}_\ell(1))\{\ell\}$.*

Example 4.1.2 There are smooth, projective, geometrically integral surfaces X over an algebraically closed field k such that the group k-scheme $\mathbf{Pic}^0_{X/k}$ is not reduced, hence not smooth. Such are the so called *non-classical Enriques surfaces* that exist when $\mathrm{char}(k) = 2$. These are minimal surfaces of Kodaira dimension 0 such that $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}_\ell) = 0$ and $\dim \mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell) = 10$ (where $\ell \neq 2$) and

$\dim \mathrm{H}^1(X, \mathcal{O}_X) = 1$. For these surfaces $\mathbf{Pic}^0_{X/k}$ is $\alpha_2 = \mathrm{Spec}(k[t]/(t^2))$ or $\mu_2 = \mathrm{Spec}(k[t]/(t^2-1))$, depending on whether the action of Frobenius on $\mathrm{H}^1(X, \mathcal{O}_X)$ is trivial or not. (The classical Enriques surfaces have $\dim \mathrm{H}^1(X, \mathcal{O}_X) = 0$, and hence their Picard scheme is smooth.) See [Dol16] for a detailed treatment and explicit examples.

Corollary 4.1.3 *Let X be a projective, geometrically integral and geometrically normal variety over a field k.*

(i) *If $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, then the groups $\mathrm{Pic}(X^{\mathrm{s}})$, $\mathrm{Pic}(\overline{X})$, $\mathrm{NS}(X^{\mathrm{s}})$ and $\mathrm{NS}(\overline{X})$ are all equal. In this case this is a finitely generated abelian group.*

(ii) *Assume $\mathrm{char}(k) = 0$. Then $\overline{X}$ has no non-trivial finite, connected, abelian étale cover if and only if $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ and $\mathrm{NS}(\overline{X})$ is torsion-free.*

Proof. We only need to prove (ii). By the Kummer sequence, the variety $\overline{X}$ has a non-trivial finite, connected, abelian étale cover if and only if $\mathrm{Pic}(\overline{X})$ has non-trivial torsion, cf. [Mil80, Cor. III.4.19]. □

Albanese variety and Albanese torsor

We continue to assume that X is a projective, geometrically integral and geometrically normal variety over a field k, so that the Picard scheme $\mathbf{Pic}_{X/k}$ exists (see §2.5). If, *in addition*, $\mathbf{Pic}_{X/k}$ represents the relative Picard functor $\mathrm{Pic}_{X/k}$, then it is a formal consequence of Yoneda's lemma that $X \times_k \mathbf{Pic}_{X/k}$ has a *universal* invertible sheaf $\mathcal{P}$. This is a sheaf with the following property: for any k-scheme T and any invertible sheaf $\mathcal{L}$ on $X \times_k T$ there exists a unique morphism of k-schemes $h : T \to \mathbf{Pic}_{X/k}$ such that $\mathcal{L} = (\mathrm{id}, h)^*\mathcal{P} \otimes p_2^*\mathcal{N}$, where $\mathcal{N}$ is an invertible sheaf on T and $p_2 : X \times_k T \to T$ is the natural projection. (See [Kle05, Ex. 9.4.3].) The sheaf $\mathcal{P}$ is unique up to tensoring with a pullback of an invertible sheaf on $\mathbf{Pic}_{X/k}$. By Corollary 2.5.8, the condition that $\mathbf{Pic}_{X/k}$ represents $\mathrm{Pic}_{X/k}$ is satisfied when X has a k-point. In this case the universal sheaf can be made unique by normalising at this point. If X is an abelian variety, then $\mathcal{P}$ normalised at 0 is the usual Poincaré sheaf.

Let $A = \mathbf{Pic}^0_{X/k,\mathrm{red}}$ be the Picard variety of X; it is an abelian variety defined over k. The dual abelian variety $A^\vee = \mathbf{Pic}^0_{A/k}$ is called the *Albanese variety* of X and is denoted by $\mathrm{Alb}_{X/k}$. If X has a k-point x_0, then the sheaf $\mathcal{P}$ normalised at x_0 gives rise to a morphism $X \to \mathrm{Alb}_{X/k}$ which sends x_0 to 0. If X does not necessarily have a k-point, we can find a K-point on X for a finite separable extension K/k. By Galois descent, the K-morphism $X \times_k K \to \mathrm{Alb}_{X/k} \times_k K$ descends to a k-morphism $X \to \mathrm{Alb}^1_{X/k}$, where $\mathrm{Alb}^1_{X/k}$ is a k-torsor of $\mathrm{Alb}_{X/k}$, called the *Albanese torsor*. This morphism $X \to \mathrm{Alb}^1_{X/k}$ is universal among the morphisms from X to torsors of abelian varieties over k. See [FGA6] (the statement of Thm. 3.3 (iii), p. 237) and [Lan83]; for a more recent reference see [Witt08].

4.2 The geometric Brauer group

Proposition 4.2.1 *Let X be a variety over a separably closed field k of characteristic exponent p. Let S be a k-scheme. The kernel of the map*

$$\mathrm{Br}(X) \longrightarrow \mathrm{Br}(X \times_k S)$$

is a p-primary group. If k is algebraically closed or if S is smooth over k, then this map is injective.

Proof. Suppose $\alpha \in \mathrm{Br}(X)$ is in the kernel of the map. We may replace S by an affine open set, say $S = \mathrm{Spec}(R)$. The k-algebra R is a direct filtering limit of k-algebras A_i of finite type. By Section 2.2.4, $\mathrm{Br}(X_R)$ is the direct limit of the Brauer groups $\mathrm{Br}(X_{A_i})$. Thus there exists a k-algebra of finite type A such that α goes to zero in $\mathrm{Br}(X_A)$. Let m be a maximal ideal of A. By Zariski's lemma, the quotient field $K = A/m$, which is a finitely generated k-algebra, is a finite extension of k. Since k is separably closed, the degree $[K : k]$ is a power of p. The homomorphism $A \to A/m = K$ induces a homomorphism $\mathrm{Br}(X_A) \to \mathrm{Br}(X_K)$. Thus α is in the kernel of the map $\mathrm{Br}(X) \to \mathrm{Br}(X_K)$. A corestriction argument (§3.8) gives that α is annihilated by $[K : k]$ which is a power of p and is 1 if k is algebraically closed.

If S is smooth over a separably closed field k, then S has a k-point, so $\mathrm{Br}(X) \to \mathrm{Br}(X \times_k S)$ is injective in this case. □

We shall soon see that if k is a field of characteristic p which is separably closed, but not algebraically closed, then the kernel of the map $\mathrm{Br}(\mathbb{A}^1_k) \to \mathrm{Br}(\mathbb{A}^1_{\bar{k}})$ contains a non-trivial p-torsion subgroup.

Proposition 4.2.2 *Let X be a variety over a separably closed field k. Let ℓ be a prime different from $\mathrm{char}(k)$. Then for any separably closed field K containing k and any $n \geq 1$ the map $\mathrm{Br}(X)[\ell^n] \to \mathrm{Br}(X_K)[\ell^n]$ is an isomorphism.*

Proof. The smooth base change theorem in étale cohomology [Mil80, VI, Cor. 4.3] gives isomorphisms

$$\mathrm{H}^i_{\text{ét}}(X, \mu_{\ell^n}) \xrightarrow{\sim} \mathrm{H}^i_{\text{ét}}(X_K, \mu_{\ell^n}), \quad i \geq 0.$$

Comparing the Kummer sequences (3.2) for X and X_K, we deduce the surjectivity of $\mathrm{Br}(X)[\ell^n] \to \mathrm{Br}(X_K)[\ell^n]$. The injectivity of this map follows from Proposition 4.2.1. □

Theorem 4.2.3 *Let X be a proper and geometrically integral variety over a separably closed field k.*

(i) *There is an embedding*

$$\mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})] \hookrightarrow \mathrm{H}^1_{\text{fppf}}(k, \mathbf{Pic}_{X/k}).$$

(ii) *If either $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ or $\mathrm{H}^2(X, \mathcal{O}_X) = 0$, then the natural map $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$ is injective.*

Proof. Let $p : X \to \mathrm{Spec}(k)$ be the structure map. The hypothesis on X implies that for any k-scheme T the map $\mathcal{O}_T \to p_*\mathcal{O}_{X_T}$ is an isomorphism, see Remark 2.5.3. It follows that the natural map

$$\mathbb{G}_{m,k} \xrightarrow{\sim} p_*\mathbb{G}_{m,X}$$

is an isomorphism of sheaves for the fppf topology on $\mathrm{Spec}(k)$.

Since the group scheme $\mathbb{G}_{m,k}$ is smooth and k is separably closed, we have $\mathrm{H}^i_{\mathrm{fppf}}(k, \mathbb{G}_m) = \mathrm{H}^i_{\text{ét}}(k, \mathbb{G}_m) = 0$ for any $i > 0$, see (2.8). By the same result we also have an isomorphism

$$\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{m,X}) \cong \mathrm{H}^2_{\mathrm{fppf}}(X, \mathbb{G}_{m,X}). \tag{4.1}$$

Since $\mathrm{H}^i_{\mathrm{fppf}}(k, p_*\mathbb{G}_{m,X}) = \mathrm{H}^i_{\mathrm{fppf}}(k, \mathbb{G}_m) = 0$ for $i > 0$, the Leray spectral sequence

$$\mathrm{H}^p_{\mathrm{fppf}}(k, R^q f_*\mathbb{G}_{m,X}) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{fppf}}(X, \mathbb{G}_{m,X})$$

gives rise to the exact sequence

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{fppf}}(k, R^1 f_*\mathbb{G}_{m,X}) \longrightarrow \mathrm{H}^2_{\mathrm{fppf}}(X, \mathbb{G}_{m,X}) \longrightarrow \mathrm{H}^0(k, R^2 f_*\mathbb{G}_{m,X}).$$

Since X is proper over a field k, the fppf sheaf $R^1 f_*\mathbb{G}_{m,X}$ is representable by a k-group scheme $\mathbf{Pic}_{X/k}$, see Theorem 2.5.7. Thus, using (4.1), we can rewrite the above exact sequence as follows:

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{fppf}}(k, \mathbf{Pic}_{X/k}) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{H}^0(k, R^2 f_*\mathbb{G}_{m,X}).$$

Since $R^2 f_*\mathbb{G}_{m,X}$ is a sheaf for the fppf topology, the last group is a subgroup of $\mathrm{H}^0(\bar{k}, R^2 f_*\mathbb{G}_{m,X})$, so we get a natural map $\mathrm{Br}(X) \to \mathrm{H}^0(\bar{k}, R^2 f_*\mathbb{G}_{m,X})$, which coincides with the composition

$$\mathrm{Br}(X) \longrightarrow \mathrm{Br}(\overline{X}) \longrightarrow \mathrm{H}^0(\bar{k}, R^2 f_*\mathbb{G}_{m,X}).$$

This formally implies statement (i).

The k-group scheme $\mathbf{Pic}_{X/k}$ is an extension of the constant group of finite type $\mathrm{NS}_{X/k}(k)$ by the connected component $\mathbf{Pic}^0_{X/k}$, see Theorem 4.1.1 (i). If either $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ or $\mathrm{H}^2(X, \mathcal{O}_X) = 0$, then $\mathbf{Pic}^0_{X/k}$ is a smooth k-group scheme by Theorem 4.1.1 (ii). Using (2.8) again, we obtain $\mathrm{H}^i_{\mathrm{fppf}}(k, \mathbf{Pic}_{X/k}) \cong \mathrm{H}^i(k, \mathbf{Pic}_{X/k}) = 0$ for all $i > 0$. □.

Let X be a variety over a field k of characteristic exponent p. Recall that $X^{\mathrm{s}} = X \times_k k_{\mathrm{s}}$, where k_{s} is a separable closure of k.

Definition 4.2.4 *The group* $\mathrm{Br}(X^{\mathrm{s}})$ *is called the* **geometric** *Brauer group of* X. *We denote by* $\mathrm{Br}^0(X^{\mathrm{s}})$ *the divisible subgroup of* $\mathrm{Br}(X^{\mathrm{s}})$.

Proposition 4.2.5 *Let* X *be a variety over a field* k *and let* n *be a positive integer coprime to* $\mathrm{char}(k)$. *Then the group* $\mathrm{Br}(X^{\mathrm{s}})[n]$ *is finite.*

Proof. The Kummer exact sequence (3.2) shows that $\mathrm{Br}(X^{\mathrm{s}})[n]$ is a quotient of $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mu_n)$, which is finite by [SGA4$\frac{1}{2}$, Finitude, Thm. 1.1]. □

Let ℓ be a prime, $\ell \neq p$. In this section we describe the ℓ-primary subgroups $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}$ and $\mathrm{Br}^0(X^{\mathrm{s}})\{\ell\}$. Let us define the *Tate module* of $\mathrm{Br}(X^{\mathrm{s}})$ as

$$T_\ell\mathrm{Br}(X^{\mathrm{s}}) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathrm{Br}(X^{\mathrm{s}})) = \varprojlim \mathrm{Br}(X^{\mathrm{s}})[\ell^n],$$

when $n\to\infty$. It is clear that $T_\ell\mathrm{Br}(X^{\mathrm{s}})$ is a torsion-free $\mathbb{Z}_\ell$-module. There are natural injective maps $T_\ell\mathrm{Br}(X^{\mathrm{s}})/\ell^n \hookrightarrow \mathrm{Br}(X^{\mathrm{s}})[\ell^n]$. By Nakayama's lemma, $T_\ell\mathrm{Br}(X^{\mathrm{s}})$ is finitely generated, so is isomorphic to $\mathbb{Z}_\ell^r$ for some non-negative integer $r \leq \dim_{\mathbb{F}_\ell} \mathrm{Br}(X^{\mathrm{s}})[\ell]$. We have an isomorphism

$$T_\ell\mathrm{Br}(X^{\mathrm{s}}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{Br}^0(X^{\mathrm{s}})\{\ell\}. \tag{4.2}$$

Let X be a smooth, proper, geometrically integral variety over k. Let $b_n = \dim \mathrm{H}^n_{\text{ét}}(X^{\mathrm{s}},\mathbb{Q}_\ell)$ be the n-th ℓ-adic *Betti number* of X^{s}. It is independent of ℓ and is equal to $\dim \mathrm{H}^n(X_{\mathbb{C}},\mathbb{Q})$ when $k^{\mathrm{s}} \subset \mathbb{C}$. The *Picard number* ρ of X^{s} is the rank of the Néron–Severi group $\mathrm{NS}(X^{\mathrm{s}}) = \mathrm{NS}(\overline{X})$.

Proposition 4.2.6 *Let X be a smooth, proper, geometrically integral variety over a field k of characteristic exponent p. Then the following statements hold.*

(i) *For a prime $\ell \neq p$ there is an exact sequence of Γ-modules*

$$0 \longrightarrow \mathrm{Br}^0(X^{\mathrm{s}})\{\ell\} \longrightarrow \mathrm{Br}(X^{\mathrm{s}})\{\ell\} \longrightarrow \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))_{\mathrm{tors}} \longrightarrow 0, \tag{4.3}$$

where

$$\mathrm{Br}^0(X^{\mathrm{s}})\{\ell\} = \big(\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))/(\mathrm{NS}(X^{\mathrm{s}})\otimes\mathbb{Z}_\ell)\big)\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2-\rho}.$$

(ii) *If $\mathrm{char}(k) = 0$, there is an exact sequence of Γ-modules*

$$0 \longrightarrow \mathrm{Br}^0(\overline{X}) \longrightarrow \mathrm{Br}(\overline{X}) \longrightarrow \bigoplus_\ell \mathrm{H}^3_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))_{\mathrm{tors}} \longrightarrow 0, \tag{4.4}$$

where $\mathrm{Br}^0(\overline{X}) \cong (\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$; the direct sum is a finite abelian group.

(iii) *When $k \subset \mathbb{C}$, the finite group $\bigoplus_\ell \mathrm{H}^3_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))_{\mathrm{tors}}$ is isomorphic to the torsion subgroup of $\mathrm{H}^3(X(\mathbb{C}),\mathbb{Z})$.*

Proof. (i) Replacing X by X^{s} in the exact sequence (3.2) obtained from the Kummer sequence, gives the exact sequence

$$0 \longrightarrow \mathrm{Pic}(X^{\mathrm{s}})/\ell^n \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mu_{\ell^n}) \longrightarrow \mathrm{Br}(X^{\mathrm{s}})[\ell^n] \longrightarrow 0.$$

By Theorem 4.1.1, the group $\mathbf{Pic}^0_{X/k}$ is a connected projective algebraic group over k, hence $A = \mathbf{Pic}^0_{X/k,\mathrm{red}}$ is an abelian variety. Since $\ell \neq p$, the multiplication by ℓ map $A\to A$ is finite étale, hence it is surjective on k_{s}-points. Thus $\mathrm{Pic}(X^{\mathrm{s}}) = \mathbf{Pic}^0_{X/k}(k_{\mathrm{s}})$ is divisible by ℓ, so we can rewrite the previous exact sequence as follows:

$$0 \longrightarrow \mathrm{NS}(X^{\mathrm{s}})/\ell^n \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mu_{\ell^n}) \longrightarrow \mathrm{Br}(X^{\mathrm{s}})[\ell^n] \longrightarrow 0. \tag{4.5}$$

By the finiteness of étale cohomology with finite coefficients [SGA4$\frac{1}{2}$, Finitude, Thm. 1.1], (4.5) is an exact sequence of *finite* abelian groups. Thus passing to the limit for $n \to \infty$ the sequence we obtain is still exact:

$$0 \longrightarrow \mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell \xrightarrow{\mathrm{cl}_\ell} \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) \longrightarrow T_\ell \mathrm{Br}(X^{\mathrm{s}}) \longrightarrow 0, \qquad (4.6)$$

where the second arrow is the definition of the ℓ-adic *cycle class map* cl_ℓ. Since $T_\ell\mathrm{Br}(X^{\mathrm{s}})$ is a free $\mathbb{Z}_\ell$-module, the ℓ-primary torsion subgroup $\mathrm{NS}(X^{\mathrm{s}})\{\ell\}$ is canonically isomorphic to $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}}$. We obtain an isomorphism of abelian groups $T_\ell\mathrm{Br}(X^{\mathrm{s}}) \cong \mathbb{Z}_\ell^{b_2-\rho}$, which, in view of the isomorphism (4.2), implies $\mathrm{Br}^0(X^{\mathrm{s}})\{\ell\} \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2-\rho}$.

If we repeat the same arguments at the level of H^3, we see that the Kummer sequence identifies $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}/\mathrm{Br}^0(X^{\mathrm{s}})\{\ell\}$ with the kernel of the map

$$\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) \longrightarrow T_\ell \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{G}_m).$$

Since the Tate module is torsion-free and the Brauer group is torsion, we get an isomorphism

$$\mathrm{Br}(X^{\mathrm{s}})\{\ell\}/\mathrm{Br}^0(X^{\mathrm{s}})\{\ell\} \xrightarrow{\sim} \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell)_{\mathrm{tors}}.$$

(ii) For an arbitrary separably closed field k_{s} a theorem of Gabber [Ga83] says that for almost all ℓ the group $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$ is torsion-free. If k has characteristic 0, this is also a consequence of the comparison theorem between étale cohomology and classical Betti cohomology, see [Mil80, Thm. III.3.12].

(iii) Since the étale cohomology groups of a scheme over k^{s} with coefficients in a torsion sheaf of order coprime to $\mathrm{char}(k)$ do not change under extension of k^{s} to a bigger separably closed field [Mil80, Cor. VI.4.3], in the case $k^{\mathrm{s}} \subset \mathbb{C}$ we have $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) = \mathrm{H}^3_{\text{ét}}(X \times_k \mathbb{C}, \mathbb{Z}_\ell(1))$. The comparison theorem [Mil80, Thm. III.3.12] says that the latter group is isomorphic to the Betti cohomology group $\mathrm{H}^3(X \times_k \mathbb{C}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell(1)$. □

Proposition 4.2.7 *Let X be a smooth, proper, geometrically integral surface over a field k. Then for every prime $\ell \neq \mathrm{char}(k)$ there is a natural isomorphism of finite Γ-modules*

$$\mathrm{Br}(X^{\mathrm{s}})\{\ell\}/\mathrm{Br}^0(X^{\mathrm{s}})\{\ell\} \cong \mathrm{Hom}(\mathrm{NS}(X^{\mathrm{s}})\{\ell\}, \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Proof. In the previous proof we pointed out a natural isomorphism of finite Γ-modules

$$\mathrm{NS}(X^{\mathrm{s}})\{\ell\} \cong \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}}.$$

In view of the exact sequence (4.3), the result follows from the perfect duality pairing for the surface X^{s}

$$\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}} \times \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}} \longrightarrow \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

coming from the Poincaré duality. □

After classical work of Godeaux and of Campedelli, surfaces X over $\mathbb{C}$ with $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, $\mathrm{H}^2(X, \mathcal{O}_X) = 0$ and $\mathrm{NS}(X)_{\mathrm{tors}} \neq 0$ have been much discussed in the literature, see [BPV84, Ch. VII, §11] and [BCGP12].

Remark 4.2.8 Proposition 4.2.6 gives a precise formula for the size of the Brauer group of a smooth projective variety X over $\mathbb{C}$. In practice, it is very hard to make these elements explicit, either as classes of Azumaya algebras over X or even to make explicit their image in $\mathrm{Br}(\mathbb{C}(X))$ as classes of central simple algebras, or as a sum of symbols – which they are according to the Merkurjev–Suslin theorem.

4.3 Algebraic and transcendental Brauer groups

For a variety X over a field k there is a natural filtration on the Brauer group

$$\mathrm{Br}_0(X) \subset \mathrm{Br}_1(X) \subset \mathrm{Br}(X),$$

which is defined as follows.

Definition 4.3.1 *Let*

$$\mathrm{Br}_0(X) = \mathrm{Im}[\mathrm{Br}(k) \to \mathrm{Br}(X)], \quad \mathrm{Br}_1(X) = \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(X^{\mathrm{s}})].$$

The subgroup $\mathrm{Br}_1(X) \subset \mathrm{Br}(X)$ *is called the* **algebraic** *Brauer group of* X*, and the quotient* $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ *is called the* **transcendental** *Brauer group of* X*.*

A particular case of the Leray spectral sequence (2.5) for the structure morphism $X \to \mathrm{Spec}(k)$ is the spectral sequence

$$E_2^{pq} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(X^{\mathrm{s}}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_m). \tag{4.7}$$

It gives rise to the functorial exact sequence of terms of low degree

$$\begin{aligned} &0 \longrightarrow \mathrm{H}^1(k, k_{\mathrm{s}}[X]^*) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}})^{\Gamma} \longrightarrow \mathrm{H}^2(k, k_{\mathrm{s}}[X]^*) \\ &\longrightarrow \mathrm{Br}_1(X) \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}})) \longrightarrow \mathrm{Ker}[\mathrm{H}^3(k, k_{\mathrm{s}}[X]^*) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)]. \end{aligned} \tag{4.8}$$

Proposition 4.3.2 *Let* X *be a variety over a field* k *such that* $k_{\mathrm{s}}[X]^* = k_{\mathrm{s}}^*$*. Then there is an exact sequence*

$$\begin{aligned} &0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}})^{\Gamma} \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}_1(X) \\ &\longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}})) \longrightarrow \mathrm{Ker}[\mathrm{H}^3(k, k_{\mathrm{s}}^*) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)]. \end{aligned} \tag{4.9}$$

This sequence is contravariant functorial in X*.*

Proof. This follows from (4.8), since by Hilbert's theorem 90 we have $\mathrm{H}^1(k, k_{\mathrm{s}}^*) = 0$. □

The assumption of Proposition 4.3.2 is satisfied when X is proper and geometrically integral over k. It also holds for $X = \mathbb{A}^n_k$.

Remark 4.3.3 1. If X has a k-point or, more generally, if X has a 0-cycle of degree 1, then each of the maps $\mathrm{Br}(k)\to\mathrm{Br}_1(X)$ and $\mathrm{H}^3(k,k_{\mathrm{s}}^*)\to\mathrm{H}^3_{\text{ét}}(X,\mathbb{G}_m)$ in (4.9) has a retraction, hence is injective. (Then $\mathrm{Pic}(X)\to\mathrm{Pic}(X_{\mathrm{s}})^\Gamma$ is an isomorphism.) Indeed, a k-point on X defines a section of the structure morphism $X\to\mathrm{Spec}(k)$. A standard restriction-corestriction argument (see Section 3.8) reduces the case when X has a 0-cycle of degree 1 to the case when X has a k-point.

2. The map $\mathrm{Br}_1(X)\to\mathrm{H}^1(k,\mathrm{Pic}(X^{\mathrm{s}}))$ is surjective when there exists a variety Y over k such that $k_{\mathrm{s}}[Y]^*=k_{\mathrm{s}}^*$ and $\mathrm{H}^1(k,\mathrm{Pic}(Y^{\mathrm{s}}))=0$ equipped with a morphism $Y\to X$. This follows by comparing (4.9) for X and Y. These conditions on Y are satisfied for proper and geometrically connected varieties Y such that $\mathrm{Pic}(Y^{\mathrm{s}})$ is a permutation Γ-module. This holds, for example, when Y is a smooth projective quadric of dimension at least 1 or a Brauer–Severi variety.

Proposition 4.3.4 *For each $n\geq 0$ the differential*

$$\mathrm{H}^n(k,\mathrm{Pic}(X^{\mathrm{s}}))\longrightarrow\mathrm{H}^{n+2}(k,k_{\mathrm{s}}[X]^*)\tag{4.10}$$

from the spectral sequence (4.7) coincides, up to sign, with the connecting map defined by the 2-extension of Γ-modules

$$0\longrightarrow k_{\mathrm{s}}[X]^*\longrightarrow k_{\mathrm{s}}(X)^*\longrightarrow\mathrm{Div}(X^{\mathrm{s}})\longrightarrow\mathrm{Pic}(X^{\mathrm{s}})\longrightarrow 0.\tag{4.11}$$

Proof. This follows from the general description of connecting maps given in [Sko07, Prop. 1.1], combined with [Sko01, Thm. 2.3.4 (a)]. □

Remark 4.3.5 The differential (4.10) can be seen as the map attached to the exact triangle

$$p_*\mathbb{G}_{m,X}\longrightarrow\tau_{[0,1]}\mathbf{R}p_*\mathbb{G}_{m,X}\longrightarrow(R^1p_*)\mathbb{G}_{m,X}[-1]$$

in the bounded below derived category $\mathcal{D}(k)$ of Γ-modules. Here $p:X\to\mathrm{Spec}(k)$ is the structure morphism, $\mathbf{R}p_*:\mathcal{D}(X)\to\mathcal{D}(k)$ is the derived functor from the bounded below derived category $\mathcal{D}(X)$ of étale sheaves on X to $\mathcal{D}(k)$, and $\tau_{[0,1]}$ is the truncation functor. Proposition 4.3.4 then follows from the fact that $\tau_{[0,1]}\mathbf{R}p_*\mathbb{G}_{m,X}$ is represented by the 2-term complex $k_{\mathrm{s}}(X)^*\to\mathrm{Div}(X^{\mathrm{s}})$, as proved in [BvH09, Lemma 2.3].

Example 4.3.6 Let k be a field of characteristic 0 which contains a primitive cubic root of 1. Let a,b,c be independent variables and let $K=k(a,b,c)$. Let $X\subset\mathbb{P}^3_K$ be the diagonal cubic surface

$$x^3+ay^3+bz^3+ct^3=0.$$

By rather involved cocycle calculations, T. Uematsu [Uem14] shows that $\mathrm{Br}(X)=\mathrm{Br}_0(X)$ by proving that the map $\mathrm{H}^1(K,\mathrm{Pic}(X^{\mathrm{s}}))\to\mathrm{H}^3(K,K_{\mathrm{s}}^*)$ is injective. In this case we have $\mathrm{H}^1(K,\mathrm{Pic}(X^{\mathrm{s}}))\simeq\mathbb{Z}/3$.

The spectral sequence (4.7) gives rise to a *complex*

$$\mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(X^{\mathrm{s}})^{\Gamma} \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(X^{\mathrm{s}})).$$

Assume $k_{\mathrm{s}}^* = k_{\mathrm{s}}[X]^*$. From the general structure of spectral sequences we see that if $\mathrm{H}^3(k, k_{\mathrm{s}}^*) = 0$ or if X has a k-point (or a 0-cycle of degree 1), then, in view of Remark 4.3.3 (1), the above complex becomes an exact sequence

$$0 \longrightarrow \mathrm{Br}_1(X) \longrightarrow \mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(X^{\mathrm{s}})^{\Gamma} \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(X^{\mathrm{s}})). \tag{4.12}$$

Thus $\mathrm{Br}(X)/\mathrm{Br}_1(X) = \mathrm{Ker}(\beta)$. For concrete calculations of the Brauer group one would like to be able to compute the map β. As an approximation to this, we now describe the following composition:

$$\mathrm{Br}^0(X^{\mathrm{s}})^{\Gamma} \hookrightarrow \mathrm{Br}(X^{\mathrm{s}})^{\Gamma} \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(X^{\mathrm{s}})) \longrightarrow \mathrm{H}^2(k, N(X^{\mathrm{s}})), \tag{4.13}$$

where $N(X^{\mathrm{s}})$ is the quotient of the Néron–Severi group $\mathrm{NS}(X^{\mathrm{s}})$ by its torsion subgroup. By the results of Section 4.2, this map coincides with β when k has characteristic 0, $\mathrm{H}^1(X, \mathcal{O}) = 0$, and the groups $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell)$ and $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell)$ are torsion-free for all primes ℓ, so our description covers many important cases. For the sake of simplicity we state the result in the case when X is a surface, referring to [CTS13b, Prop. 4.1] for the general case.

Let X be a smooth, projective, geometrically integral surface over a field k of characteristic 0. Assume that k is a finitely generated subfield of $\mathbb{C}$. We have seen that the Néron–Severi group does not change when a separably closed ground field is extended to a larger separably closed field, hence we have an isomorphism $N(X^{\mathrm{s}}) \xrightarrow{\sim} N(X_{\mathbb{C}})$. Let us write $\mathrm{H}^2(X_{\mathbb{C}})$ for the quotient of $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ by its torsion subgroup. For a surface X the Poincaré duality gives rise to a perfect (unimodular) pairing

$$\mathrm{H}^2(X_{\mathbb{C}}) \times \mathrm{H}^2(X_{\mathbb{C}}) \longrightarrow \mathbb{Z}$$

given by the cup-product. By the Hodge index theorem, the restriction of this pairing to $N(X_{\mathbb{C}})$ has a non-zero discriminant. A classical argument based on the exponential exact sequence shows that $N(X_{\mathbb{C}})$ is a saturated subgroup of $\mathrm{H}^2(X_{\mathbb{C}})$, in the sense that the quotient is torsion-free.

Let $T(X_{\mathbb{C}})$ be the *lattice of transcendental cycles* of $X_{\mathbb{C}}$ defined as the orthogonal complement to $N(X_{\mathbb{C}})$ in $\mathrm{H}^2(X_{\mathbb{C}})$ with respect to the cup-product pairing. Thus $T(X_{\mathbb{C}})$ is a saturated subgroup of $\mathrm{H}^2(X_{\mathbb{C}})$, and $N(X_{\mathbb{C}}) \cap T(X_{\mathbb{C}}) = 0$. Write

$$N(X_{\mathbb{C}})^* = \mathrm{Hom}(N(X_{\mathbb{C}}), \mathbb{Z}), \qquad T(X_{\mathbb{C}})^* = \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Z}).$$

The cup-product gives rise to the injective maps

$$N(X_{\mathbb{C}}) \hookrightarrow N(X_{\mathbb{C}})^*, \qquad T(X_{\mathbb{C}}) \hookrightarrow T(X_{\mathbb{C}})^*.$$

By the unimodularity of the pairing on $\mathrm{H}^2(X_{\mathbb{C}})$ we have canonical isomorphisms of finite abelian groups

$$N(X_{\mathbb{C}})^*/N(X_{\mathbb{C}}) = \mathrm{H}^2(X_{\mathbb{C}})/(N(X_{\mathbb{C}}) \oplus T(X_{\mathbb{C}})) = T(X_{\mathbb{C}})^*/T(X_{\mathbb{C}}).$$

We deduce a natural exact sequence

$$0 \longrightarrow N(X^{\mathrm{s}}) \longrightarrow N(X^{\mathrm{s}})^* \longrightarrow T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

By the comparison theorem between classical and étale cohomology we have an isomorphism $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \otimes \mathbb{Z}_\ell \cong \mathrm{H}^2(\overline{X}, \mathbb{Z}_\ell(1))$, compatible with the cycle class map and the cup-product, for any prime ℓ. Thus $T(X_{\mathbb{C}}) \otimes \mathbb{Z}_\ell$ is the orthogonal complement to $\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell$ in $\mathrm{H}^2(\overline{X}, \mathbb{Z}_\ell(1))$. In particular, $T(X_{\mathbb{C}}) \otimes \mathbb{Z}_\ell$ is naturally a Γ-module, so that the previous 4-term exact sequence is an exact sequence of Γ-modules.

Since $N(X_{\mathbb{C}})$ is the orthogonal complement to $T(X_{\mathbb{C}})$ in $\mathrm{H}^2(X_{\mathbb{C}})$, we obtain $T(X_{\mathbb{C}})^* = \mathrm{H}^2(X_{\mathbb{C}})/N(X_{\mathbb{C}})$. Tensoring with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ we get

$$\mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \big(\mathrm{H}^2(X_{\mathbb{C}})/N(X_{\mathbb{C}})\big) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = \frac{\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))}{\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell.$$

From the description of $\mathrm{Br}^0(X^{\mathrm{s}})$ given in Proposition 4.2.6 (i) we now obtain a canonical isomorphism of Γ-modules

$$\mathrm{Br}^0(X^{\mathrm{s}}) = \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z})$$

and an exact sequence of Γ-modules

$$0 \longrightarrow N(X^{\mathrm{s}}) \longrightarrow N(X^{\mathrm{s}})^* \longrightarrow T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{Br}^0(\overline{X}) \longrightarrow 0. \tag{4.14}$$

The following proposition formally resembles Proposition 4.3.4.

Proposition 4.3.7 *Let X be a smooth, projective, geometrically integral surface over a field k of characteristic* 0*. The composed map* (4.13) *coincides, up to sign, with the connecting map*

$$\mathrm{Br}^0(\overline{X})^\Gamma \longrightarrow \mathrm{H}^2(k, N(X^{\mathrm{s}}))$$

defined by the 2*-extension of* Γ*-modules* (4.14).

Proof. See [CTS13b, Prop. 4.1]. □

Remark 4.3.8 This remark is a continuation of Remark 4.3.5 and uses the same notation. Let X be a smooth, projective, geometrically integral surface over a subfield of $\mathbb{C}$ such that $\mathrm{Pic}(X^{\mathrm{s}})$ is torsion-free. Then the 2-term complex

$$N(X^{\mathrm{s}})^* \longrightarrow T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z},$$

which is the middle part of (4.14), represents $\tau_{[1,2]}\mathbf{R}p_*\mathbb{G}_{m,X}[1]$ in the bounded below derived category of Γ-modules. This explains the previous proposition, because the relevant differential in the spectral sequence coincides with the map attached to the exact triangle

$$(R^1p_*)\mathbb{G}_{m,X}[-1] \longrightarrow \tau_{[1,2]}\mathbf{R}p_*\mathbb{G}_{m,X} \longrightarrow (R^2p_*)\mathbb{G}_{m,X}[-2].$$

See [GS, Prop. 1.2] for details.

In the rest of this section we prove that the transcendental Brauer group $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ has finite index in $\mathrm{Br}(X^{\mathrm{s}})^\Gamma$, at least when the characteristic of the ground field k is 0.

Lemma 4.3.9 *Let $L \subset k_{\mathrm{s}}$ be a finite, separable extension of a field k of degree n. Write $\Gamma_k = \mathrm{Gal}(k_{\mathrm{s}}/k)$ and $\Gamma_L = \mathrm{Gal}(k_{\mathrm{s}}/L)$. Let X be a k-scheme and let $X_L = X \times_k L$. The following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{Br}(X) & \xrightarrow{\mathrm{res}_{L/k}} & \mathrm{Br}(X_L) & \xrightarrow{\mathrm{cores}_{L/k}} & \mathrm{Br}(X) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_L} & & \downarrow{\scriptstyle \alpha} \\ \mathrm{Br}(X^{\mathrm{s}})^{\Gamma_k} & \hookrightarrow & \mathrm{Br}(X^{\mathrm{s}})^{\Gamma_L} & \xrightarrow{\sigma} & \mathrm{Br}(X^{\mathrm{s}})^{\Gamma_k} \end{array}$$

Here $\sigma(x) = \sum \sigma_i(x)$, where $\sigma_i \in \Gamma_k$ are coset representatives of Γ_k/Γ_L. The composition of maps in each row of the diagram is the multiplication by n.

Proof. We have an isomorphism $L \otimes_k k_{\mathrm{s}} \tilde{\longrightarrow} k_{\mathrm{s}}^{\oplus n}$ whose components correspond to the n distinct embeddings of L into k_{s}. By changing the base from X to X^{s} we obtain the commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^p_{\text{ét}}(X, \mathbb{G}_m) & \xrightarrow{\mathrm{res}_{L/k}} & \mathrm{H}^p_{\text{ét}}(X_L, \mathbb{G}_m) & \xrightarrow{\mathrm{cores}_{L/k}} & \mathrm{H}^p_{\text{ét}}(X, \mathbb{G}_m) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^p_{\text{ét}}(X^{\mathrm{s}}, \mathbb{G}_m) & \hookrightarrow & \mathrm{H}^p_{\text{ét}}(X^{\mathrm{s}}, \mathbb{G}_m)^{\oplus n} & \longrightarrow & \mathrm{H}^p_{\text{ét}}(X^{\mathrm{s}}, \mathbb{G}_m) \end{array}$$

where the maps in the bottom row are the diagonal embedding and the sum. The representation of the Galois group Γ_k in $\mathrm{H}^p_{\text{ét}}(X^{\mathrm{s}}, \mathbb{G}_m)^{\oplus n}$ is induced from the natural representation of Γ_L in $\mathrm{H}^p_{\text{ét}}(X^{\mathrm{s}}, \mathbb{G}_m)$. Passing to Γ_k-invariant subgroups, and taking $p = 2$, we obtain the statement of the lemma. □

Theorem 4.3.10 [CTS13b] *Let X be a smooth, projective and geometrically integral variety over a field k of characteristic 0. Then the cokernel of the natural map $\alpha : \mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^\Gamma$ is finite. In particular, the image of $\mathrm{Br}(X)$ in $\mathrm{Br}(\overline{X})$ is finite if and only if the group $\mathrm{Br}(\overline{X})^{\Gamma_k}$ is finite.*

Proof. By Proposition 4.2.6 (ii) the group $\mathrm{Br}(\overline{X})[n]$ is finite for any positive integer n. Hence it is enough to show that $\mathrm{Coker}(\alpha)$ has finite exponent.

Suppose that $k \subset L \subset \bar{k}$ is a finite extension of k such that $[L : k] = n$. By Lemma 4.3.9 restriction and corestriction induce the maps

$$\mathrm{Coker}(\alpha) \longrightarrow \mathrm{Coker}(\alpha_L) \longrightarrow \mathrm{Coker}(\alpha)$$

whose composition is the multiplication by n. Thus the kernel of the map $\mathrm{Coker}(\alpha) \to \mathrm{Coker}(\alpha_L)$ is annihilated by n, and to show that $\mathrm{Coker}(\alpha)$ has finite exponent it is enough to show that $\mathrm{Coker}(\alpha_L)$ has finite exponent.

Therefore without loss of generality we can replace k by any finite extension. In particular, we can assume that $X(k) \neq \emptyset$ and Γ_k acts trivially on the Néron–Severi group $\mathrm{NS}(\overline{X})$. Since $X(k) \neq \emptyset$, we have the exact sequence (4.12)

$$0 \longrightarrow \mathrm{Br}_1(X) \longrightarrow \mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(\overline{X})^\Gamma \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})).$$

Thus it is enough to show that $\mathrm{Im}(\beta)$ has finite exponent. We do this by considering finitely many curves on X and restricting our maps to each of these curves. This is a meaningful strategy because $\mathrm{Br}(\overline{C}) = 0$ by Tsen's theorem (Theorem 1.2.12).

More precisely, $\mathrm{NS}(\overline{X})/_{\mathrm{tors}}$ is a finitely generated free abelian group, so we can choose finitely many, say m, curves in $\overline{X}$ such that the intersection pairing with the classes of these curves defines an injective group homomorphism $\iota : \mathrm{NS}(\overline{X})/_{\mathrm{tors}} \hookrightarrow \mathbb{Z}^m$. By taking normalisation we obtain m morphisms from smooth projective curves defined over $\bar{k}$ to $\overline{X}$. We replace k by a finite extension over which all of these curves are defined.

By successively applying the Bertini theorem for hyperplane sections of smooth projective varieties [Jou84] we find a smooth and connected curve in $\overline{X}$. By replacing the field k by a finite extension we can assume that we have a *smooth* and geometrically connected curve $C_0 \subset X$ defined over k. We assume that C_0 is one of the curves from our finite family of curves equipped with finite morphisms to X.

A morphism $f : C \to X$, where C is a smooth, projective and geometrically integral curve over k gives rise to the commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(\overline{X})^\Gamma & \xrightarrow{\beta_X} & \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) \\ \downarrow f^* & & \downarrow f^* \\ 0 = \mathrm{Br}(\overline{C})^\Gamma & \xrightarrow{\beta_C} & \mathrm{H}^2(k, \mathrm{Pic}(\overline{C})) \end{array}$$

We have thus established

Claim 1. *For any morphism $f : C \to X$ the group $\mathrm{Im}(\beta_X)$ is contained in the kernel of the right vertical map in the diagram.*

We have the exact sequence of Γ_k-modules (4.16):

$$0 \longrightarrow \mathrm{Pic}^0(\overline{C}) \longrightarrow \mathrm{Pic}(\overline{C}) \longrightarrow \mathrm{NS}(\overline{C}) \longrightarrow 0.$$

Hence we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} & & \mathrm{H}^2(k, \mathrm{Pic}^0(\overline{X})) & \to & \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) & \to & \mathrm{H}^2(k, \mathrm{NS}(\overline{X})) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathrm{H}^2(k, \mathrm{Pic}^0(\overline{C})) & \to & \mathrm{H}^2(k, \mathrm{Pic}(\overline{C})) & \to & \mathrm{H}^2(k, \mathrm{NS}(\overline{C})) \end{array} \tag{4.15}$$

The zero in the bottom row is due to the fact that $\mathrm{H}^1(k, \mathbb{Z}) = 0$.

A combination of the Bertini theorem and Zariski's connectedness theorem (see [SGA1, Cor. 2.11, p. 210]) implies that a connected finite étale cover of

$\overline{X}$ restricts to a connected cover of $\overline{C}_0$. In particular, the map of abelian varieties $\mathrm{Pic}^0_{X/k}\to\mathrm{Pic}^0_{C_0/k}$ has trivial kernel. By the Poincaré reducibility theorem [Mum74, §19, Thm. 1] there exists an abelian subvariety $A\subset\mathrm{Pic}^0_{C_0/k}$ such that the natural map

$$\mathrm{Pic}^0_{X/k}\times A\longrightarrow\mathrm{Pic}^0_{C_0/k}$$

is an isogeny of abelian varieties over k, that is, a surjective morphism with finite kernel. It follows that the kernel of $\mathrm{H}^2(k,\mathrm{Pic}^0(\overline{X}))\to\mathrm{H}^2(k,\mathrm{Pic}^0(\overline{C}_0))$ has finite exponent. From diagram (4.15) we now obtain the following statement.

Claim 2. *The kernel of the composite map*

$$\mathrm{H}^2(k,\mathrm{Pic}^0(\overline{X}))\longrightarrow\mathrm{H}^2(k,\mathrm{Pic}(\overline{X}))\longrightarrow\mathrm{H}^2(k,\mathrm{Pic}(\overline{C}_0))$$

has finite exponent.

In view of (4.15), Claims 1 and 2, to complete the proof it is enough to show that the map of Γ_k-modules

$$\mathrm{NS}(\overline{X})\longrightarrow\bigoplus_{i=1}^m\mathrm{NS}(\overline{C}_i)=\mathbb{Z}^m$$

induces a map $\xi:\mathrm{H}^2(k,\mathrm{NS}(\overline{X}))\to\mathrm{H}^2(k,\mathbb{Z}^m)$ whose kernel has finite exponent. The map ξ is the composition of two maps:

$$\mathrm{H}^2(k,\mathrm{NS}(\overline{X}))\xrightarrow{\xi_1}\mathrm{H}^2(k,\mathrm{NS}(\overline{X})/_{\mathrm{tors}})\xrightarrow{\xi_2}\mathrm{H}^2(k,\mathbb{Z}^m).$$

It is enough to show that the kernel of each of these has finite exponent.

From the cohomology sequence attached to the exact sequence of Γ_k-modules

$$0\longrightarrow\mathrm{NS}(\overline{X})_{\mathrm{tors}}\longrightarrow\mathrm{NS}(\overline{X})\longrightarrow\mathrm{NS}(\overline{X})/_{\mathrm{tors}}\longrightarrow 0$$

we deduce that $\mathrm{Ker}(\xi_1)$ is annihilated by the exponent of the finite group $\mathrm{NS}(\overline{X})_{\mathrm{tors}}$.

There exists a homomorphism of abelian groups $\sigma:\mathbb{Z}^m\to\mathrm{NS}(\overline{X})/_{\mathrm{tors}}$ such that the composition $\sigma\circ\iota$ is the multiplication by a positive integer on $\mathrm{NS}(\overline{X})/_{\mathrm{tors}}$. This integer annihilates $\mathrm{Ker}(\xi_2)$. □

Remark 4.3.11 This proof can be used to produce an explicit upper bound for the size of the cokernel of $\alpha:\mathrm{Br}(X)\to\mathrm{Br}(\overline{X})^\Gamma$, see [CTS13b, Thm. 2.2]. When $\mathrm{H}^1(X,\mathcal{O}_X)=0$ or k is a number field, Proposition 4.3.7 can also be used to give upper bounds for this cokernel, see [CTS13b, Thm. 4.2, 4.3]. In some cases, for example in the case of diagonal quartic surfaces over $\mathbb{Q}$, Proposition 4.3.7 allows one to completely determine the image of $\mathrm{Br}(X)$ in $\mathrm{Br}(\overline{X})^\Gamma$, see [GS].

4.4 Projective varieties with $\mathrm{H}^i(X,\mathcal{O}_X)=0$

Theorem 4.4.1 *Let X be a smooth, projective and geometrically integral variety over a field k. Assume that $\mathrm{H}^1(X,\mathcal{O}_X)=0$ and $\mathrm{NS}(\overline{X})$ is torsion-free. Then $\mathrm{H}^1(k,\mathrm{Pic}(\overline{X}))$ and $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ are finite groups.*

Proof. From the exact sequence (4.9) we see that the quotient $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ is a subgroup of $\mathrm{H}^1(k,\mathrm{Pic}(\overline{X}))$. The result then follows from Proposition 4.1.3 and the finiteness of $\mathrm{H}^1(k,M)$ for any finitely generated torsion-free abelian group M. □

Theorem 4.4.2 *Let X be a smooth, projective and geometrically integral variety over a field k of characteristic 0. Assume that $\mathrm{H}^1(X,\mathcal{O}_X)=0$, $\mathrm{H}^2(X,\mathcal{O}_X)=0$ and the Néron–Severi group $\mathrm{NS}(\overline{X})$ is torsion-free. Then we have the following properties.*

(i) *The groups $\mathrm{Br}(\overline{X})$ and $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ are finite.*

(ii) $\mathrm{Br}(\overline{X})=0$ *if and only if $\mathrm{H}^3_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))_{\text{tors}}=0$ for every prime ℓ. In this case $\mathrm{Br}(X)=\mathrm{Br}_1(X)$.*

(iii) *If $\dim X=2$, then $\mathrm{Br}(\overline{X})=0$ and $\mathrm{Br}_1(X)=\mathrm{Br}(X)$.*

Proof. By Hodge theory the condition $\mathrm{H}^2(X,\mathcal{O}_X)=0$ implies $\rho=b_2$. Now Proposition 4.2.6 (ii) and the comparison theorems for étale and classical cohomology show that $\mathrm{Br}(\overline{X})$ is finite and isomorphic to $\oplus_\ell \mathrm{H}^3_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))_{\text{tors}}$. Statements (i) and (ii) now follow from Theorem 4.4.1. Statement (iii) follows from (ii) and Proposition 4.2.7. □

Corollary 4.4.3 *Let X be a smooth, projective, geometrically integral variety over a field k of characteristic 0 which is either a complete intersection of dimension at least 2, or a $K3$ surface. Then $\mathrm{H}^1(k,\mathrm{Pic}(\overline{X}))$ and $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ are finite groups.*

Proof. In both cases $\mathrm{Pic}(\overline{X})$ is torsion free. □

A similar statement is true for rationally connected varieties (see Definition 13.1.1).

Corollary 4.4.4 *Let X be a rationally connected variety over a field k of characteristic 0. Then $\mathrm{H}^1(k,\mathrm{Pic}(\overline{X}))$ and $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ are finite groups.*

Proof. In this case $\mathrm{Pic}(\overline{X})$ is torsion free and $\mathrm{Br}(\overline{X})$ is finite. □

Corollary 4.4.5 *Let $X\subset\mathbb{P}^n_k$ be a smooth complete intersection of dimension at least 3 over a field k of characteristic 0. Then the natural map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is an isomorphism.*

Proof. For such a variety X, by a theorem of Max Noether, the restriction map

$$\mathbb{Z}=\mathrm{Pic}(\mathbb{P}^n_{\bar{k}})=\mathrm{Pic}(\mathbb{P}^n_{\bar{k}})\longrightarrow\mathrm{Pic}(\overline{X})$$

is an isomorphism. The map $\mathrm{Pic}(X)\to\mathrm{Pic}(\overline{X})^\Gamma$ is surjective, since the analogous statement holds for $\mathbb{P}^n_k$. From exact sequence (4.9) we conclude that $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is injective. On the other hand, $\mathrm{H}^1(X,\mathcal{O}_X)=0$, $\mathrm{H}^2(X,\mathcal{O}_X)=0$, and there is no torsion in $\mathrm{H}^3(\overline{X},\mathbb{Z}_\ell)$ for any prime number ℓ. By Theorem 4.4.2, the map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is surjective. □

Corollary 4.4.5 also holds over a field of characteristic $p>0$, provided one restricts attention to the prime-to-p torsion subgroup, see [PV04, Prop. A.1].

4.5 Curves

If C be a smooth, projective, geometrically integral curve over a field k, then $\mathrm{NS}(C^{\mathrm{s}})=\mathbb{Z}$ and the natural morphism $\mathbf{Pic}_{C/k}\to\mathbb{Z}$ is given by the degree map on divisors. For an integer n let $\mathbf{Pic}^n_{C/k}$ be the component of degree n. Then the abelian variety $\mathbf{Pic}^0_{C/k}$ is the Jacobian J of the curve C so that there is an exact sequence

$$0\longrightarrow J(k_{\mathrm{s}})\longrightarrow \mathrm{Pic}(C^{\mathrm{s}})\longrightarrow \mathbb{Z}\longrightarrow 0. \tag{4.16}$$

The variety $\mathbf{Pic}^n_{C/k}$ is a k-torsor for $\mathbf{Pic}^0_{C/k}$. For $g=\dim \mathrm{H}^1(C,\mathcal{O}_C)\geq 1$ there is a natural embedding $C\hookrightarrow \mathbf{Pic}^1_{C/k}$, so $\mathbf{Pic}^1_{C/k}$ is the Albanese torsor of C. (Recall that the Jacobian is principally polarised, hence isomorphic to its dual abelian variety.) The cohomological exact sequence attached to (4.16) gives an exact sequence

$$0\longrightarrow J(k)\longrightarrow \mathrm{Pic}(C^{\mathrm{s}})^\Gamma\longrightarrow \mathbb{Z}\longrightarrow \mathrm{H}^1(k,J)\longrightarrow \mathrm{H}^1(k,\mathrm{Pic}(C^{\mathrm{s}}))\longrightarrow 0.$$

The group $\mathrm{H}^1(k,J)$ classifies k-torsors for J. The homomorphism $\mathbb{Z}\to\mathrm{H}^1(k,J)$ sends $n\in\mathbb{Z}$ to the class of the torsor $\mathbf{Pic}^n_{C/k}$.

Theorem 4.5.1 *Let C be a quasi-projective curve over a field k. Then the following statements hold.*

(i) *If $\alpha\in\mathrm{Br}(C)$ vanishes at each schematic point of C, then $\alpha=0$.*

(ii) *If k is algebraically closed, then $\mathrm{Br}(C)=0$.*

(iii) *If k is separably closed of characteristic $p>0$, then $\mathrm{Br}(C)$ is a p-primary torsion group.*

(iv) *If k is separably closed and C is proper over k, then $\mathrm{Br}(C)=0$.*

(v) *If k is finite and C is proper over k, then $\mathrm{Br}(C)=0$.*

(vi) *If k is not perfect, then $\mathrm{Br}(\mathbb{A}^1_k)\neq 0$. If k is separably closed, then $\mathrm{Br}(\mathbb{A}^1_k)=0$ if and only if k is algebraically closed.*

(vii) *The natural map $\mathrm{Br}(k)\to\mathrm{Br}(\mathbb{P}^1_k)$ is an isomorphism.*

(viii) *If k is perfect, then the natural map $\mathrm{Br}(k)\to\mathrm{Br}(\mathbb{A}^1_k)$ is an isomorphism.*

(ix) *If the prime ℓ is distinct from the characteristic exponent of k, then the map $\mathrm{Br}(k)\{\ell\}\to\mathrm{Br}(\mathbb{A}^1_k)\{\ell\}$ is an isomorphism.*

Proof. (i) The normalisation $\widetilde{C}$ of C is a finite union of regular curves. Statement (i) follows from Theorem 3.5.4, Propositions 7.2.4 and 7.2.1.

(ii) By Tsen's theorem (Theorem 1.2.12) the Brauer group of a function field in one variable over an algebraically closed field is zero. The result then follows from (i).

(iii) By a version of Tsen's theorem over a separably closed field (Proposition 3.8.2), the Brauer group of a function field in one variable over a separably closed field of characteristic $p > 0$ is p-primary. The Brauer group of a separably closed field is zero. The result now follows from (i).

(iv) This follows from (ii) and Theorem 4.2.3 since we have $\mathrm{H}^2(C, \mathcal{O}_C) = 0$ because C is curve.

(v) By (i) and the triviality of the Brauer group of a finite field $\mathrm{Br}(k) = 0$, it is enough to prove that $\mathrm{Br}(C) = 0$, where C is a regular, proper, geometrically integral curve over a finite field. The exact sequence (4.9) gives an isomorphism

$$\mathrm{Ker}[\mathrm{Br}(C) \longrightarrow \mathrm{Br}(C^{\mathrm{s}})] \tilde{\longrightarrow} \mathrm{H}^1(k, \mathrm{Pic}(C^{\mathrm{s}})).$$

By (ii), we have $\mathrm{Br}(C^{\mathrm{s}}) = 0$. Now consider the exact sequence (4.16):

$$0 \longrightarrow J(k_{\mathrm{s}}) \longrightarrow \mathrm{Pic}(C^{\mathrm{s}}) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the Galois module $J(k_{\mathrm{s}})$ is the group of k_{s}-points of the jacobian J of C. By Lang's theorem on the first cohomology group of a finite field with values in a connected algebraic group, we have $\mathrm{H}^1(k, J) = 0$. But $\mathrm{H}^1(k, \mathbb{Z}) = 0$, so we deduce $\mathrm{H}^1(k, \mathrm{Pic}(C^{\mathrm{s}})) = 0$. Hence $\mathrm{Br}(C) = 0$.

(vi) If k is algebraically closed, then (vi) is a particular case of (ii). Suppose k has characteristic $p > 0$ and is not perfect. Then there is an element $c \in k \smallsetminus k^p$. It gives rise to a non-zero class in $\mathrm{H}^1_{\mathrm{fppf}}(k, \mu_p)$ and hence in $\mathrm{H}^1_{\mathrm{fppf}}(\mathbb{A}^1_k, \mu_p)$. The étale Artin–Schreier covering of $\mathbb{A}^1_k = \mathrm{Spec}(k[x]) \to \mathbb{A}^1_k = \mathrm{Spec}(k[t])$ given by $x^p - x = t$ gives a non-zero element of $\mathrm{H}^1_{\text{ét}}(\mathbb{A}^1_k, \mathbb{Z}/p) = \mathrm{H}^1_{\mathrm{fppf}}(\mathbb{A}^1_k, \mathbb{Z}/p)$. This finite étale cover extends to a finite cover $\mathbb{P}^1_k \to \mathbb{P}^1_k$ which is totally ramified of degree p above the point at infinity of $\mathbb{P}^1_k$. We claim that the cup-product of these two classes is a non-zero element of $\mathrm{H}^2_{\mathrm{fppf}}(\mathbb{A}^1_k, \mu_p) = \mathrm{Br}(\mathbb{A}^1_k)[p]$. For this it is enough to prove that the class of the corresponding cyclic algebra is non-zero in $\mathrm{Br}(k(t))$, for which we need to show that $c \in k \subset k(t)$ is not a norm of an element from $k(x)$. For this, one looks at the completion at the point at infinity. If c were a norm, then its image in the residue field, which is just k, would be a p-th power. [SerCL, Ch. V, §3, Prop. 5 (i)].

(vii) For $C = \mathbb{P}^1_k$, we have an isomorphism of $\mathrm{Pic}(C^{\mathrm{s}})$ with the trivial Γ-module $\mathbb{Z}$ given by the degree map. The map $\mathrm{Pic}(C) \to \mathrm{Pic}(C^{\mathrm{s}}) = \mathbb{Z}$ is an isomorphism. By (iv), $\mathrm{Br}(C^{\mathrm{s}}) = 0$. Since $\mathrm{H}^1(k, \mathbb{Z}) = 0$, the exact sequence (4.9) gives an isomorphism $\mathrm{Br}(k) \tilde{\longrightarrow} \mathrm{Br}(\mathbb{P}^1_k)$.

(viii) Since the affine line has a k-point, we obtain from (4.9) that the natural map $\mathrm{Br}(k) \to \mathrm{Br}_1(\mathbb{A}^1_k)$ is an isomorphism. Since k is perfect, k_{s} is algebraically closed, hence $\mathrm{Br}(\mathbb{A}^1_{k_{\mathrm{s}}}) = 0$ by (ii). Thus $\mathrm{Br}(\mathbb{A}^1_k) = \mathrm{Br}_1(\mathbb{A}^1_k)$.

(ix) This follows from (iii) and (4.9). □

Remark 4.5.2 If a smooth, projective, geometrically integral curve C has a k-point or, more generally, a zero-cycle of degree 1, then (4.16) splits. In this case (4.9) gives an isomorphism $\mathrm{Pic}(C) = \mathrm{Pic}(C^{\mathrm{s}})^{\Gamma}$ and, in view of Theorem 4.5.1 (iv), a split exact sequence

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(C) \longrightarrow \mathrm{H}^1(k, J) \longrightarrow 0.$$

4.6 The Picard and Brauer groups of a product

In this section we discuss the Picard group and the Brauer group of the product of two varieties over a field.

Theorem 4.6.1 *Let X and Y be proper and geometrically integral varieties over a separably closed field k. Write $p_X : X \times_k Y \to X$ and $p_Y : X \times_k Y \to Y$ for the natural projections. Let n be a positive integer coprime to* $\mathrm{char}(k)$*. Then the pullback maps*

$$p_X^* : \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n) \longrightarrow \mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n), \quad p_Y^* : \mathrm{H}^i_{\text{ét}}(Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)$$

give rise to canonical isomorphism

$$\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \oplus \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n) \tilde{\longrightarrow} \mathrm{H}^1_{\text{ét}}(X \times_k Y, \mathbb{Z}/n). \tag{4.17}$$

The maps p_X^* and p_Y^*, together with the map

$$\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \tag{4.18}$$

that sends $a \otimes b$ to $p_X^*(a) \cup p_Y^*(b)$, give rise to a canonical isomorphism

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}/n) \oplus \mathrm{H}^2_{\text{ét}}(Y, \mathbb{Z}/n) \oplus \big(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)\big) \tilde{\longrightarrow} \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n). \tag{4.19}$$

It is clear that if k is a separable closure of a subfield $k_0 \subset k$, then p_X^*, p_Y^* and $p_X^*(x) \cup p_Y^*(y)$ respect the action of the Galois group $\mathrm{Gal}(k/k_0)$. Thus (4.17) and (4.19) are isomorphisms of $\mathrm{Gal}(k/k_0)$-modules.

Proof. We have an obvious commutative diagram

$$\begin{array}{ccc} Y & \xleftarrow{p_Y} & X \times_k Y \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle p_X} \\ \mathrm{Spec}(k) & \xleftarrow{\pi_X} & X \end{array}$$

The field k is separably closed, hence $\mathrm{H}^i(k, M) = 0$ for any abelian group M and any $i \geq 1$.

Let us choose base points $x_0 : \mathrm{Spec}(k) \to X$ and $y_0 : \mathrm{Spec}(k) \to Y$. The composition of $(\mathrm{id}, y_0) : X \to X \times_k Y$ with p_X is the identity on X, hence p_X^* sends $\mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n)$ isomorphically onto a direct summand of $\mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)$, for any $i \geq 0$.

Since X is connected, the map $\pi_X^* : \mathrm{H}^0_{\text{ét}}(k,\mathbb{Z}/n) = \mathbb{Z}/n \to \mathrm{H}^0_{\text{ét}}(X,\mathbb{Z}/n)$ is an isomorphism with section x_0^*. The k-variety Y is geometrically connected, hence p_X has connected fibres, thus we have an isomorphism of étale X-sheaves $\mathbb{Z}/n \tilde{\longrightarrow} p_{X*}(\mathbb{Z}/n)$. We also obtain that $p_X^* : \mathrm{H}^0_{\text{ét}}(X,\mathbb{Z}/n) \to \mathrm{H}^0(X\times_k Y,\mathbb{Z}/n)$ is an isomorphism with section $(\mathrm{id},y_0)^*$.

The proper base change theorem [Mil80, Cor. VI.2.3] implies that the constant étale X-sheaf $\pi_X^*\mathrm{H}^i_{\text{ét}}(Y,\mathbb{Z}/n)$ is canonically isomorphic to $R^ip_{X*}(\mathbb{Z}/n)$. Thus we have the Leray spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_{\text{ét}}(X,\mathrm{H}^q_{\text{ét}}(Y,\mathbb{Z}/n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X\times_k Y,\mathbb{Z}/n). \tag{4.20}$$

The standard properties of spectral sequences imply that the composition

$$\mathrm{H}^i_{\text{ét}}(X,\mathbb{Z}/n) \tilde{\longrightarrow} \mathrm{H}^i_{\text{ét}}(X,\mathrm{H}^0_{\text{ét}}(Y,\mathbb{Z}/n)) = E^{i,0} \longrightarrow \mathrm{H}^i_{\text{ét}}(X\times_k Y,\mathbb{Z}/n)$$

coincides with p_X^*. The functoriality of the spectral sequence (4.20) in X gives rise to a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^i_{\text{ét}}(X\times_k Y,\mathbb{Z}/n) & \longrightarrow & E^{0,i} = \mathrm{H}^0_{\text{ét}}(X,\mathrm{H}^i_{\text{ét}}(Y,\mathbb{Z}/n)) \\
\downarrow{\scriptstyle (x_0,\mathrm{id})^*} & & \downarrow{\scriptstyle x_0^*\ \cong} \\
\mathrm{H}^i_{\text{ét}}(Y,\mathbb{Z}/n) & \xrightarrow{=} & E^{0,i} = \mathrm{H}^0_{\text{ét}}(k,\mathrm{H}^i_{\text{ét}}(Y,\mathbb{Z}/n))
\end{array}$$

Hence the composition

$$\mathrm{H}^i_{\text{ét}}(X\times_k Y,\mathbb{Z}/n) \longrightarrow E^{0,i} = \mathrm{H}^0_{\text{ét}}(X,\mathrm{H}^i_{\text{ét}}(Y,\mathbb{Z}/n)) = \mathrm{H}^i_{\text{ét}}(Y,\mathbb{Z}/n)$$

coincides with the pullback $(x_0,\mathrm{id})^*$.

For $i=1$ we deduce from the spectral sequence the split exact sequence

$$0 \longrightarrow \mathrm{H}^1_{\text{ét}}(X,\mathbb{Z}/n) \xrightarrow{p_X^*} \mathrm{H}^1_{\text{ét}}(X\times_k Y,\mathbb{Z}/n) \xrightarrow{(x_0,\mathrm{id})^*} \mathrm{H}^1_{\text{ét}}(Y,\mathbb{Z}/n) \longrightarrow 0$$

with section p_Y^*. This gives (4.17).

Let us denote by

$$\widetilde{\mathrm{H}}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/n) \subset \mathrm{H}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/n)$$

the intersection of kernels of $(x_0,\mathrm{id})^*$ and $(\mathrm{id},y_0)^*$. By the same argument as above we have a direct sum decomposition

$$\mathrm{H}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/n) = \widetilde{\mathrm{H}}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/n) \oplus \mathrm{H}^2_{\text{ét}}(X,\mathbb{Z}/n) \oplus \mathrm{H}^2_{\text{ét}}(Y,\mathbb{Z}/n),$$

where the two last summands are the images of the injective maps p_X^* and p_Y^*, respectively. Moreover, the spectral sequence (4.20) also gives an exact sequence

$$0\to\widetilde{\mathrm{H}}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/n)\to\mathrm{H}^1_{\text{ét}}(X,\mathrm{H}^1_{\text{ét}}(Y,\mathbb{Z}/n))\to\mathrm{H}^3_{\text{ét}}(X,\mathbb{Z}/n) \xrightarrow{p_X^*} \mathrm{H}^3_{\text{ét}}(X\times_k Y,\mathbb{Z}/n).$$

The last map here is injective, hence $\widetilde{\mathrm{H}}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/n) \cong \mathrm{H}^1_{\text{ét}}(X,\mathrm{H}^1_{\text{ét}}(Y,\mathbb{Z}/n))$.

Using that $\mathrm{H}^1(k, \mathbb{Z}/n) = 0$ we see that the image of the map (4.18) belongs to the kernels of $(x_0, \mathrm{id})^*$ and $(\mathrm{id}, y_0)^*$, so (4.18) factors through a map

$$\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)). \tag{4.21}$$

For the proof that this is an isomorphism, see [SZ14, Thm. 2.6]. □

Remark 4.6.2 To show that the two groups in (4.21) are isomorphic one argues as follows. (We continue to assume that k is separably closed and X is proper and geometrically integral.) Let G be a finite commutative group k-scheme of order coprime to $\mathrm{char}(k)$. Let $\hat{G}$ be the Cartier dual of G. By definition, $\hat{G} = \mathrm{Hom}(G, \mathbb{G}_{m,k})$ in the category of commutative group k-schemes. The natural pairing

$$\mathrm{H}^1_{\text{ét}}(X, G) \times \hat{G} \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_{m,X}) = \mathrm{Pic}(X),$$

gives rise to a canonical isomorphism

$$\mathrm{H}^1_{\text{ét}}(X, G) \xrightarrow{\sim} \mathrm{Hom}(\hat{G}, \mathrm{Pic}(X)). \tag{4.22}$$

The map in (4.22) associates to a class of a G-torsor $\mathcal{T} \to X$ its 'type'. This map is defined when $\mathrm{char}(k)$ is coprime to $|G|$ without assuming k separably closed (see [Sko01, Theorem 2.3.6]), but if k is separably closed, then it is an isomorphism. (In this case, without loss of generality, we can assume $G = \mu_n$ and $\hat{G} = \mathbb{Z}/n$. Since $\mathrm{H}^0_{\text{ét}}(X, \mathbb{G}_m) = k^*$, an isomorphism $\mathrm{H}^1_{\text{ét}}(X, \mu_n) \xrightarrow{\sim} \mathrm{Pic}(X)[n]$ is provided by the Kummer sequence.) Applying (4.22) to $G = \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)$ and taking into account that $\mathrm{Hom}(\hat{G}, \mu_n)$ is canonically isomorphic to G, we get a canonical isomorphism

$$\mathrm{H}^1_{\text{ét}}(X, \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n).$$

For the proof of Theorem 4.6.1 one needs to show, in addition, that this isomorphism is the inverse of the map defined in terms of the cup-product.

Proposition 4.6.3 *Let X and Y be smooth, projective, geometrically integral varieties over a separably closed field k. The projection maps p_X and p_Y induce an isomorphism*

$$\mathbf{Pic}^0_{X/k} \oplus \mathbf{Pic}^0_{Y/k} \xrightarrow{\sim} \mathbf{Pic}^0_{X \times_k Y/k}. \tag{4.23}$$

Proof. Since X and Y are smooth and k is separably closed, both X and Y have k-points. By Corollary 2.5.8, the relative Picard functor $\mathrm{Pic}_{X/k}$ is represented by a commutative group k-scheme $\mathbf{Pic}_{X/k}$. By Theorem 4.1.1 the connected component of 0 in $\mathbf{Pic}_{X/k}$ is a projective and connected (but not necessarily reduced) group k-scheme $\mathbf{Pic}^0_{X/k}$. The same holds for Y and for $X \times_k Y$, which satisfies the same assumptions as X and Y. The natural morphism

$$\mathbf{Pic}_{X/k} \oplus \mathbf{Pic}_{Y/k} \longrightarrow \mathbf{Pic}_{X \times_k Y/k} \tag{4.24}$$

given by $(a,b)\mapsto p_X^*(a)+p_Y^*(b)$ has a retraction which sends a line bundle L on $X\times_k Y$ to $(\mathrm{id},y_0)^*L\oplus(x_0,\mathrm{id})^*L$. In particular, it identifies $\mathbf{Pic}_{X/k}\oplus\mathbf{Pic}_{Y/k}$ with a direct summand of $\mathbf{Pic}_{X\times_k Y/k}$. Restricting (4.24) to the connected components of 0 gives an isomorphism of $\mathbf{Pic}^0_{X/k}\times\mathbf{Pic}^0_{Y/k}$ with a direct summand of $\mathbf{Pic}^0_{X\times_k Y/k}$. Let us denote by C the kernel of the restriction of $\big((\mathrm{id},y_0)^*,(x_0,\mathrm{id})^*\big)$ to $\mathbf{Pic}^0_{X\times_k Y/k}$. Then we have

$$\mathbf{Pic}^0_{X\times_k Y/k}=C\oplus\mathbf{Pic}^0_{X/k}\oplus\mathbf{Pic}^0_{Y/k}.$$

As a surjective image of a connected group k-scheme, C is connected. By the Künneth formula [Stacks, Lemma 0BED]

$$\mathrm{H}^1(X\times_k Y,\mathcal{O})\cong\mathrm{H}^1(X,\mathcal{O})\oplus\mathrm{H}^1(Y,\mathcal{O})$$

we see that (4.24) induces an isomorphism of tangent spaces at 0. Thus the tangent space to C at 0 is trivial, hence $C=0$. $\square$

Let $A=\mathbf{Pic}^0_{X/k,\mathrm{red}}$ and $B=\mathbf{Pic}^0_{Y/k,\mathrm{red}}$ be the Picard varieties of X and Y, respectively. A line bundle L on $X\times_k Y$ gives rise to a morphism $Y\to\mathbf{Pic}_{X/k}$. If L restricts trivially to $X\times y_0$ and $x_0\times Y$, then, since Y is reduced and connected, this morphism factors through a morphism $Y\to A$ sending y_0 to 0. By the seesaw principle [Mum74, Ch. II, §5, Cor. 6], this last morphism is zero if and only if $L=0$.

The dual abelian variety $B^\vee$ is the Albanese variety of Y; there is a canonical Albanese morphism $\mathrm{Alb}_{Y,y_0}:Y\to B^\vee$ such that $\mathrm{Alb}_{Y,y_0}(y_0)=0$, see Section 4.1. By the universal property of the Albanese variety, the morphism $Y\to A$ is uniquely written as the composition $\phi\circ\mathrm{Alb}_{Y,y_0}:Y\to B^\vee$, where $\phi:B^\vee\to A$ is a map of abelian varieties. We have $\phi=0$ if and only if $L=0$. Conversely, any $\phi:B^\vee\to A$ gives rise to a line bundle on $X\times_k Y$ that restricts trivially to $x_0\times Y$ and $X\times y_0$, namely, to the pullback via the morphism

$$(\mathrm{Alb}_{X,x_0},\phi\circ\mathrm{Alb}_{Y,y_0}):X\times_k Y\longrightarrow A^\vee\times_k A$$

of the Poincaré line bundle $\mathcal{P}$ on $A^\vee\times_k A$, see Section 4.1. Thus we obtain a split exact sequence of abelian groups

$$0\longrightarrow\mathrm{Pic}(X)\oplus\mathrm{Pic}(Y)\longrightarrow\mathrm{Pic}(X\times_k Y)\longrightarrow\mathrm{Hom}(B^\vee,A)\longrightarrow 0,\qquad(4.25)$$

where the second map is (p_X^*,p_Y^*). The third map does not depend on the choice of x_0 and y_0. This implies that if k is a separable closure of a subfield k_0, then (4.25) is an exact sequence of $\mathrm{Gal}(k/k_0)$-modules. Note that this exact sequence is split when x_0 and y_0 are k_0-points, but in general it is not necessarily split.

Proposition 4.6.4 *Let X and Y be smooth, projective and geometrically integral varieties over a field k such that $\mathrm{H}^3(k,k_s^*)=0$, for example, a number field. Then the cokernel of the natural map*

$$\mathrm{Br}_1(X)\oplus\mathrm{Br}_1(Y)\longrightarrow\mathrm{Br}_1(X\times Y)$$

is finite.

Proof. This is an immediate consequence of exact sequences (4.25) and (4.9), and the fact that for A and B as above, the group $\mathrm{Hom}(B^\vee, A)$ is a finitely generated free abelian group. □

Proposition 4.6.5 *Let X and Y be smooth, projective and geometrically integral varieties over a field k with separable closure $k_{\rm s}$ and Galois group $\Gamma = \mathrm{Gal}(k_{\rm s}/k)$. Let $A = \mathbf{Pic}^0_{X/k,\mathrm{red}}$ and $B = \mathbf{Pic}^0_{Y/k,\mathrm{red}}$ be the Picard varieties of X and Y, respectively. We have a commutative diagram of Γ-modules with exact rows and columns, where the exact sequence in the bottom row is split:*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & A(k_{\rm s})\oplus B(k_{\rm s}) & = & A(k_{\rm s})\oplus B(k_{\rm s}) & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \mathrm{Pic}(X^{\rm s})\oplus \mathrm{Pic}(Y^{\rm s}) & \to & \mathrm{Pic}(X^{\rm s}\times_k Y^{\rm s}) & \to & \mathrm{Hom}((B^\vee)^{\rm s}, A^{\rm s}) & \to & 0 \\ & & \downarrow & & \downarrow & & || & & \\ 0 & \to & \mathrm{NS}(X^{\rm s})\oplus \mathrm{NS}(Y^{\rm s}) & \to & \mathrm{NS}(X^{\rm s}\times_k Y^{\rm s}) & \to & \mathrm{Hom}((B^\vee)^{\rm s}, A^{\rm s}) & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

If $(X\times_k Y)(k)\neq\emptyset$, then the exact sequence in the middle row is also split.

Proof. The upper row of the diagram comes from (4.23) and the middle row comes from (4.25). It remains to prove that the bottom row is split as a sequence of Γ-modules. This follows from the fact that the class of the line bundle $(\mathrm{Alb}_{\mathrm{X},\mathrm{x}_0}, \phi\circ \mathrm{Alb}_{\mathrm{Y},\mathrm{y}_0})^*P$ in $\mathrm{NS}(X^{\rm s}\times Y^{\rm s})$ does not depend on the choice of x_0 and y_0. The last statement of the proposition is clear: it is enough to choose $(x_0, y_0)\in (X\times_k Y)(k)$. □

Remark 4.6.6 If A_1 and A_2 are abelian varieties, then $\mathrm{Hom}(A_1^{\rm s}, A_2^{\rm s})$ is a free abelian group of finite rank. Thus the bottom row of the diagram shows that

$$\mathrm{NS}(X^{\rm s}\times Y^{\rm s})_{\rm tors}\cong \mathrm{NS}(X^{\rm s})_{\rm tors}\oplus \mathrm{NS}(Y^{\rm s})_{\rm tors}.$$

The bottom row of the diagram gives an isomorphism of Γ-modules

$$\mathrm{NS}(X^{\rm s}\times Y^{\rm s})/n\cong \mathrm{NS}(X^{\rm s})/n\oplus \mathrm{NS}(Y^{\rm s})/n\oplus \mathrm{Hom}((B^\vee)^{\rm s}, A^{\rm s})/n.$$

Let n be coprime to $\mathrm{char}(k)$. From the isomorphism (4.19) and the Kummer exact sequences for $X^{\rm s}$, $Y^{\rm s}$ and $X^{\rm s}\times Y^{\rm s}$ we deduce a canonical isomorphism of Γ-modules

$$\mathrm{Br}(X^{\rm s}\times Y^{\rm s})[n]\cong \mathrm{Br}(X^{\rm s})[n]\oplus \mathrm{Br}(Y^{\rm s})[n]\oplus B(X,Y)_n,$$

where $B(X,Y)_n$ is the quotient of $\mathrm{H}^1_{\text{ét}}(X^{\rm s},\mathbb{Z}/n)\otimes \mathrm{H}^1_{\text{ét}}(Y^{\rm s},\mathbb{Z}/n)(1)$ by the image of $\mathrm{Hom}((B^\vee)^{\rm s}, A^{\rm s})$. Indeed, $\mathrm{Hom}((B^\vee)^{\rm s}, A^{\rm s})$ is the kernel of the pullback of $\mathrm{NS}(X^{\rm s}\times Y^{\rm s})$ to $x_0\times Y^{\rm s}$ and $X^{\rm s}\times y_0$, and so is sent by the class map to $\mathrm{H}^1_{\text{ét}}(X^{\rm s},\mathbb{Z}/n)\otimes \mathrm{H}^1_{\text{ét}}(Y^{\rm s},\mathbb{Z}/n)(1)$.

Corollary 4.6.7 *Let X and Y be smooth, projective and geometrically integral varieties over a field k of characteristic zero. Then the natural map of Γ-modules*

$$\mathrm{Br}(X^{\mathrm{s}}) \oplus \mathrm{Br}(Y^{\mathrm{s}}) \longrightarrow \mathrm{Br}(X^{\mathrm{s}} \times Y^{\mathrm{s}})$$

is split injective.

To obtain a closed formula for $\mathrm{Br}(X^{\mathrm{s}} \times Y^{\mathrm{s}})$ we impose a condition on the torsion in the Néron–Severi groups of X^s and Y^{s}.

Corollary 4.6.8 *Let X and Y be smooth, projective and geometrically integral varieties over a field k. Let $A = \mathbf{Pic}^0_{X/k,\mathrm{red}}$ and $B = \mathbf{Pic}^0_{Y/k,\mathrm{red}}$ be the Picard varieties of X and Y, respectively. Let n be a positive integer coprime to $\mathrm{char}(k)$. If $\mathrm{Pic}(X^{\mathrm{s}})[n] \neq 0$ and $\mathrm{Pic}(Y^{\mathrm{s}})[n] \neq 0$, then assume also that n is coprime to $|\mathrm{NS}(X^{\mathrm{s}})_{\mathrm{tors}}| \cdot |\mathrm{NS}(Y^{\mathrm{s}})_{\mathrm{tors}}|$. Then we have a canonical isomorphism of Γ-modules*

$$\mathrm{Br}(X^{\mathrm{s}} \times Y^{\mathrm{s}})[n] \cong \mathrm{Br}(X^{\mathrm{s}})[n] \oplus \mathrm{Br}(Y^{\mathrm{s}})[n] \oplus \mathrm{Hom}(B^\vee[n], A[n]) / \big(\mathrm{Hom}((B^\vee)^{\mathrm{s}}, A^{\mathrm{s}})/n\big).$$

Proof. From the isomorphism $\mathrm{H}^1_{\text{ét}}(X^{\mathrm{s}}, \mu_n) = \mathrm{Pic}(X^{\mathrm{s}})[n]$ we see that this group is an extension of $\mathrm{NS}(X^{\mathrm{s}})[n]$ by $A[n]$. In our assumptions $\mathrm{H}^1_{\text{ét}}(X^{\mathrm{s}}, \mu_n) \otimes \mathrm{H}^1_{\text{ét}}(Y^{\mathrm{s}}, \mu_n) \cong A[n] \otimes B[n]$. Using the non-degeneracy of the Weil pairing $B[n] \times B^\vee[n] \to \mu_n$ we identify $B[n]$ with $\mathrm{Hom}(B^\vee[n], \mu_n)$, and obtain an isomorphism of $\mathrm{H}^1_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y^{\mathrm{s}}, \mathbb{Z}/n)(1)$ with $\mathrm{Hom}(B^\vee[n], A[n])$. ☐

Remark 4.6.9 The map $\mathrm{Hom}((B^\vee)^{\mathrm{s}}, A^{\mathrm{s}}) \to \mathrm{Hom}(B^\vee[n], A[n])$ in Corollary 4.6.8 comes from the first Chern class map. Assume $\mathrm{char}(k) = 0$. Then this map is the *negative* of the natural map defined by the action of homomorphisms on n-torsion points. It is enough to consider the case when X and Y coincide with their respective Albanese varieties $A^\vee$ and $B^\vee$. For the verification in this case we refer the reader to [OSZ, Lemma 2.6] (based on the Appell–Humbert theorem), which should be applied to the abelian variety $A^\vee \times B^\vee$.

Corollary 4.6.8 can be used to compute the Brauer group of a product of two elliptic curves and the attached Kummer variety. Here we restrict ourselves to one example, referring to [SZ12] for general results and more explicit examples.

Example 4.6.10 [SZ12, Prop. 4.1, Example A1] Let E be an elliptic curve over a number field k such that the representation of Γ in $E[\ell]$ is a surjection $\Gamma \to \mathrm{GL}(E[\ell])$ for every prime ℓ. Let E' be an elliptic curve with complex multiplication over k, which has a k-point of order 6. Then for $A = E \times_k E'$ we have $\mathrm{Br}(\overline{A})^\Gamma = 0$. For example, one can take $k = \mathbb{Q}$, the elliptic curve E with equation $y^2 = x^3 + 6x + 2$ of conductor $2^6 3^3$, and the elliptic curve E' with equation $y^2 = x^3 + 1$.

Chapter 5

Birational invariance

For a scheme X and a positive integer n the structure morphism $\mathbb{A}^n_X \to X$ induces an injective map $\mathrm{Br}(X) \to \mathrm{Br}(\mathbb{A}^n_X)$. Similarly, $\mathbb{P}^n_X \to X$ induces an injective map $\mathrm{Br}(X) \to \mathrm{Br}(\mathbb{P}^n_X)$. In Section 5.1 we give conditions on X under which these maps are isomorphisms.

In Section 5.2 we discuss the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(K/k) \subset \mathrm{Br}(K)$ of a field K finitely generated over a subfield k. The definition of $\mathrm{Br}_{\mathrm{nr}}(K/k)$ only uses the discrete valuations of K that are trivial on k, so this group depends only on the extension of fields $k \subset K$. When K is the function field of an integral variety over k, the group $\mathrm{Br}_{\mathrm{nr}}(K/k)$ is a birational invariant that can be used even when one does not have an explicit smooth projective model X/k with function field K at one's disposal. If we have such a model X then there is an isomorphism $\mathrm{Br}(X) \simeq \mathrm{Br}_{\mathrm{nr}}(K/k)$. We also recall that the Galois module $\mathrm{Pic}(X^{\mathrm{s}})$ up to addition of a permutation module is a birational invariant. Another birational invariant of smooth projective varieties X is the Chow group $\mathrm{CH}_0(X)$ of zero-cycles. In Section 5.3 we define a natural pairing between $\mathrm{CH}_0(X)$ and $\mathrm{Br}(X)$ with values in $\mathrm{Br}(k)$. This is used to give a proof of Mumford's theorem that the Chow group of degree 0 of a smooth complex surface with $\mathrm{H}^2(X, \mathcal{O}_X) \neq 0$ is not algebraically representable by the complex points of an abelian variety.

5.1 Affine and projective spaces

Theorem 5.1.1 *Let X be a connected regular scheme. Let K be its function field. For any prime ℓ distinct from the characteristic exponent of K, and any integer $n \geq 0$, the natural map of torsion groups $\mathrm{Br}(X) \to \mathrm{Br}(\mathbb{A}^n_X)$ induces an isomorphism on ℓ-torsion subgroups.*

Proof. If X is regular and connected, so is $\mathbb{A}^1_X$. Induction thus reduces the proof to the case $n = 1$. Using a section of $\mathbb{A}^1_X \to X$, we produce a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(\mathbb{A}^1_X) & \hookrightarrow & \mathrm{Br}(\mathbb{A}^1_K) \\ \downarrow\uparrow & & \downarrow\uparrow \\ \mathrm{Br}(X) & \hookrightarrow & \mathrm{Br}(K) \end{array}$$

where the downwards pointing arrows are induced by the restriction to the section and the upwards pointing arrows are induced by structure morphisms. To prove the result, it is thus enough to prove that for a field K of characteristic different from ℓ, the map $\mathrm{Br}(K)\{\ell\}\to\mathrm{Br}(\mathbb{A}^1_K)\{\ell\}$ is an isomorphism: this is Theorem 4.5.1 (ix). □

Remark 5.1.2 We have already seen in Theorem 4.5.1 (vi) that when k is separably closed but not algebraically closed, then $\mathrm{Br}(\mathbb{A}^1_k)\neq 0$ and hence $\mathrm{Br}(\mathbb{A}^n_k)\neq 0$ for all $n\geq 1$. Moreover, if k is an algebraically closed field of characteristic $p>0$ and $n\geq 2$ is an integer, then $\mathrm{Br}(\mathbb{A}^n_k)\neq 0$ [KOS76, Prop. 5.3], [Hür81, Thm. 4.4, Cor. 6.5]. These papers build upon earlier work of Zelinsky and Yuan (see [KO74b]).

The following theorem was proved by D. Saltman [Sal85] in terms of the unramified Brauer group.

Theorem 5.1.3 *For any field k the natural map $\mathrm{Br}(k)\to\mathrm{Br}(\mathbb{P}^n_k)$ is an isomorphism.*

Proof We proceed by induction in n. In the case $n=1$ this is Theorem 4.5.1 (vii). Suppose that $n\geq 2$ and we have the isomorphism $\mathrm{Br}(k)\xrightarrow{\sim}\mathrm{Br}(\mathbb{P}^{n-1}_k)$.

Let $\psi: W\to\mathbb{P}^n_k$ be the blowing-up of $\mathbb{P}^n_k$ in a k-point P. The projection of $\mathbb{P}^n_k\smallsetminus P$ onto $\mathbb{P}^{n-1}_k$ extends to a morphism $\pi: W\to\mathbb{P}^{n-1}_k$ which is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}_k$ with a section. To see this we choose coordinates on $\mathbb{P}^n_k$ so that $P=(1:0:\ldots:0)$. The restriction of $\pi: W\to\mathbb{P}^{n-1}_k$ to the open set $\mathbb{P}^n_k\smallsetminus P$ sends $(x_0:\ldots:x_n)$ to $(x_1:\ldots:x_n)$. Then the morphism $\sigma:\mathbb{P}^{n-1}_k\to W$ defined by $\sigma(x_1:\ldots:x_n)=(0:x_1:\ldots:x_n)$ is a section of π.

Let $K=k(\mathbb{P}^{n-1}_k)$ be the field of functions on $\mathbb{P}^{n-1}_k$. The section σ gives rise to a K-point s of the generic fibre of π, hence this generic fibre is isomorphic to the projective line $\mathbb{P}^1_K$. Proposition 3.5.4 implies that the restriction to the generic fibre of π defines an injective map $\mathrm{Br}(W)\hookrightarrow\mathrm{Br}(\mathbb{P}^1_K)$. The closed embedding of the section $\sigma(\mathbb{P}^{n-1}_k)$ into W defines a map $\mathrm{Br}(W)\to\mathrm{Br}(\mathbb{P}^{n-1}_k)$. Similarly, we have a restriction to the generic point $\mathrm{Br}(\mathbb{P}^{n-1}_k)\hookrightarrow\mathrm{Br}(K)$ and the map $\mathrm{Br}(\mathbb{P}^1_K)\to\mathrm{Br}(K)$ induced by the restriction to the K-point s of $\mathbb{P}^1_K$. We obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(W) & \hookrightarrow & \mathrm{Br}(\mathbb{P}^1_K) \\ \downarrow\uparrow & & \downarrow\uparrow \\ \mathrm{Br}(\mathbb{P}^{n-1}_k) & \hookrightarrow & \mathrm{Br}(K) \end{array}$$

where the upwards pointing arrows are induced by π and the structure morphism $\mathbb{P}^1_K\to\mathrm{Spec}(K)$. By Theorem 4.5.1 (vii) we know that the vertical arrows in the right hand part of the diagram are isomorphisms which are inverse to each other.

The diagram shows that the map $\mathrm{Br}(\mathbb{P}_k^{n-1})\to\mathrm{Br}(W)$ is an isomorphism. The induction assumption now implies that the natural map $\mathrm{Br}(k)\to\mathrm{Br}(W)$ is an isomorphism.

The contraction $\psi : W\to\mathbb{P}_k^n$ is a birational morphism of smooth varieties. The restriction of ψ to some non-empty open subset $U \subset W$ is an isomorphism. By Proposition 3.5.4 the restriction map $\mathrm{Br}(\mathbb{P}_k^n)\to\mathrm{Br}(U)$ is injective. Since it factors through $\psi^* : \mathrm{Br}(\mathbb{P}_k^n)\to\mathrm{Br}(W)$, we see that ψ^* is injective. It is clear that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(\mathbb{P}_k^n) & \stackrel{\psi^*}{\longrightarrow} & \mathrm{Br}(W) \\ \uparrow & & \uparrow \\ \mathrm{Br}(k) & = & \mathrm{Br}(k) \end{array}$$

We know that the right hand vertical map is an isomorphism. This implies that the left hand vertical map is an isomorphism too. □

Corollary 5.1.4 *Let X be a regular, connected scheme. For any positive integer n the canonical projection $\pi : \mathbb{P}_X^n\to X$ induces an isomorphism*

$$\pi^* : \mathrm{Br}(X)\xrightarrow{\sim}\mathrm{Br}(\mathbb{P}_X^n).$$

Proof Fix a section of $\mathbb{P}_X^n\to X$. Let K denote the function field of X. As in the proof of the previous theorem we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(\mathbb{P}_X^n) & \hookrightarrow & \mathrm{Br}(\mathbb{P}_{k(X)}^n) \\ \downarrow\uparrow & & \downarrow\uparrow \\ \mathrm{Br}(X) & \hookrightarrow & \mathrm{Br}(k(X)) \end{array}$$

where the downwards pointing arrows are induced by the restriction to the section and the upwards pointing arrows are induced by structure morphisms. By Theorem 5.1.3, the vertical arrows in the right hand part of the diagram are mutually inverse isomorphisms. The statement of the corollary now follows from the diagram. □

5.2 The unramified Brauer group

The following definition goes back to D. Saltman.

Definition 5.2.1 *Let $k \subset K$ be an extension of fields such that K is finitely generated over k. The* **unramified Brauer group** *of K over k is the subgroup*

$$\mathrm{Br}_{\mathrm{nr}}(K/k) \subset \mathrm{Br}(K)$$

which is the intersection of images of the natural maps $\mathrm{Br}(A)\to\mathrm{Br}(K)$, where A is a discrete valuation ring with field of fractions K such that $k \subset A$.

Proposition 5.2.2 *Let k be a field. Let X be a regular, proper, integral variety over k with function field $k(X)$. The natural inclusion $\mathrm{Br}(X) \subset \mathrm{Br}(k(X))$ induces an isomorphism $\mathrm{Br}(X)\xrightarrow{\sim}\mathrm{Br}_{\mathrm{nr}}(k(X)/k)$.*

Proof. This is a special case of Proposition 3.7.8. □

In spite of this proposition, there are good reasons for using the unramified Brauer group of function fields. Since its definition only involves discrete valuation rings, usually one can easily check that some specific elements of $\mathrm{Br}(K)$ belong to $\mathrm{Br}_{\mathrm{nr}}(K/k)$. The group is visibly a birational invariant of algebraic varieties over k. One may use it when no smooth projective model is available – or even known to exist (positive characteristic).

The unramified Brauer group is functorial in the following sense.

Proposition 5.2.3 *Let $K \subset L$ be finitely generated fields over k. The restriction map $\mathrm{Br}(K) \to \mathrm{Br}(L)$ induces a map $\mathrm{Br}_{\mathrm{nr}}(K/k) \to \mathrm{Br}_{\mathrm{nr}}(L/k)$.*

Proof. Let $v : L \to \mathbb{Z}$ be a discrete valuation with valuation ring B such that $k \subset B$. The restriction of v to K can be trivial or non-trivial. In the first case $K \subset B$, hence $\mathrm{Br}(K) \to \mathrm{Br}(L)$ factors through $\mathrm{Br}(B)$. In the second case, $A = B \cap K$ is a discrete valuation ring with field of fractions K. The restriction to $\mathrm{Br}(L)$ of an element in the image of $\mathrm{Br}(A) \to \mathrm{Br}(K)$ is in the image of $\mathrm{Br}(B) \to \mathrm{Br}(L)$. □

Proposition 5.2.4 *Let k be a field and let K and L be finitely generated field extensions of k. If L is a purely transcendental extension of K, then the natural map $\mathrm{Br}(K) \to \mathrm{Br}(L)$ induces an isomorphism $\mathrm{Br}_{\mathrm{nr}}(K/k) \to \mathrm{Br}_{\mathrm{nr}}(L/k)$.*

Proof. It is enough to consider the case $L = K(\mathbb{P}^1_K) = K(t)$, where t is an independent variable. Let $\beta \in \mathrm{Br}_{\mathrm{nr}}(L/k)$. Then $\beta \in \mathrm{Br}_{\mathrm{nr}}(L/K)$, but this group is equal to $\mathrm{Br}(\mathbb{P}^1_K)$ by Proposition 5.2.2. The natural map $\mathrm{Br}(K) \to \mathrm{Br}(\mathbb{P}^1_K)$ is an isomorphism; in positive characteristic, this is not obvious, see Theorem 4.5.1. Thus β comes from a unique $\alpha \in \mathrm{Br}(K)$, so it is enough to show that $\alpha \in \mathrm{Br}_{\mathrm{nr}}(K/k)$.

Let us check that α belongs to the image of $\mathrm{Br}(A) \to \mathrm{Br}(K)$, where $A \subset K$ is a discrete valuation ring with fraction field K such that $k \subset A$. Let π be a uniformising parameter of A. Let $B \subset L$ be the 2-dimension local ring at the closed point of $\mathrm{Spec}(A[t])$ defined be the ideal (π, t). By purity of the Brauer group for 2-dimensional regular rings (which is a classical result), $\beta \in \mathrm{Br}_{\mathrm{nr}}(L/k)$ is the image of an element $\gamma \in \mathrm{Br}(B) \subset \mathrm{Br}(L)$. The value of γ at $t = 0$ is an element of $\mathrm{Br}(A)$ whose image in $\mathrm{Br}(K)$ is α. Since this holds for any such A, we conclude that $\alpha \in \mathrm{Br}_{\mathrm{nr}}(K/k)$. □

Corollary 5.2.5 *Let k be a field and let X and Y be integral varieties over k. If X and Y are stably k-birationally equivalent over k, then*

$$\mathrm{Br}_{\mathrm{nr}}(k(X)/k) \simeq \mathrm{Br}_{\mathrm{nr}}(k(Y)/k).$$

In particular, if X is stably k-rational, then $\mathrm{Br}(k) \cong \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$. □

Proposition 5.2.2 then gives

Corollary 5.2.6 *Let k be a field, and let X and Y be smooth, proper, integral varieties over k. If there exist integers n and m such that $X \times_k \mathbb{P}^n_k$ is birationally equivalent to $Y \times_k \mathbb{P}^m_k$, then* $\mathrm{Br}(X) \simeq \mathrm{Br}(Y)$. □

A characterisation of unramified classes by evaluation at points

We start with the following useful lemma from [Duc98]. Such constructions were previously used by Merkurjev and Suslin.

Lemma 5.2.7 (Ducros) *For any field k of characteristic 0 there exists a field extension L of cohomological dimension at most 1 such that k is algebraically closed in L.*

Proof. Recall that $\mathrm{cd}(k) \leq 1$ if and only if $\mathrm{Br}(k') = 0$ for every finite extension $k \subset k'$.

If $\mathrm{cd}(k) \leq 1$, we take $L = k$. Otherwise there exist non-trivial Severi–Brauer varieties W over some finite extensions k'/k. Choose one Severi–Brauer variety W in each k'-isomorphism class and consider the Weil restriction of scalars $R_{k_i/k}(W \times_k k_i)$. The finite products of these varieties form a filtering inductive system of geometrically integral varieties over k; their fields of functions are extensions $k \subset K$ such that k is algebraically closed in K. Passing to the inductive limit we obtain a field extension $k \subset k_1$ such that k is algebraically closed in k_1. Define $k_n = (k_{n-1})_1$ for $n \geq 2$. Let L be the inductive limit of k_n as $n \to \infty$. On the one hand, k is algebraically closed in L. On the other hand, any variety $R_{L'/L}(V \times_L L')$, where V is a Severi–Brauer variety over a finite extension L' of L, is defined already over some k_n. Any integral variety has a rational point over its field of functions, so $R_{L'/L}(V \times_L L')$ has a k_{n+1}-point which is also an L-point. Then V has an L'-point, and so is trivial over L'. This proves that $\mathrm{cd}(L) \leq 1$. □

The following lemma and theorem are due to O. Wittenberg (private communication). A partial earlier result in this direction is [Mer02, Prop. 3.4].

Lemma 5.2.8 *Let X be a smooth geometrically integral variety over a field k. For any $\alpha \in \mathrm{Br}(X)$ and any point $P : \mathrm{Spec}(k((t))) \to X$ there exists a point $P' : \mathrm{Spec}(k((t))) \to X$ such that the last map is dominant and $\alpha(P) = \alpha(P')$.*

Proof. A k-morphism $P' : \mathrm{Spec}(k((t))) \to X$ is dominant if it induces an inclusion of the fields of functions $k(X) \subset k((t))$.

Let $x \in X$ be the image of the k-morphism $P : \mathrm{Spec}(k((t))) \to X$. Since X is smooth over k, there exist an open subset $U \subset X$ containing x and an étale morphism $f : U \to \mathbb{A}^d_k$. Let $Q = f(P) \in \mathbb{A}^d_k(k((t)))$. The field $k((t))$ is of infinite transcendence degree over k. One can choose a $k((t))$-point Q' in $\mathbb{A}^d_k$ as close as we wish to Q in the topology of the field $k((t))$ such that the d coordinates of Q' are algebraically independent over k. Moreover, by the implicit function theorem (Theorem 9.5.1) over the field $k((t))$, we can choose Q' with the additional property that Q' lifts to a $k((t))$-point P' in U which is as close as we wish to P. Corollary 3.4.4 applied over $k((t))$ (see the proof of Proposition 9.5.2) then ensures the equality $\alpha(P') = \alpha(P)$ in $\mathrm{Br}(k((t)))$. Since the coordinates of Q' are algebraically independent over k, the morphism $Q' : \mathrm{Spec}(k((t))) \to \mathbb{A}^d_k$ induces a k-embedding of the fields of functions $k(x_1, \ldots, x_d) \subset k((t))$. But $Q' = f(P')$ and f is dominant, hence this embedding factors as $k(x_1, \ldots, x_d) \subset k(X) \subset k((t))$, which shows that $P' : \mathrm{Spec}(k((t))) \to X$ is dominant. □

Theorem 5.2.9 (Wittenberg) *Let k be a field of characteristic zero and let X be a smooth geometrically integral variety over k. Let $\alpha \in \mathrm{Br}(X) \subset \mathrm{Br}(k(X))$. The following conditions are equivalent.*

(i) $\alpha \in \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$.

(ii) *For any field extension L/k and any $P \in X(L((t)))$, the value $\alpha(P)$ is in the image of $\mathrm{Br}(L) \to \mathrm{Br}(L((t)))$.*

(iii) *For any field extension L/k with $\mathrm{cd}(L) \leq 1$ and any $P \in X(L((t)))$ we have $\alpha(P) = 0$ in $\mathrm{Br}(L((t)))$.*

Proof. It is clear that (ii) implies (iii). Let us prove that (iii) implies (ii). Choose an embedding $L \subset L'$ as in Lemma 5.2.7. We have a commutattive diagram with exact rows (3.10)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Br}(L') & \longrightarrow & \mathrm{Br}(L'((t))) & \longrightarrow & \mathrm{H}^1(L', \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Br}(L) & \longrightarrow & \mathrm{Br}(L((t))) & \longrightarrow & \mathrm{H}^1(L, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

Here the vertical arrows are restriction maps; the right hand map is injective because L is algebraically closed in L'. This diagram implies the statement of (ii).

Let us prove that (ii) implies (i). Let $A \subset k(X)$ be a discrete valuation ring which contains k. Let κ be the residue field of A. By Cohen's theorem, the completion of A is isomorphic to $\kappa[[t]]$. We have $k \subset \kappa$, hence we have a k-embedding $k(X) \subset \kappa((t))$ such that $A = k(X) \cap \kappa[[t]]$. This gives a commutattive diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Br}(\kappa[[t]]) & \longrightarrow & \mathrm{Br}(\kappa((t))) & \longrightarrow & \mathrm{H}^1(\kappa, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow {\scriptstyle =} & & \\
0 & \longrightarrow & \mathrm{Br}(A) & \longrightarrow & \mathrm{Br}(k(X)) & \longrightarrow & \mathrm{H}^1(\kappa, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

Here the top row is (3.10), and the bottom row comes from Proposition 3.6.4. The assumption of (ii) applied to $L = \kappa$ implies that the image of α in $\mathrm{Br}(\kappa((t)))$ goes to zero in $\mathrm{H}^1(\kappa, \mathbb{Q}/\mathbb{Z})$. By the diagram this implies $\alpha \in \mathrm{Br}(A)$. Thus (ii) implies (i).

Let us prove that (i) implies (ii). If X is projective, by the valuative criterion of properness we have $X(L((t))) = X(L[[t]])$. Hence $P \in X(L[[t]])$, and thus $\alpha(P) \in \mathrm{Br}(L[[t]]) = \mathrm{Br}(L)$, proving (ii).

Let us now drop the assumption that X is projective (and avoid the resolution of singularities). By Lemma 5.2.8 we may assume that the morphism $P : \mathrm{Spec}(L((t))) \to X$ is dominant while keeping the value of $\alpha(P) \in \mathrm{Br}((L((t)))$. Then we have a k-embedding $k(X) \subset L((t))$. By the functoriality of the unramified Brauer group we have $\alpha(P) \in \mathrm{Br}(L[[t]]) = \mathrm{Br}(L)$. $\square$

Examples of unramified classes

Here are three types of unramified Brauer classes which will be used to construct counter-examples to the Hasse principle in Section 12.5.

Example 5.2.10 Let k be a field of characteristic not equal to 2, and let $a \in k^*$. Let $P(x) \in k[x]$ be a separable polynomial such that $P(x) = Q(x)R(x)$, where $Q(x)$ is a polynomial of even degree. Let X be a smooth projective variety birationally equivalent to the smooth, affine, geometrically integral surface with equation

$$y^2 - az^2 = P(x).$$

Let us show that *the class of the quaternion algebra* $\alpha = (a, Q(x)) \in \mathrm{Br}(k(X))$ *is unramified*. Proposition 5.2.2 then implies that $\alpha \in \mathrm{Br}(X)$.

Let $R \subset k(X)$ be a discrete valuation ring such that $k \subset R$. Let κ be the residue field of R and let $v : k(X)^* \to \mathbb{Z}$ be the valuation. By formula (1.16), the residue $\partial_v(\alpha) \in \kappa^*/\kappa^{*2}$ is the class of $a^{v(Q(x))}$ in κ^*/κ^{*2}.

If a is a square in κ, then $\partial_v(\alpha) = 1$. If $v(x) < 0$, then $v(Q(x))$ is even, hence $\partial_v(\alpha) = 1$. If a is a not a square in κ and $v(x) \geq 0$, then the equality $y^2 - az^2 = Q(x)R(x) \in k(X)^*$ implies that $v(y^2 - az^2)$ is even, thus $v(Q(x)) + v(R(x))$ is even. The polynomials $Q(x)$ and $R(x)$ are coprime, hence there exist polynomials $a(x)$ and $b(x)$ such that $a(x)Q(x) + b(x)R(x) = 1$. Since $v(x) \geq 0$, if $v(Q(x))$ was odd hence positive we would have $v(R(x)) = 0$, but then $v(Q(x)) + v(R(x))$ would be odd. This proves that α is unramified.

Suppose that a is not a square in k and that neither $Q(x)$ nor $R(x)$ is of the form $c(S(x)^2 - aT(x)^2)$ with $c \in k^*$ and $S(x), T(x) \in k[x]$. Then α *is not in the image of the map* $\mathrm{Br}(k) \to \mathrm{Br}(k(X))$.

Indeed, the assumption is equivalent to: neither $(a, Q(x))$ nor $(a, R(x))$ belongs to $\mathrm{Br}(k) \subset \mathrm{Br}(k(x))$. By Proposition 6.2.1, the kernel of the restriction map $\mathrm{Br}(k(x)) \to \mathrm{Br}(k(X))$ is generated by the quaternion algebra $(a, P(x))$. This implies that if $\alpha = (a, Q(x)) \in \mathrm{Br}(k(X))$ is in the image of $\mathrm{Br}(k)$, then in $\mathrm{Br}(k(x))$ either $(a, Q(x))$ or $(a, R(x))$ is in $\mathrm{Br}(k) \subset \mathrm{Br}(k(x))$.

Note that the assumption that a separable polynomial $M(x) \in k[x]$ is not of the form $c(S(x)^2 - aT(x)^2)$ is equivalent to the existence of a root x_0 of $M(x)$ such that a is not a square in the field $k(x_0)$.

Example 5.2.11 (Reichardt–Lind) Let k be a field, $\mathrm{char}(k) \neq 2$, let $a, b \in k^*$. Let X be a smooth projective curve birationally equivalent to the affine curve

$$ay^2 = x^4 - b.$$

The class of the quaternion algebra $(y, b) \in \mathrm{Br}(k(X))$ is unramified, hence by Proposition 5.2.2 belongs to $\mathrm{Br}(X)$.

Let $R \subset k(X)$ be a discrete valuation ring such that $k \subset R$. Let κ be the residue field of R and let $v : k(X)^* \to \mathbb{Z}$ be the valuation. By (1.16), the residue $\partial_v(\alpha)$ is the class of $b^{v(y)}$ in κ^*/κ^{*2}. If b is a square in κ or of $v(y)$ is even, the residue is 1. Assume b is not a square in κ. If $v(x) < 0$ then $v(x^4 - b)$ is a multiple of 4, hence so is $v(ay^2)$, hence $v(y)$ is even. Assume $v(x) \geq 0$. Since b is not a square in κ, we have $v(x^4 - b) = 0$. Thus $v(ay^2) = 0$ hence $v(y) = 0$.

When b is not a square in k, there does not seem to exist a simple criterion for (y, b) to be in the image of $\mathrm{Br}(k)$.

Exercise 5.2.12 [CT14] Let k be a field, $\mathrm{char}(k) \neq 2$, let $a, b, c \in k^*$. Let X be a smooth projective variety birationally equivalent to the affine variety with equation

$$(x^2 - ay^2)(z^2 - bt^2)(u^2 - abw^2) = c.$$

Computing residues for any valuation of $K(X)$ trivial on k, one checks that the quaternion algebra $(x^2 - ay^2, b)$ is unramified, hence is an element of $\mathrm{Br}(X)$.

Projecting to affine space $\mathbb{A}^4_k$ with coordinates (z, t, u, w), we represent $k(X)$ as the function field of a conic over $k(\mathbb{A}^4)$. Using this and Witt's theorem, one shows that if none of a, b, ab is a square in k, then $\alpha = (x^2 - ay^2, b) \in \mathrm{Br}(k(X))$ is not in the image of $\mathrm{Br}(k)$. See [CT14, Thm. 4.1] for details.

Explicit examples of unramified classes in the Brauer group of the function field of a variety over the complex field will be given in Section 10.6 (the Artin–Mumford example). See also Sections 11.1.2 and 11.2.1.

Galois action on the Picard group

Proposition 5.2.13 *Let X and Y be smooth, projective, geometrically integral varieties over a field k. If X and Y are stably k-birationally equivalent, then there exist finitely generated permutation Γ-modules P_1 and P_2 and an isomorphism of Γ-modules*

$$\mathrm{Pic}(X^{\mathrm{s}}) \oplus P_1 \cong \mathrm{Pic}(Y^{\mathrm{s}}) \oplus P_2.$$

This gives an isomorphism $\mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}})) \cong \mathrm{H}^1(k, \mathrm{Pic}(Y^{\mathrm{s}}))$.

If X is stably k-rational, then the Γ-module $\mathrm{Pic}(X^{\mathrm{s}})$ is stably a permutation Γ-module: there are finitely generated permutation Γ-modules P_1 and P_2 such that $\mathrm{Pic}(X^{\mathrm{s}}) \oplus P_1 \cong P_2$.

If there exists a smooth, projective, geometrically integral variety Z over k such that $X \times_k Z$ is k-rational, then the Γ-module $\mathrm{Pic}(X^{\mathrm{s}})$ is a direct summand of a permutation module.

For an elegant proof due to Moret-Bailly, see [CTS87a, Prop. 2.A.1].

Suppose $X(k) \neq \emptyset$. In this case, in view of $\mathrm{Br}_1(X)/\mathrm{Br}(k) = \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}}))$, this proposition gives another proof of the birational invariance of $\mathrm{Br}_1(X)$. But in special cases the birational invariant given by the Γ-module $\mathrm{Pic}(X^{\mathrm{s}})$ up to addition of a permutation module is finer than $\mathrm{Br}_1(X)$, see [CTS77, §8].

Proposition 5.2.14 *Let k be a field. Let $\mathcal{C}$ be a class of smooth, projective, geometrically integral varieties X over K, where K varies over arbitrary field extensions of k. Suppose that $\mathcal{C}$ is stable under field extensions, and that for each variety X in $\mathcal{C}$ one has $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. If one of the following statements* *holds for all varieties X/K in $\mathcal{C}$ which have the additional property $X(K) \neq \emptyset$, then it holds for all X/K in $\mathcal{C}$:*

(i) *the* $\mathrm{Gal}(K_s/K)$*-module* $\mathrm{Pic}(X_{K_s})$ *is a permutation module;*

(ii) *the* $\mathrm{Gal}(K_s/K)$*-module* $\mathrm{Pic}(X_{K_s})$ *is a direct summand of a permutation module;*

(iii) $\mathrm{H}^1(K, \mathrm{Pic}(X_{K_s})) = 0$.

Proof. (Sketch) Let $F = K(X)$. The F-variety $X_F = X \times_K F$ has an F-point. The hypothesis $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ implies that $\mathbf{Pic}^0_{X/k} = 0$, so $\mathbf{Pic}_{X/k}$ is a twisted constant group k-scheme split by a separable closure k_s of k, see Theorem 4.1.1. This implies that the natural maps

$$\mathrm{Pic}(X_{k_s}) \tilde{\longrightarrow} \mathrm{Pic}(X_{k_s(X)}) \tilde{\longrightarrow} \mathrm{Pic}(X_{F_s})$$

are isomorphisms. For more details, see [CTS87a, Thm. 2.B.1]. □

5.3 Zero-cycles and the Brauer group

In this section we collect some results about the relations between the Brauer group $\mathrm{Br}(X)$ of a variety X over a field k and another birational invariant of smooth and proper varieties, the Chow group of zero-cycles $\mathrm{CH}_0(X)$. The basic reference for the Chow group is the first chapter of Fulton's book [Ful98].

Let $Z_0(X)$ be the free abelian group whose generators are the closed points of X. The elements of $Z_0(X)$ are called 0*-cycles*. In other words, a 0-cycle is a finite sum $\sum_P n_P P$, where P is a closed point and $n_P \in \mathbb{Z}$. A 0-cycle is called *effective* if $n_P \geq 0$ for all P. The degree map

$$\deg_k : Z_0(X) \longrightarrow \mathbb{Z}$$

sends a 0-cycle $\sum_i n_i P_i$ to $\sum_i n_i [k(P_i) : k]$.

A morphism of varieties $f : X \to Y$ gives rise to a natural homomorphism

$$f_* : Z_0(X) \longrightarrow Z_0(Y)$$

sending the closed point $P \in X$ to $[k(P) : k(f(P))] f(P)$. The degree map is compatible with morphisms of varieties over k.

A 0-cycle on a normal integral curve C is called rationally equivalent to zero if it is the divisor $\mathrm{div}_C(g)$ of a non-zero rational function $g \in k(C)^*$. The *Chow group* of 0-cycles on X is defined as the quotient of $Z_0(X)$ by the group generated by the elements $\phi_*(\mathrm{div}_C(g))$, for all proper morphisms $\phi : C \to X$ where C is a normal integral curve over k and all $g \in k(C)^*$.

Let k be a field, let X a variety over k and let $Y \subset X$ be a finite subscheme. Then $Y = \mathrm{Spec}(A)$, where $A = \prod_{i=1}^m A_i$, each A_i being a local k-algebra. For $i = 1, \ldots, m$, let k_i be the residue field of A_i and let $n_i = \dim_k(A_i)/[k_i : k]$. For each i, the composition $\mathrm{Spec}(k_i) \to \mathrm{Spec}(A_i) \to \mathrm{Spec}(A) \to X$ defines a closed point $P_i \in Y$ with residue field k_i. The *zero-cycle associated to* $Y \subset X$ is by definition the formal sum $\sum_{i=1}^m n_i P_i$.

If $f : X \to Y$ is a proper morphism, then $f_* : Z_0(X) \to Z_0(Y)$ induces a map

$$f_* : \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(Y).$$

In particular, if X is a proper variety over k, then the structure morphism $X \to \mathrm{Spec}(k)$ induces a degree map $\deg_k : \mathrm{CH}_0(X) \to \mathbb{Z}$. Define

$$A_0(X) = \mathrm{Ker}[\deg_k : \mathrm{CH}_0(X) \longrightarrow \mathbb{Z}].$$

By the functoriality of the Brauer group, an element $\alpha \in \mathrm{Br}(X)$ can be evaluated at a closed point $P : \mathrm{Spec}(k(P)) \to X$. We denote this value by $\alpha(P) \in \mathrm{Br}(k(P))$. Define

$$\langle \alpha, P \rangle = \mathrm{cores}_{k(P)/k}(\alpha(P)) \in \mathrm{Br}(k).$$

By linearity this extends to a pairing

$$\mathrm{Br}(X) \times Z_0(X) \longrightarrow \mathrm{Br}(k). \tag{5.1}$$

This pairing is functorial in X. Namely, let $f : X \to Y$ be a morphism of varieties over k, let $\alpha \in \mathrm{Br}(Y)$ and let $z \in Z_0(X)$. Using that the composition of restriction $\mathrm{res}_{k(P)/k(f(P))} : \mathrm{Br}(k(f(P)) \to \mathrm{Br}(k(P))$ with corestriction $\mathrm{cores}_{k(P)/k(f(P))} : \mathrm{Br}(k(P)) \to \mathrm{Br}(k(f(P)))$ is multiplication by $[k(P) : k(f(P))]$, we obtain

$$\langle f^*(\alpha), z \rangle = \langle \alpha, f_*(z) \rangle.$$

Lemma 5.3.1 *Let k be a field, X a k-variety and $Y = \mathrm{Spec}(A) \subset X$ a finite subscheme. Let $[Y] \in Z_0(X)$ be the associated zero-cycle. For any $\alpha \in \mathrm{Br}(X)$, one has*

$$\langle \alpha, [Y] \rangle = \mathrm{cores}_{A/k}(\alpha_Y) \in \mathrm{Br}(k).$$

Proof. For the identity map $Y = X$, this is Lemma 3.8.5. The general case follows from the functoriality of the pairing. □

Proposition 5.3.2 *Let X be a proper variety over a field k. Then the pairing (5.1) induces a bilinear pairing*

$$\mathrm{Br}(X) \times \mathrm{CH}_0(X) \longrightarrow \mathrm{Br}(k). \tag{5.2}$$

This pairing is functorial with respect to proper morphisms.

Proof. Let $C \to X$ be a morphism from a proper normal integral curve C over k. Let $f : C \to \mathbb{P}^1_k$ be a dominant morphism. This is a finite locally free morphism of constant rank. Let $z_0 \in Z_0(C)$, respectively $z_1 \in Z_0(C)$, be the 0-cycle on C associated to the finite scheme $\mathrm{Spec}(A_0) = f^{-1}(p_0)$, respectively to $\mathrm{Spec}(A_1) = f^{-1}(p_1)$, where p_0 and p_1 are distinct k-points in $\mathbb{P}^1_k$. Let $\alpha \in \mathrm{Br}(X)$. By Lemma 5.3.1 we have

$$\langle \alpha, z_i \rangle = \mathrm{cores}_{A_i/k}(\alpha_Y) \in \mathrm{Br}(k)$$

for $i = 0, 1$. By Proposition 3.8.1, $\mathrm{cores}_{A_i/k}(\alpha_Y) = \langle \mathrm{cores}_{C/\mathbb{P}^1}(\alpha), p_i\rangle$. The map $\mathrm{Br}(k) \to \mathrm{Br}(\mathbb{P}^1_k)$ is an isomorphism (Theorem 5.1.3). Thus $\mathrm{cores}_{C/\mathbb{P}^1}(\alpha) \in \mathrm{Br}(\mathbb{P}^1_k)$ is a constant class, hence $\langle \alpha, z_0\rangle = \langle \alpha, z_1\rangle$. □

The following definition was given in [ACTP17].

Definition 5.3.3 *A projective variety X over a field k is called* **universally CH_0-trivial** *if for any field extension $k \subset K$ the degree map $\deg_K : \mathrm{CH}_0(X_K) \to \mathbb{Z}$ is an isomorphism.*

For example, if X is smooth, projective and rational over k, then X is universally CH_0-trivial.

Theorem 5.3.4 *Let X be a smooth, projective, geometrically integral variety over a field k.*

(i) *Assume that X is universally CH_0-trivial. Then for every field extension K of k the natural map $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ is an isomorphism.*

(ii) *Assume that for every field extension $k \subset K$, the group $A_0(X_K)$ is a torsion group. Then there exists a positive integer N such that for every field extension $k \subset K$ the quotient $\mathrm{Br}(X_K)/\mathrm{Br}(K)$ is annihilated by N.*

(iii) *Let $k = \mathbb{C}$. Suppose that there exist a smooth, projective, integral curve Y over $\mathbb{C}$ and a morphism $f : Y \to X$ such that $f_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is surjective. Then $\mathrm{Br}(X)$ is a finite group.*

Proof. (i) It is enough to prove the statement over k. Since X is universally CH_0-trivial, it has a zero-cycle z of degree 1. The map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is injective because evaluating at z gives a section. Now let $\alpha \in \mathrm{Br}(X)$. Take $F = k(X)$ to be the function field of X. The pairing (5.2)

$$\mathrm{Br}(X_F) \times \mathrm{CH}_0(X_F) \longrightarrow \mathrm{Br}(F)$$

gives rise to the pairing

$$\mathrm{Br}(X) \times \mathrm{CH}_0(X_F) \longrightarrow \mathrm{Br}(F).$$

Let η be the generic point of X. It is clear that $\langle \alpha, \eta\rangle_F$ is the image of α under the natural map $\mathrm{Br}(X) \to \mathrm{Br}(F)$. Since X is smooth, this map is injective. By hypothesis $z_F - \eta = 0$ in $\mathrm{CH}_0(X_F)$, hence $\langle \alpha, z\rangle_F = \langle \alpha, \eta\rangle_F \in \mathrm{Br}(F)$. Therefore, $\langle \alpha, \eta\rangle_F$ is the image of $\langle \alpha, z\rangle \in \mathrm{Br}(k)$ under the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(F)$, hence $\alpha \in \mathrm{Br}(X)$ is the image of $\langle \alpha, z\rangle \in \mathrm{Br}(k)$ under the map $\mathrm{Br}(k) \to \mathrm{Br}(X)$.

(ii) Let P be a closed point of X. Let η be the generic point of X and let $F = k(X)$ be the field of fractions. By assumption there is a positive integer N such that $N(\deg_k(P)\eta - P_F) = 0 \in \mathrm{CH}_0(X_F)$. Arguing as above, we see that for any $\alpha \in \mathrm{Br}(X)$ we have $N(\deg_k(P)\alpha - \langle \alpha, P\rangle) = 0 \in \mathrm{Br}(X)$, hence $\mathrm{Br}(X)/\mathrm{Br}(k)$ is annihilated by $N\deg_k(P)$. The proof shows that the same statement, with the same factor $N\deg_k(P)$, holds for X_K over any field extension K of k.

(iii) As $\mathbb{C}$ is an algebraically closed field of infinite transcendance degree over $\mathbb{Q}$, there exists an algebraically closed field $F \subset \mathbb{C}$ of finite transcendence degree over $\mathbb{Q}$ such that Y and X descend to varieties Y_0 and X_0 over F, that is, $X \cong X_0 \otimes_F \mathbb{C}$ and $Y \cong Y_0 \otimes_F \mathbb{C}$, and $f : Y \to X$ descends to an F-morphism $f_0 : Y_0 \to X_0$.

We first claim that for any such field F, the map $\mathrm{CH}_0(Y_0) \to \mathrm{CH}_0(X_0)$ is surjective. Let z_0 be a zero-cycle on X_0. By assumption, over $\mathbb{C}$ there exists a zero-cycle $\sum_i n_i w_i$ on Y, finitely many smooth projective integral curves C_j, morphisms $\theta_j : C_j \to X$, and rational functions $g_j \in \mathbb{C}(C_j)^*$ such that the equality

$$z_{0,\mathbb{C}} = f_*(\sum_i n_i w_i) + \sum_j \theta_{j,*}(\mathrm{div}_{C_j}(g_j))$$

holds in the group of zero-cycles $Z_0(X)$. This equality involves only finitely many terms. One may thus realise all its constituents over a field $L \subset \mathbb{C}$ which is of finite type over F. This field L itself is the field of fractions of a regular F-algebra A of finite type. After suitable localisation, the displayed equality holds over such an A. Since F is algebraically closed, the F-rational points are Zariski dense on $\mathrm{Spec}(A)$, thus we can specialise the above equality to an equality over F. We obtain an equality of cycles on X_0. In this specialisation process, the zero-cycle $z \in Z_0(X_0)$ specialises to itself. This proves the claim.

Let us now consider $K = F(X_0)$, which we may embed into $\mathbb{C}$, and let η be the generic point of X_0 over F. By the previous claim applied to the algebraic closure of K in $\mathbb{C}$, there exists a finite extension L/K such that η_L is in the image of $\mathrm{CH}_0(Y_{0,L}) \to \mathrm{CH}_0(X_{0,L})$. A restriction-corestriction argument implies that there exists a positive integer N such that $N\eta \in \mathrm{CH}_0(X_{0,K})$ is in the image of $\mathrm{CH}_0(Y_{0,K})$, that is, we have an equality

$$N\eta = f_{0,*}(z)$$

for some $z \in \mathrm{CH}_0(Y_{0,K})$. By functoriality of the pairing between Chow groups of 0-cycles and Brauer groups (Proposition 5.3.2), for any $\alpha \in \mathrm{Br}(X_0)$ we get

$$\langle \alpha, N\eta \rangle = \langle \alpha, f_{0,*}(z) \rangle = \langle f_0^*(\alpha), z \rangle \in \mathrm{Br}(K).$$

But $f_0^*(\alpha) \in \mathrm{Br}(Y_0)$ and $\mathrm{Br}(Y_0) = 0$ since Y_0 is a curve over the algebraically closed field F (Theorem 4.5.1). Thus $N\langle \alpha, \eta \rangle = 0 \in \mathrm{Br}(K)$. But

$$\langle \alpha, \eta \rangle \in \mathrm{Br}(K) = \mathrm{Br}(F(X_0))$$

is the image of $\alpha \in \mathrm{Br}(X_0)$ under the injective map $\mathrm{Br}(X_0) \to \mathrm{Br}(F(X_0))$. We thus have $N\,\mathrm{Br}(X_0) = 0$.

The map $\mathrm{Br}(X_0) \to \mathrm{Br}(X_0 \times_F \mathbb{C})$ is an isomorphism by Proposition 4.2.2. We thus conclude that $N\,\mathrm{Br}(X) = 0$ for the original X over $\mathbb{C}$. Proposition 4.2.6 now gives that $\mathrm{Br}(X)$ is the finite group $\mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$. □

Remark 5.3.5 Under the assumptions of Theorem 5.3.4 (iii), using Theorem 4.2.6, one gets $b_2 = \rho$, which by Hodge theory is equivalent to $\mathrm{H}^2(X, \mathcal{O}_X) = 0$. Under the same hypotheses, one can actually show that $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for all $i \geq 2$. The proof of Theorem 5.3.4 (iii) given above is due to Salberger. It is a

Brauer theoretic version of a theorem of Bloch, itself inspired by a theorem of Mumford: the Chow group of 0-cycles of a smooth, complex, projective surface X with $p_g(X) \neq 0$ is not representable. Bloch's argument was much developed by Bloch and Srinivas, and then by many other authors.

Chapter 6

Severi–Brauer varieties and hypersurfaces

Isomorphism classes of Severi–Brauer varieties are in bijection with isomorphism classes of central simple algebras. This leads to many intricate relations. In Section 6.1 we briefly recall the basic properties of Severi–Brauer varieties. Any such variety is birationally equivalent to a principal homogeneous space of a torus. We give a precise version of this statement. We then discuss morphisms from an arbitrary variety to a Severi–Brauer variety. In Section 6.2 we deal with another simple class of projective homogenous varieties, namely smooth projective quadrics. For a variety X of either type, the restriction map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is surjective and the kernel is a finite cyclic group with a natural generator. The knowledge of this kernel will be used to establish the non-vanishing of specific Brauer classes of the function field of certain conic bundles over $\mathbb{P}^2_{\mathbb{C}}$ (see Section 10.5). Recently, in connection with arithmetic investigations of integral points, the computation of the Brauer group of open algebraic varieties has become of interest. In Section 6.3 we give a few examples of such computations.

6.1 Severi–Brauer varieties

Definition and basic properties

The following definition is due to F. Châtelet.

Definition 6.1.1 *Let n be a positive integer. A Severi–Brauer variety of dimension $n-1$ over k is a twisted form of the projective space $\mathbb{P}^{n-1}_k$. Equivalently, this is a k-variety X such that there exist a field extension $k\subset K$ and an isomorphism of K-varieties $X\times_k K\simeq\mathbb{P}^{n-1}_K$.*

The automorphism group of the projective space $\mathbb{P}^{n-1}_k$ is the algebraic group $\mathrm{PGL}_{n,k}$. By the Skolem–Noether theorem, it coincides with the automorphism group of the matrix algebra $M_{n,k}$. Galois descent (see Section 1.3.2) then gives a bijection between the isomorphism classes of twisted forms of $\mathbb{P}_k^{n-1}$ and the isomorphism classes of twisted forms of $M_{n,k}$, which are precisely the central simple algebras of degree n over k. Thus we obtain canonical bijections of pointed sets

$$\mathrm{SB}_{n-1,k} \cong \mathrm{H}^1(k, \mathrm{PGL}_{n,k}) \cong \mathrm{Az}_{n,k}$$

and a map of pointed sets $\mathrm{H}^1(k, \mathrm{PGL}_{n,k}) \to \mathrm{Br}(k)$. For a Severi–Brauer variety X of dimension $n-1$ we denote by $[X] \in \mathrm{Br}(k)$ the image of the isomorphism class of X under the composite map

$$\mathrm{SB}_{n-1,k} \cong \mathrm{H}^1(k, \mathrm{PGL}_{n,k}) \longrightarrow \mathrm{Br}(k).$$

Recall that the map $\mathrm{Az}_{n,k} \cong \mathrm{H}^1(k, \mathrm{PGL}_{n,k}) \to \mathrm{Br}(k)$ associates to a central simple algebra A of degree n its class $[A] \in \mathrm{Br}(k)$, as discussed in Section 1.3.3.

For a central simple k-algebra A of degree n define $X(A)$ to be the k-scheme of right ideals of A of rank n. More precisely, for any commutative k-algebra R, the set $X(A)(R)$ is the set of right ideals of the matrix algebra $A \otimes_k R$ which are projective R-modules of rank n and are direct summands of the R-module $A \otimes_k R$, see [KMRT, Ch. I, §1.C]. This is a closed subscheme of the Grassmannian variety of n-dimensional subspaces of the k-vector space A.

Theorem 6.1.2 *Let X be a variety over k. The following properties are equivalent.*

(i) *X is a Severi–Brauer variety of dimension $n-1$.*

(ii) *There is an isomorphism $\overline{X} \simeq \mathbb{P}_{\bar{k}}^{n-1}$.*

(iii) *There is an isomorphism $X^{\mathrm{s}} \simeq \mathbb{P}_{k_{\mathrm{s}}}^{n-1}$.*

(iv) *There is a central simple k-algebra A of degree n such that $X \simeq X(A)$.*

The central simple algebra A in (iv) is well defined up to isomorphism. If $X = X(A)$, then $[X] = [A] \in \mathrm{Br}(k)$.

For the proof of this theorem see [Lic68], [Art82], [KMRT, Ch. I, §1.C], [GS17, Ch. 5], [Kol], [Po18, §4.5.1].

Given a variety X as in (i), one recovers the central simple k-algebra A in (iv) in the following direct manner (Quillen, Szabó, Kollár, see [Kol]). Let T_X be the tangent bundle of X and let Ω_X^1 be the cotangent bundle. It is known that the coherent cohomology group $\mathrm{H}^1(X, \Omega_X^1)$ is a 1-dimensional vector space over k. (This can be computed over $\bar{k}$, where $\overline{X} \simeq \mathbb{P}_{\bar{k}}^n$.) We have

$$\mathrm{H}^1(X, \Omega_X^1) = \mathrm{H}^1(X, \mathcal{H}om(T_X, \mathcal{O}_X)) = \mathrm{Ext}_X^1(T_X, \mathcal{O}_X).$$

Up to multiplying the maps by non-zero scalars in k^*, this defines a unique non-split extension of vector bundles

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{F}(X) \longrightarrow T_X \longrightarrow 0.$$

This is a twisted version of the classical exact sequence on $\mathbb{P}_k^{n-1}$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow \mathcal{O}(1)_{\mathbb{P}^{n-1}}^{\oplus n} \longrightarrow T_{\mathbb{P}^{n-1}} \longrightarrow 0.$$

Then $A = \mathrm{End}_X(\mathcal{F}(X))$ is a central simple k-algebra such that $X = X(A)$.

Let X be a Severi–Brauer variety. The Picard group

$$\mathrm{Pic}(X^{\mathrm{s}}) \simeq \mathrm{Pic}(\mathbb{P}^{n-1}_{k_{\mathrm{s}}}) = \mathbb{Z}$$

is generated by the class L_X of an ample line bundle of degree 1. The class of the canonical bundle $\omega_X \in \mathrm{Pic}(X)$ is $-nL_X$. The action of Γ on $\mathrm{Pic}(X^{\mathrm{s}})$ is trivial, so $L_X \in \mathrm{Pic}(X^{\mathrm{s}})^\Gamma$ and $\mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}})) = 0$. Next, $\mathrm{Br}(X^{\mathrm{s}}) = \mathrm{Br}(\mathbb{P}^{n-1}_{k_{\mathrm{s}}}) = 0$. Thus the exact sequence (4.9)

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}})^\Gamma \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}_1(X) \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}}))$$

takes the following form:

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}})^\Gamma \xrightarrow{\partial_X} \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X) \longrightarrow 0, \tag{6.1}$$

where $\mathrm{Pic}(X^{\mathrm{s}}) = \mathbb{Z}$. The kernel of $\mathrm{Br}(k)\to\mathrm{Br}(X)$, which coincides with the kernel of $\mathrm{Br}(k)\to\mathrm{Br}(k(X))$, is a finite cyclic group annihilated by n. Let $\alpha_X = \partial_X(L_X)$ be the image of L_X in $\mathrm{Br}(k)$.

If $X = X(A)$, then α_X equals the class $[A] \in \mathrm{Br}(k)$ of the central simple algebra A. For a proof, see [Lic68, p. 1217] and [GS17, Thm. 5.4.11]. This is a refinement of an earlier result of Amitsur that the kernel of $\mathrm{Br}(k)\to\mathrm{Br}(k(X))$ is the finite cyclic group generated by $[X]$.

Proposition 6.1.3 (F. Châtelet) *Let $X = X(A)$ be a Severi–Brauer variety. The following properties are equivalent:*

(i) $X(k) \neq \emptyset$*;*

(ii) $\alpha_X = 0$*;*

(iii) $X \simeq \mathbb{P}^{n-1}_k$*;*

(iv) *there is a k-algebra isomorphism $A \simeq M_n(k)$.*

Proof. Condition (i) implies that the map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is injective, thus the map $\mathrm{Pic}(X)\to\mathrm{Pic}(X^{\mathrm{s}})^\Gamma$ is surjective. This implies (ii), which itself implies that there is a well-defined line bundle $L \in \mathrm{Pic}(X)$ which over k_{s} is isomorphic to L_X. The line bundle L on X defines a k-morphism to the projective space $\mathbb{P}^n_k$ which becomes an isomorphism over k_{s}, hence is an isomorphism over k. This gives (iii), which trivially gives (i). The equivalence of (ii) and (iv) follows from the equality $\alpha_X = [A]$ mentioned above. □

This proposition is a particular case of the following more general statement.

Proposition 6.1.4 *Severi–Brauer varieties X_1 and X_2 over k of the same dimension are isomorphic over k if and only if $\alpha_{X_1} = \alpha_{X_2} \in \mathrm{Br}(k)$.*

Proof. For an even more general result of M. Artin see [GS17, Prop. 5.3.2]. □

Proposition 6.1.5 *Let X_1 and X_2 be Severi–Brauer varieties over k. The following properties are equivalent.*

(i) *α_{X_1} and α_{X_2} generate the same cyclic subgroup of* $\mathrm{Br}(k)$*;*

(ii) *X_1 and X_2 are stably birationally equivalent, i.e., there exist projective spaces $\mathbb{P}^r_k$ and $\mathbb{P}^s_k$ such that $X_1 \times_k \mathbb{P}^r_k$ is birationally equivalent to $X_2 \times_k \mathbb{P}^s_k$.*

Proof. See [GS17, Cor. 5.4.2, Remark 5.4.3]. □

It is an open question whether stably birationally equivalent Severi–Brauer varieties of the same dimension are birationally equivalent.

Torsors under tori as birational models of Severi–Brauer varieties

The following statement does not seem to be in the literature.

Proposition 6.1.6 *Let A be a central simple algebra of degree n over a field k and let $X = X(A)$ be the associated $(n-1)$-dimensional Severi–Brauer variety. Let K be a maximal commutative étale k-subalgebra of A. The action of K on A by left multiplication defines a maximal k-torus $T \subset \mathrm{PGL}_A$ which is $(n-1)$-dimensional and fits into the exact sequence*

$$1 \longrightarrow \mathbb{G}_{m,k} \longrightarrow R_{K/k}(\mathbb{G}_{m,K}) \longrightarrow T \longrightarrow 1. \tag{6.2}$$

The natural action of PGL_A on X restricts to an action of T on X, which has a dense open orbit $E \subset X$ consisting of the points of X with trivial stabilisers in T. Then E is a k-torsor for T. Moreover, the connecting map defined by the exact sequence (6.2) sends the class $[E] \in \mathrm{H}^1(k,T)$ to the class $[A] \in \mathrm{Br}(k)$.

Proof. Let $c : \Gamma \to \mathrm{PGL}_{n,k}(k_\mathrm{s})$ be a 1-cocycle such that A is the twisted form of the matrix algebra $M_n(k)$ by c. Twisting by c we obtain $X = (\mathbb{P}^n_k)_c$ and the inner form $\mathrm{PGL}_A = (\mathrm{PGL}_{n,k})_c$. After twisting, the left action of $\mathrm{PGL}_{n,k}$ on $\mathbb{P}^{n-1}_k$ becomes a left action of PGL_A on X.

For a maximal commutative étale k-subalgebra $K \subset A$ and the associated maximal k-torus $T \subset \mathrm{PGL}_A$, the open subset $E \subset X$ consists of k_s-points with trivial stabilisers in $T(k_\mathrm{s})$. Since $K \otimes_k k_\mathrm{s}$ is conjugate in $M_n(k_\mathrm{s})$ to the subalgebra of diagonal matrices, E^s is the open subset of $X^\mathrm{s} \cong \mathbb{P}^{n-1}_{k_\mathrm{s}}$ whose complement is the union of coordinate hyperplanes. Hence E^s is a torsor for T^s. This implies that E is a k-torsor for T.

By a corollary of Steinberg's theorem [PR91, Prop. 6.19], there is an embedding $\phi : T \hookrightarrow \mathrm{PGL}_{n,k}$ for which there is a 1-cocycle $\tilde{c} : \Gamma \to T(k_\mathrm{s})$ such that $c = \phi_*(\tilde{c})$. This implies that (6.2) sends the class $[\tilde{c}] \in \mathrm{H}^1(k,T)$ to $[A] \in \mathrm{Br}(k)$. On the other hand, the action of $T \subset \mathrm{PGL}_A$ on X is obtained by twisting the action of $\phi(T) \subset \mathrm{PGL}_{n,k}$ on $\mathbb{P}^{n-1}_k$ by $\tilde{c}$. (Conjugation of $\mathrm{PGL}_{n,k}(k_\mathrm{s})$ by an element of $\phi(T)(k_\mathrm{s})$ induces the trivial action on $\phi(T)(k_\mathrm{s}) \subset \mathrm{PGL}_{n,k}(k_\mathrm{s})$.) The open orbit of $\phi(T)$ in $\mathbb{P}^{n-1}_k$ is a trivial k-torsor for $\phi(T)$. Hence E is the twisted form of a trivial torsor by $\tilde{c}$, thus $[E] = [\tilde{c}] \in \mathrm{H}^1(k,T)$. □

The following special case is better known, though it is sometimes stated in the weaker form of a stable birational equivalence.

Proposition 6.1.7 *Let X be the Severi–Brauer variety attached to a cyclic algebra $D_k(\chi, a)$. Let $K \subset k_\mathrm{s}$ be the invariant subfield of $\mathrm{Ker}(\chi) \subset \Gamma$. Then X contains a dense open subset isomorphic to the k-torsor for the norm 1 torus $R^1_{K/k}(\mathbb{G}_{m,K})$ given by $\mathrm{N}_{K/k}(x) = a$.*

Proof. Write $A = D_k(\chi, a)$. We note that $K \subset A$ is a maximal commutative étale k-subalgebra and $T = R_{K/k}(\mathbb{G}_{m,K})/\mathbb{G}_{m,k} \subset \mathrm{PGL}_A$ is the associated maximal k-torus. By Proposition 6.1.6, the Severi–Brauer variety X contains a dense open subset isomorphic to a k-torsor E for T such that the class $[E] \in \mathrm{H}^1(k, T)$ goes to

$$[A] = (\chi, a) \in \mathrm{Br}(K/k) = \mathrm{H}^2(G, K^*)$$

under the isomorphism $\mathrm{H}^1(k, T) \xrightarrow{\sim} \mathrm{Br}(K/k)$ provided by the connecting map of (6.2).

Having fixed a generator σ of $\mathrm{Gal}(K/k) \simeq \mathbb{Z}/n$, we construct an isomorphism of k-tori $T \xrightarrow{\sim} R^1_{K/k}(\mathbb{G}_{m,K})$ as follows. The map $K^* \to K^*$ sending x to $\sigma(x)/x$ commutes with $\mathrm{Gal}(K/k)$ and hence induces an automorphism ϕ of the k-torus $R_{K/k}(\mathbb{G}_{m,K})$. It is clear that $\mathrm{Ker}(\phi)$ is $\mathbb{G}_{m,k}$ naturally embedded in $R_{K/k}(\mathbb{G}_{m,K})$. By Hilbert's theorem 90 for a cyclic extension, $\mathrm{Coker}(\phi) = \mathbb{G}_{m,k}$ and the surjective map $R_{K/k}(\mathbb{G}_{m,K}) \to \mathbb{G}_{m,k}$ is induced by the norm $\mathrm{N}_{K/k}$. We obtain an exact sequence of k-tori

$$1 \longrightarrow \mathbb{G}_{m,k} \longrightarrow R_{K/k}(\mathbb{G}_{m,K}) \xrightarrow{\phi} R_{K/k}(\mathbb{G}_{m,K}) \longrightarrow \mathbb{G}_{m,k} \longrightarrow 1. \tag{6.3}$$

Hence ϕ induces an isomorphism $\varphi : T \xrightarrow{\sim} R^1_{K/k}(\mathbb{G}_{m,K})$, which thus depends on the choice of the generator σ.

Recall that every k-torsor of $R^1_{K/k}(\mathbb{G}_{m,K})$ is isomorphic to the closed subset $Z_c \subset R_{K/k}(\mathbb{G}_{m,K})$ given by $\mathrm{N}_{K/k}(x) = c$ for some $c \in k^*$. Indeed, Shapiro's lemma and Hilbert's theorem 90 imply that $\mathrm{H}^1(k, R_{K/k}(\mathbb{G}_{m,K})) = \mathrm{H}^1(K, \mathbb{G}_{m,K}) = 0$. The exact sequence of tori

$$1 \longrightarrow R^1_{K/k}(\mathbb{G}_{m,K}) \longrightarrow R_{K/k}(\mathbb{G}_{m,K}) \longrightarrow \mathbb{G}_{m,k} \longrightarrow 1 \tag{6.4}$$

gives an isomorphism

$$k^*/\mathrm{N}_{K/k}(K^*) = \hat{\mathrm{H}}^0(G, K^*) \xrightarrow{\sim} \mathrm{H}^1(k, R^1_{K/k}(\mathbb{G}_{m,K})).$$

Every element of this group is represented by some $c \in k^*$. The exact sequence (6.4) shows that the inverse image of c in $R_{K/k}(\mathbb{G}_{m,K})$, which we called Z_c, is a k-torsor for $R^1_{K/k}(\mathbb{G}_{m,K})$ whose class is represented by c.

We want to show that the isomorphism φ induces an isomorphism $E \xrightarrow{\sim} Z_a$, which is equivalent to $\varphi_*[E] = [Z_a]$. We have $(\chi, a) = a \cup \partial(\chi)$, see (1.5). Thus it remains to show that the following diagram of isomorphisms commutes:

$$\begin{array}{ccccc}
[E] \in & \mathrm{H}^1(k, T) & \xrightarrow[\cong]{\varphi_*} & \mathrm{H}^1(k, R^1_{K/k}(\mathbb{G}_{m,K})) & \ni [Z_a] \\
 & \downarrow \cong & & \cong \uparrow & \\
(\chi, a) \in & \mathrm{H}^2(G, K^*) & \xleftarrow[\cong]{\cup\partial(\chi)} & \hat{\mathrm{H}}^0(G, K^*) & \ni a
\end{array}$$

It suffices to show that the connecting map attached to (6.3) is the cup-product with the generator $\partial(\chi)$ of $\mathrm{H}^2(G, \mathbb{Z}) \simeq \mathbb{Z}/n$. For this it is enough to show that the connecting map associated to the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G] \xrightarrow{\sigma - 1} \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

sends $1 \in \mathbb{Z}$ to $\partial(\chi)$. This sequence is (and also is dual to) a truncated piece of the standard free resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$, thus the induced map $\mathbb{Z}^G \to \mathrm{H}^2(G, \mathbb{Z})$ is the canonical surjection $\mathbb{Z} \to \mathbb{Z}/n$ sending 1 to $\partial(\chi)$. Decomposing this 2-extension into a Yoneda product of two short exact sequences, one shows that this map coincides with the connecting map $\mathbb{Z}^G \to \mathrm{H}^2(G, \mathbb{Z})$. □

Morphisms to Severi–Brauer varieties

Let k be a field. Let Y be a Severi–Brauer variety and let X be an arbitrary k-scheme. A morphism $f : X \to Y$ gives rise to a map of Γ-modules $f^* : \mathrm{Pic}(Y^{\mathrm{s}}) \to \mathrm{Pic}(X^{\mathrm{s}})$ and a distinguished class $f^*(L_Y) \in \mathrm{Pic}(X^{\mathrm{s}})^\Gamma$. Moreover, we have a map of Γ-modules

$$\mathrm{H}^0(Y^{\mathrm{s}}, L_Y) \longrightarrow \mathrm{H}^0(X^{\mathrm{s}}, f^*(L_Y)).$$

The image of this map is a finite dimensional, Γ-invariant, k_{s}-vector subspace V of $\mathrm{H}^0(X^{\mathrm{s}}, f^*(L_Y))$. Since f is a morphism, the natural map $V \otimes_{k_{\mathrm{s}}} \mathcal{O}_{X^{\mathrm{s}}} \to f^*(L_Y)$ is surjective: the line bundle $f^*(L_Y) \in \mathrm{Pic}(Y^{\mathrm{s}})$ is generated by the vector subspace of sections $V \subset \mathrm{H}^0(X^{\mathrm{s}}, f^*(L_Y))$.

There is a converse to this observation.

Proposition 6.1.8 *Let k be a field. Let X be a k-scheme and let $L \in \mathrm{Pic}(X^{\mathrm{s}})^\Gamma$. Let $V \subset \mathrm{H}^0(X^{\mathrm{s}}, L)$ be a finite dimensional, Γ-invariant, non-zero k_{s}-vector subspace such that the map $V \otimes_{k_{\mathrm{s}}} \mathcal{O}_{X^{\mathrm{s}}} \to L$ is surjective. Let $n = \dim(V)$. Then there is an $(n-1)$-dimensional Severi–Brauer variety Y over k and a k-morphism $f : X \to Y$ such that $f^*(L_Y) = L \in \mathrm{Pic}(X^{\mathrm{s}})$ and the map $f^* : \mathrm{H}^0(Y^{\mathrm{s}}, L_Y) \to \mathrm{H}^0(X^{\mathrm{s}}, L)$ is injective with image V.*

Proof. Under the assumption that X is proper over k, and $V = \mathrm{H}^0(X^{\mathrm{s}}, L)$, the above proposition is established in [Lie17, Thm. 3.4]. The proof by descent extends to the above more general statement. □

Let X be a smooth, quasi-projective, geometrically integral variety over a field k, such that $k_{\mathrm{s}}[X^{\mathrm{s}}]^* = k_{\mathrm{s}}^*$. By Proposition 4.3.2, we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}})^\Gamma \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}_1(X),$$

and this exact sequence is functorial contravariant with respect to such k-varieties. Let ∂_X denote the map $\mathrm{Pic}(X^{\mathrm{s}})^\Gamma \longrightarrow \mathrm{Br}(k)$. If $X(k) \neq \emptyset$, then $\mathrm{Br}(k) \to \mathrm{Br}_1(X)$ has a retraction, hence $\partial_X = 0$. More generally, if X has index d, i.e. has a zero-cycle of degree d, then $d\,\partial_X(L) = 0$ for all $L \in \mathrm{Pic}(X^{\mathrm{s}})^\Gamma$.

We want to understand restrictions on the order of $\partial_X(L)$ in the general case. By abuse of notation, let us use the same notation for a line bundle L on X^{s} and its class in $\mathrm{Pic}(X^{\mathrm{s}})$.

If Y is a Severi–Brauer variety of dimension $n-1$, then the image in $\mathrm{Pic}(Y^{\mathrm{s}})$ of the canonical bundle $\omega_Y \in \mathrm{Pic}(Y)$ is the opposite of $L_Y^{\otimes n}$. This implies

$$n\,\partial_Y(L_Y) = 0.$$

Part (i) of the following proposition is stated in various degrees of generality by S. Lichtenbaum [Lic68, Lic69]. Part (ii) was recently suggested by A. Kuznetsov.

Proposition 6.1.9 *Let X be a smooth, projective, geometrically integral variety over a field k and let $L \in \mathrm{Pic}(X^{\mathrm{s}})^{\Gamma}$.*

(i) *If there exists a Γ-equivariant vector subspace $V \subset \mathrm{H}^0(X^{\mathrm{s}}, L)$ of dimension $n \geq 1$, then $n\,\partial_X(L) = 0$.*

(ii) *Let $\chi(L)$ be the coherent Euler–Poincaré characteristic of L on X^{s}. Then*

$$\chi(L)\,\partial_X(L) = 0.$$

Proof. Let us prove (i). Suppose first that $n = 1$. Then there exists a unique effective Cartier divisor D on X^{s} with $\mathcal{O}_{X^{\mathrm{s}}}(D) \simeq L$ which is the zero set of a generator of the one-dimensional vector space V. This divisor is Γ-invariant. Hence it comes from $\mathrm{Div}(X)$, hence L comes from $\mathrm{Pic}(X)$, hence $\partial_X(L) = 0$.

Suppose $n \geq 1$. The linear system $V \subset \mathrm{H}^0(X^{\mathrm{s}}, L)$ may have a fixed component. As above, it corresponds to a fixed effective divisor D in $\mathrm{Div}(X)$. One then considers $M := L \otimes \mathcal{O}_{X^{\mathrm{s}}}(-D) \in \mathrm{Pic}(X^{\mathrm{s}})$. We may identify V with a Γ-invariant vector subspace of $\mathrm{H}^0(X^{\mathrm{s}}, M)$. Since there is now no fixed component, this defines a k-morphism $g : U \to Y$, where $U \subset X$ is an open set which contains all codimension 1 points of the smooth variety X, so that $k_{\mathrm{s}}[U^{\mathrm{s}}]^* = k_{\mathrm{s}}^*$, and Y is a Severi–Brauer variety of dimension $n-1$, equipped with its natural line bundle $L_Y \in \mathrm{Pic}(Y^{\mathrm{s}})$. We have $n\,\partial_Y(L_Y) = 0$.

The inverse image $g^*(L_Y) \in \mathrm{Pic}(U^{\mathrm{s}})$ coincides with the restriction of the line bundle $M^{\mathrm{s}} \in \mathrm{Pic}(U^{\mathrm{s}})$. By functoriality we conclude $n\,\partial_U(M) = 0$. Since U contains all the codimension 1 points of X, the restriction map $\mathrm{Pic}(X^{\mathrm{s}}) \to \mathrm{Pic}(U^{\mathrm{s}})$ is an isomorphism. By functoriality again we have $n\,\partial_X(M) = 0$. Now we have $\partial(\mathcal{O}_{X^{\mathrm{s}}}(D)) = 0$ since D is defined over k. Since ∂ is additive, and we have $L = M \otimes \mathcal{O}_{X^{\mathrm{s}}}(D)$, we conclude $n\,\partial_X(L) = 0$. This proves (i).

Let us prove (ii). Let $\mathcal{O}(1) \in \mathrm{Pic}(X)$ be a very ample sheaf. By the Riemann–Roch theorem, there exists a polynomial $P(t) \in \mathbb{Q}[t]$, depending only on X, such that for any line bundle $L \in \mathrm{Pic}(X^{\mathrm{s}})$ we have $\chi(L(m)) = P(m)$. Let s be a positive integer such that $s\,P(t) \in \mathbb{Z}[t]$.

Let $L \in \mathrm{Pic}(X^{\mathrm{s}})^{\Gamma}$. By a result of Serre, there exists an integer $m_0 = m_0(L)$ such that for any integer $m \geq m_0$, the line bundle $L(m)$ is very ample and satisfies $\mathrm{H}^i(X^{\mathrm{s}}, L(m)) = 0$ for $i > 0$. For any such m, we have $\chi(L(m)) = h^0(X^{\mathrm{s}}, L(m))$. By (i), we deduce $\chi(L(m))\,\partial_X(L(m)) = 0$. Since ∂ is additive and $\mathcal{O}(1) \in \mathrm{Pic}(X)$, this gives $\chi(L(m))\,\partial_X(L) = 0$. We have $\chi(L(m)) - \chi(L) = R(m)/s$ with $R(t) \in \mathbb{Z}[t]$ a polynomial with no constant term and depending only on X. Let r be an integer such that $r\,\partial_X(L) = 0$. Let $n \geq m_0$ be a multiple of rs. Then $\chi(L(n)) - \chi(L)$ is an integer and a multiple of r. Thus $(\chi(L(n)) - \chi(L))\,\partial_X(L) = 0$. We now get $\chi(L)\,\partial_X(L) = 0$. □

As an application of Severi–Brauer varieties we now justify the claim of Remark 1.2.15.

Proposition 6.1.10 *Let K be a henselian discretely valued field. Let $\widehat{K}$ be the completion of K. Then the natural map $\mathrm{Br}(K) \to \mathrm{Br}(\widehat{K})$ is an isomorphism.*

Proof. By [BLR90, III, §6, Cor. 10, p. 82], if X is a smooth variety over K, then $X(K)$ is a dense subset of $X(\widehat{K})$. Let α be an element of the kernel of

$\mathrm{Br}(K)\to\mathrm{Br}(\widehat{K})$. Choose a Severi–Brauer variety X over K such that the class of X in $\mathrm{Br}(K)$ is α. Then $X(\widehat{K})\neq\emptyset$, hence $X(K)\neq\emptyset$ and this implies $\alpha=0$.

Conversely, let $\beta\in\mathrm{Br}(\widehat{K})$. There is a positive integer n such that β is the image of β_1 under the map $\mathrm{H}^1(\widehat{K},\mathrm{PGL}_n)\to\mathrm{H}^2(\widehat{K},\mathbb{G}_m)$. Choose a closed embedding of algebraic K-groups $\mathrm{PGL}_n\hookrightarrow\mathrm{GL}_N$. Then $X=\mathrm{GL}_N/\mathrm{PGL}_n$ is a smooth variety over K. Applying [SerCG, Ch. I, §5.4, Prop. 36] and using Hilbert's Theorem 90, we obtain the following commutative diagram of pointed sets with exact rows:

$$\begin{array}{ccccccc}
\mathrm{GL}_N(K) & \longrightarrow & X(K) & \longrightarrow & \mathrm{H}^1(K,\mathrm{PGL}_n) & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{GL}_N(\widehat{K}) & \longrightarrow & X(\widehat{K}) & \longrightarrow & \mathrm{H}^1(\widehat{K},\mathrm{PGL}_n) & \longrightarrow & 0
\end{array}$$

Choose a lifting $\beta_2\in X(\widehat{K})$ of β_1. By the implicit function theorem (Theorem 9.5.1), $\mathrm{GL}_N(\widehat{K})\beta_2$ is an open subset of $X(\widehat{K})$ in the topology induced by the topology of $\widehat{K}$. Since $X(K)$ is dense in $X(\widehat{K})$, we can find an $\alpha_2\in X(K)$ and a $g\in\mathrm{GL}_N(\widehat{K})$ such that $g\beta_2=\alpha_2$. Since $g\beta_2$ goes to $\beta_1\in\mathrm{H}^1(\widehat{K},\mathrm{PGL}_n)$ (see [SerCG, Ch. I, p. 55]), the image $\alpha_1\in\mathrm{H}^1(K,\mathrm{PGL}_n)$ of α_2 goes to β_1. This implies that the image $\alpha\in\mathrm{Br}(K)$ of α_1 goes to $\beta\in\mathrm{Br}(\widehat{K})$. □

6.2 Projective quadrics

Let C be a smooth, projective, geometrically integral curve of genus 0 over a field k. Since C is smooth, it has a k_{s}-point and hence $C^{\mathrm{s}}\cong\mathbb{P}^1_{k_{\mathrm{s}}}$, cf. Remark 1.1.11 (3). The anticanonical line bundle of C is very ample of degree 2, so it gives an embedding of C into $\mathbb{P}^2_k$ as a smooth conic. From the isomorphism $C^{\mathrm{s}}\cong\mathbb{P}^1_{k_{\mathrm{s}}}$ we also see that the degree map gives an isomorphism of $\mathrm{Pic}(C^{\mathrm{s}})$ with the trivial Γ-module $\mathbb{Z}$, hence $\mathrm{H}^1(k,\mathrm{Pic}(C^{\mathrm{s}}))=0$. Since $\mathrm{Br}(C^{\mathrm{s}})=0$ by Theorem 4.5.1 (iv), the exact sequence (4.9) can be written as

$$0\longrightarrow\mathrm{Pic}(C)\longrightarrow\mathrm{Pic}(C^{\mathrm{s}})^{\Gamma}\longrightarrow\mathrm{Br}(k)\longrightarrow\mathrm{Br}(C)\longrightarrow 0. \qquad (6.5)$$

Proposition 6.2.1 *Let k be a field,* $\mathrm{char}(k)\neq 2$. *Let C be a smooth conic over k. Let Q be the quaternion algebra over k associated to C. Then the image of a generator of* $\mathrm{Pic}(C^{\mathrm{s}})^{\Gamma}\cong\mathbb{Z}$ *in* $\mathrm{Br}(k)$ *is the class of Q, so that the natural map* $\mathrm{Br}(k)\to\mathrm{Br}(C)$ *is surjective with the kernel generated by the class of Q.*

Proof. By Remark 1.1.11 (3) or by the Riemann–Roch theorem, a smooth conic C has a k-point if and only if $C\cong\mathbb{P}^1_k$. In this case the natural map $\mathrm{Pic}(C)\to\mathrm{Pic}(C^{\mathrm{s}})$ is visibly an isomorphism. The natural map $\mathrm{Br}(k)\to\mathrm{Br}(\mathbb{P}^1_k)$ is an isomorphism by Theorem 4.5.1 (vii). On the other hand, Q is split over k by Proposition 1.1.7, so the class of Q in $\mathrm{Br}(k)$ is zero.

If C has no k-point, then Q is a division algebra by Proposition 1.1.7, so the class $[Q]\in\mathrm{Br}(k)$ is non-zero. By Exercise 1.1.12 (4), the class $[Q]$ lies in the kernel of the natural map $\mathrm{Br}(k)\to\mathrm{Br}(k(C))$. This map factors through the natural map $\mathrm{Br}(C)\to\mathrm{Br}(k(C))$, which is injective by Proposition 3.5.4. We conclude that $[Q]$ is a non-zero element in the kernel of the natural map $\mathrm{Br}(k)\to\mathrm{Br}(C)$. To finish the proof it remains to show that the cokernel of $\mathrm{Pic}(C)\to\mathrm{Pic}(C^{\mathrm{s}})$ is annihilated by 2. This follows from the fact that the degree map identifies $\mathrm{Pic}(C^{\mathrm{s}})$ with $\mathbb{Z}$ and the canonical class of C is an element of $\mathrm{Pic}(C)$ of degree -2. □

There is a version of this proposition over a field of characteristic 2.

Remark 6.2.2 Since $\mathrm{Br}(C^{\mathrm{s}})=0$ and $\mathrm{H}^1(k,\mathrm{Pic}(C^{\mathrm{s}}))=0$, the Leray spectral sequence (4.7) shows that the homomorphism $\mathrm{H}^3(k,k_{\mathrm{s}}^*)\to\mathrm{H}^3(C,\mathbb{G}_m)$ is injective.

Proposition 6.2.3 *Let k be a field, $\mathrm{char}(k)\neq 2$. Let $X\subset\mathbb{P}^n_k$, $n\geq 2$, be a smooth projective quadric.*

(a) *The map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is surjective.*

(b) *For $n=2$, let X be given by*

$$x^2-ay^2-bt^2=0,$$

where $a,b\in k^*$. The map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is an isomorphism if and only if $X(k)\neq\emptyset$. If $X(k)=\emptyset$, then $\mathrm{Ker}[\mathrm{Br}(k)\to\mathrm{Br}(X)]=\mathbb{Z}/2$ is generated by the class of the quaternion algebra (a,b).

(c) *For $n=3$, let X be given by*

$$x^2-ay^2-bz^2+dabt^2=0,$$

where $a,b,d\in k^*$. The class of d in k^*/k^{*2} is uniquely determined by X. If d is not a square, then $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is an isomorphism. If d is a square, then X is birationally equivalent to $\mathbb{P}^1_k\times C$, where C is the conic $x^2-ay^2-bt^2=0$. In this case, the map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is an isomorphism if and only if $X(k)\neq\emptyset$. If $X(k)=\emptyset$, then

$$\mathrm{Ker}[\mathrm{Br}(k)\to\mathrm{Br}(X)]=\mathrm{Ker}[\mathrm{Br}(k)\to\mathrm{Br}(C)]=\mathbb{Z}/2$$

is generated by the class of the quaternion algebra (a,b).

(d) *For $n\geq 4$, the map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is an isomorphism.*

Proof. Statement (b) was proved in Proposition 6.2.1.

A smooth quadric of dimension at least 1 with a rational point is birationally equivalent to the projective space. By Theorem 5.2.6 we have $\mathrm{Br}(X^{\mathrm{s}})=0$, hence $\mathrm{Br}_1(X)=\mathrm{Br}(X)$. Thus statement (a) will follow once we prove that $\mathrm{H}^1(k,\mathrm{Pic}(X^{\mathrm{s}}))=0$ for all $n\geq 2$.

Let us prove (d). For $n\geq 4$ an easy direct proof shows that the restriction map $\mathrm{Pic}(\mathbb{P}^n_{k_{\mathrm{s}}})\to\mathrm{Pic}(X^{\mathrm{s}})$ is an isomorphism. Indeed, the homogeneous equation of X^{s} can be written as $x_0x_1+q(x_2,\ldots,x_n)=0$, where q is a non-degenerate quadratic form in $n-1\geq 3$ variables. The hyperpane $x_0=0$ cuts out the integral divisor D given by $x_0 = q(x_2, \ldots, x_n) = 0$ in $\mathbb{P}^n_{k_s}$. The complement $X^s \smallsetminus D$ is isomorphic to the affine space $\mathbb{A}^{n-1}_{k_s}$. From the exact sequence

$$0 = k_s[\mathbb{A}^{n-1}_{k_s}]^*/k_s^* \longrightarrow \mathbb{Z}[D] \longrightarrow \mathrm{Pic}(X^s) \longrightarrow \mathrm{Pic}(\mathbb{A}^{n-1}_{k_s}) = 0$$

we conclude that $\mathbb{Z} = \mathrm{Pic}(\mathbb{P}^n_{k_s}) \to \mathrm{Pic}(X^s)$ is an isomorphism. Now a commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(\mathbb{P}^n_{k_s}) & \xrightarrow{\cong} & \mathrm{Pic}(X^s) \\ {\scriptstyle\cong}\uparrow & & \uparrow \\ \mathrm{Pic}(\mathbb{P}^n) & \longrightarrow & \mathrm{Pic}(X) \end{array}$$

implies that $\mathrm{Pic}(X) \to \mathrm{Pic}(X^s)$ is an isomorphism. In particular, in this case we have $\mathrm{H}^1(k, \mathrm{Pic}(X^s)) = 0$. Now the statement of (d) follows from the exact sequence (4.9).

Let us prove (c). (Quadric surfaces were already discussed by F. Châtelet in the 1940s, see [CTS93, Thm. 2.5].) In this case $X^s \cong \mathbb{P}^1_{k_s} \times \mathbb{P}^1_{k_s}$, hence $\mathrm{Pic}(X^s) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, where e_i is the inverse image of a k_s-point under the projection to the i-th factor, for $i = 1, 2$. These are the two rulings on the quadric surface X^s. The Galois group Γ preserves the integral basis $\{e_1, e_2\}$. The class of the hyperplane section is $e_1 + e_2$, which is thus in the image of $\mathrm{Pic}(X)$. The Galois action on $\{e_1, e_2\}$ is trivial if d is a square. If d is not a square, the action of Γ factors through its image $\mathrm{Gal}(k(\sqrt{d})/k)$; the generator of this group permutes e_1 and e_2. Using Shapiro's lemma we see that in all cases we have $\mathrm{H}^1(k, \mathrm{Pic}(X^s)) = 0$. The basic exact sequence (4.9) then becomes

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^s)^\Gamma \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X) \longrightarrow 0.$$

If d is not a square, then $\mathrm{Pic}(X^s)^\Gamma$ is generated by $e_1 + e_2$, hence the map $\mathrm{Pic}(X) \to \mathrm{Pic}(X^s)^\Gamma$ is surjective in this case and thus the map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is an isomorphism. If d is a square, it is easy to see that X is isomorphic to $C \times C$. The diagonal $C \hookrightarrow C \times C$ is a section of the projection $C \times C \to C$, whose generic fibre is thus isomorphic to $\mathbb{P}^1_{k(C)}$. It follows that X is birationally equivalent to the product of $\mathbb{P}^1_k \times C$. By Theorem 5.2.6 the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is the same as the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(C)$ described in (b).

Finally, statement (a) is now established for all $n \geq 2$. □

6.3 Some affine hypersurfaces

Proposition 6.3.1 *Let k be a field of characteristic 0. Let $X \subset \mathbb{P}^n_k$ be a smooth hypersurface and let $Z \subset X$ be a smooth hyperplane section. If $n \geq 4$, then the natural map $\mathrm{Br}(k) \to \mathrm{Br}(X \smallsetminus Z)$ is an isomorphism.*

Proof. As usual we write $\overline{X} = X \times_k \bar{k}$ and $\overline{Z} = Z \times_k \bar{k}$, where $\bar{k}$ is an algebraic closure of k. Since $n \geq 4$ the restriction map $\mathrm{Pic}(\mathbb{P}^n_{\bar{k}}) \to \mathrm{Pic}(\overline{X})$ is an isomorphism by a theorem of Lefschetz, so $\mathrm{Pic}(\overline{X}) = \mathbb{Z}[\overline{Z}]$. Thus every divisor class on $\overline{X}$ is a multiple of $[\overline{Z}]$, which implies that $\overline{Z}$ is integral. Let $U = X \setminus Z$. If a rational function $f \in \bar{k}(X)^*$ is regular and invertible on $\overline{U}$, then $\mathrm{div}(f)$ is a multiple of an ample divisor $\overline{Z}$, hence $\mathrm{div}(f) = 0$. This shows that $\bar{k}[U]^* = \bar{k}^*$.

The natural restriction map $\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{U})$ is surjective because $\overline{X}$ is smooth. The kernel of this map is the cyclic subgroup generated by $[\overline{Z}]$, hence the exact sequence

$$0 \longrightarrow \mathbb{Z}[\overline{Z}] \longrightarrow \mathrm{Pic}(\overline{X}) \longrightarrow \mathrm{Pic}(\overline{U}) \longrightarrow 0$$

shows that $\mathrm{Pic}(\overline{U}) = 0$.

Since $n \geq 4$, we have $\mathrm{H}^1(\overline{Z}, \mathbb{Q}/\mathbb{Z}) = 0$.

Since $n \geq 4$, we have $\mathrm{Br}(\overline{X}) = 0$. From the exact sequence

$$0 \longrightarrow \mathrm{Br}(\overline{X}) \longrightarrow \mathrm{Br}(\overline{U}) \longrightarrow \mathrm{H}^1(\overline{Z}, \mathbb{Q}/\mathbb{Z})$$

we conclude that $\mathrm{Br}(\overline{U}) = 0$. Now the exact sequence (4.9) gives the required statement. □

The following proposition is taken from [CTX09, §5.8].

Proposition 6.3.2 *Let k be a field,* $\mathrm{char}(k) \neq 2$. *Let $f(x, y, z)$ be a non-degenerate quadratic form and let $a \in k^*$. Let X be the affine quadric defined by the equation $f(x, y, z) = a$. Assume that $X(k) \neq \emptyset$ and $-a \cdot \mathrm{discr}(f) \notin k^{*2}$. Then* $\mathrm{Br}(X)/\mathrm{Br}(k) = \mathbb{Z}/2$.

In [CTX09, §5.8] there is an explicit algorithm to compute a generator of $\mathrm{Br}(X)/\mathrm{Br}(k) = \mathbb{Z}/2$ from a k-point on X. There is a misprint in *loc. cit.*, so we give a corrected description here. The algorithm generates a function ρ whose divisor $\mathrm{div}(\rho)$ is a norm for the extension K/k and such that the class of the quaternion algebra $(\rho, d) \in \mathrm{Br}(k(X))$ belongs to $\mathrm{Br}(X)$ and generates $\mathrm{Br}(X)/\mathrm{Br}(k)$. Let $Y \subset \mathbb{P}^3_k$ be the smooth projective quadric given by the homogeneous equation

$$f(x, y, z) = at^2.$$

Let $M \in Y(k)$. Let $l_1(x, y, z, t)$ be a linear form with coefficients in k defining the tangent plane to Y at M. There then exist linear forms l_2, l_3, l_4 and a constant $c \in k^*$ such that

$$f(x, y, z) - at^2 = l_1 l_2 + c(l_3^2 - d l_4^2).$$

The linear forms l_i for $i = 1, 2, 3, 4$ are linearly independent. Conversely, if we have such an identity, then $l_1 = 0$ is an equation for the tangent plane at the k-point $l_1 = l_3 = l_4 = 0$. Define $\rho = l_1(x, y, z, t)/t \in k(X)$ and let $\alpha = (\rho, d) \in \mathrm{Br}(k(X))$. We have

$$(l_1(x, y, z, t)/t, d) = (-c l_2(x, y, z, t)/t, d) \in \mathrm{Br}(k(X)).$$

Thus α is unramified on X away from the plane at infinity $t = 0$, and the finitely many closed points given by $l_1 = l_2 = 0$. By the purity theorem for the Brauer group of smooth varieties we see that this class is unramified on the affine quadric X, i.e. belongs to $\mathrm{Br}(X) \subset \mathrm{Br}(k(X))$. The complement of X in Y is the smooth projective conic C given by $f(x,y,z) = 0$. An easy computation shows that the residue of α at the generic point of this conic is the class of d in

$$k^*/k^{*2} = \mathrm{H}^1(k, \mathbb{Z}/2) \subset \mathrm{H}^1(k(C), \mathbb{Z}/2) \subset \mathrm{H}^1(k(C), \mathbb{Q}/\mathbb{Z})$$

(note that k is algebraically closed in $k(C)$). Since d is not a square in k, this class is not trivial. Thus $\alpha \in \mathrm{Br}(X)$ does not lie in the image of $\mathrm{Br}(k)$, and hence generates $\mathrm{Br}(X)/\mathrm{Br}(k)$. Note that at any k-point of X, either l_1 or l_2 is not zero. The map $X(k) \to \mathrm{Br}(k)$ associated to α can thus be computed by means of the map $X(k) \to k^*/\mathrm{N}_{K/k}(K^*)$ given by either the function $\rho = l_1(x,y,z,t)/t$ or the function $-cl_2(x,y,z,t)/t$.

The following result was established by T. Uematsu [Uem16] via an explicit cocycle computation.

Proposition 6.3.3 *Let a, b, c be independent variables over $\mathbb{C}$. Let $K = \mathbb{C}(a,b,c)$. Let $X \subset \mathbb{A}^3_K$ be the affine quadric*

$$x^2 + ay^2 + bz^2 + c = 0.$$

Then $\mathrm{Br}(X)/\mathrm{Br}(K) = 0$.

Exercise 6.3.4 Prove Proposition 6.3.3 without cocycle computations. *Hint*: Go over to the quadratic extension $K(\sqrt{b})/K$ where the quadric acquires a rational point. Then use [CTX09, §5.8].

The following propositions are left as exercises for the reader. They extend some of the computations in [Gun13].

Proposition 6.3.5 *Let k be a field. Let $Q(x) \in k[x]$ be a separable polynomial with $Q(0) \neq 0$. Let $X \subset \mathbb{A}^3_k$ be the affine surface $yz = xQ(x)$. Let $F \subset X$ be the closed subset defined by $y = Q(x) = 0$, and let $V = X \smallsetminus F$. Then*

(i) $V \cong \mathbb{A}^2_k$, *hence* $\mathrm{Pic}(V) = 0$.

(ii) $k[X]^* = k[V]^* = k^*$.

(iii) $\mathrm{Pic}(X)$ *is a finitely generated torsion-free abelian group.*

(iv) *Assume* $\mathrm{char}(k) = 0$. *Then* $\mathrm{Br}(k) \tilde{\longrightarrow} \mathrm{Br}(V)$ *and* $\mathrm{Br}(k) \tilde{\longrightarrow} \mathrm{Br}(X)$.

Let us give the proof of (i). The function $f(x,y,z) = x/y = z/Q(x)$ is defined everywhere on V. We have a morphism $V \to \mathbb{A}^2$ given by $(u,v) = (f(x,y,z), y)$. The image of the morphism $\mathbb{A}^2 \to \mathbb{A}^3$ given by $(u,v) \mapsto (x,y,z) = (uv, v, uQ(u,v))$ is in V. The two morphisms are the inverses of each other.

Proposition 6.3.6 *Let k be a field of characteristic 0. Let $a \in k^*$ and let $P(x) \in k[x]$ be a separable polynomial. Let $X \subset \mathbb{A}^3_k$ be the affine surface with equation $y^2 - az^2 = P(x)$. Then the quotient* $\mathrm{Br}(X)/\mathrm{Im}(\mathrm{Br}(k))$ *is a finite group.*

Remark 6.3.7 Note that the finiteness of $\mathrm{Br}(X)/\mathrm{Im}(\mathrm{Br}(k))$ for X as above is a general algebraic result. By contrast, the finiteness of $\mathrm{Br}(Y)/\mathrm{Im}(\mathrm{Br}(\mathbb{Q}))$, where $Y \subset \mathbb{A}^3_{\mathbb{Q}}$ is given by $x^3 + y^3 + z^3 = a$ with $a \in \mathbb{Q}^*$, uses arithmetic arguments [CTW12]. The point here is that the 'curve at infinity' in this case is a curve of genus one.

Given a k-point of X, can one compute explicit elements of $\mathrm{Br}(X)$ which generate the quotient of $\mathrm{Br}(X)$ modulo the image of $\mathrm{Br}(k)$?

Proposition 6.3.8 *Let k be a field of characteristic 0. Let $P(x) \in k[x]$ be a separable irreducible polynomial of degree d such that $K = k[x]/P(x)$ is a cyclic extension of k. Let $X \subset \mathbb{A}^3_k$ be the affine surface with equation $yz = P(x)$. Then $\mathrm{Br}(X)/\mathrm{Br}(k) \cong \mathbb{Z}/d$. The cyclic algebra over $k(X)$ defined by $A = (K/k, \sigma, y)$ lies in $\mathrm{Br}(X)$ and generates $\mathrm{Br}(X)/\mathrm{Br}(k)$.*

Proposition 6.3.9 *Let k be a field of characteristic 0. Let $P(x) \in k[x]$ be a separable polynomial. Write $P(x) = \prod_{i=1}^n P_i(x)$ as a product of irreducible polynomials. Let $X \subset \mathbb{A}^3_k$ be the affine surface with equation $y^2 - az^2 = P(x)$, where $a \in k^*$. For each $i = 1, \ldots, n$ the quaternion algebra $A_i = (a, P_i(x)) \in \mathrm{Br}(k(X))$ lies in $\mathrm{Br}(X)$.*

In connection with applications to the integral Brauer–Manin obstruction, the Brauer groups of many quasi-projective varieties has been computed in recent years. See [CTX09], [CTW12], [CTHa12], [JaSc17], [BrKo], [BrLy], [Harp1], [Harp2], [Mit18], [Berg], [LM18], [CTWX18].

Chapter 7

Singular schemes and varieties

This chapter collects and in some cases rectifies a number of results in the literature on the Brauer groups of singular schemes.

The Brauer group of a field is a torsion group, but this is not always so for schemes. Let X be an integral variety over a field k of characteristic zero and let $k(X)$ be the function field of X. If X is geometrically locally factorial, for example smooth, Theorem 3.5.4 says that the restriction map $\mathrm{Br}(X)\to\mathrm{Br}(k(X))$ is injective, in particular $\mathrm{Br}(X)$ is a torsion group. If, moreover, X is smooth over k, then, by Corollary 3.7.3, there is an exact sequence

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(k(X)) \longrightarrow \oplus_{x\in X^{(1)}}\mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z}).$$

Thus there is a purity theorem for $\mathrm{Br}(k(X))$: unramified classes in $\mathrm{Br}(k(X))$ lie in the subgroup $\mathrm{Br}(X)\subset\mathrm{Br}(k(X))$. It is natural to ask whether and to what extent the above results fail for a singular variety over k.

In Section 7.1 we give examples of non-reduced or reducible varieties X such that the Brauer group $\mathrm{Br}(X)$ is not a torsion group. Sections 7.2 and 7.3 treat schemes of dimension 1, and Section 7.4 integral normal schemes with isolated singular points. Here the reader will find an example of an integral normal surface X over $\mathbb{C}$ such that $\mathrm{Br}(X)$ contains an element of infinite order which lies in the kernel of $\mathrm{Br}(X)\to\mathrm{Br}(\mathbb{C}(X))$. Brauer groups of singular complete intersections and projective cones are subjects of Sections 7.5 and 7.6, respectively. The last section contains some more examples.

These examples leave the following question open: if X is an integral normal variety over a field k of positive characteristic, is $\mathrm{Br}(X)$ a torsion group?

For some singular varieties X, the exact computation of $\mathrm{Br}(X)$, for example by comparison with the Brauer group of a desingularisation, turns out to be of interest in connection with arithmetic investigations [HS14], [BrLo].

7.1 The Brauer–Grothendieck group is not always a torsion group

In this section we give elementary examples of schemes X for which $\mathrm{Br}(X)$ is not a torsion group. In some of these examples the scheme is non-reduced, and in some others it is reduced, but not irreducible.

A non-reduced scheme

Let Y be a variety over a field k. Let $X = Y \times_k k[\varepsilon]$ where $\varepsilon^2 = 0$. Let $i : Y = X_{\mathrm{red}} \to X$ be the closed immersion. Since the k-algebra homomorphism $k[\varepsilon] \to k$ has a section, there is a morphism $s : X \to Y$ such that $s \circ i = \mathrm{id}$. In particular, the map $i^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is surjective.

We have an exact sequence of sheaves for the étale topology on X

$$0 \longrightarrow i_*\mathcal{O}_Y \longrightarrow \mathbb{G}_{m,X} \longrightarrow i_*\mathbb{G}_{m,Y} \longrightarrow 0,$$

where the first map sends x to $1 + \epsilon x$. Since the functor i_* is exact for the étale topology [Mil80, Cor. II.3.6], we obtain a long exact sequence of abelian groups

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y) \longrightarrow \mathrm{H}^2(Y, \mathcal{O}_Y) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(Y) \longrightarrow \mathrm{H}^3(Y, \mathcal{O}_Y)$$

We thus obtain an exact sequence

$$0 \longrightarrow \mathrm{H}^2(Y, \mathcal{O}_Y) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(Y) \longrightarrow \mathrm{H}^3(Y, \mathcal{O}_Y).$$

If $\mathrm{H}^2(Y, \mathcal{O}_Y) \neq 0$, then the kernel of the reduction map $\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathrm{red}})$ is a non-zero finite dimensional vector space over k.

If $\mathrm{H}^2(Y, \mathcal{O}_Y) \neq 0$ and $\mathrm{char}(k) = 0$, then the kernel of the reduction map $\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathrm{red}})$ is a positive-dimensional vector space over a field of characteristic zero, in particular $\mathrm{Br}(X)$ is not a torsion group. From the above exact sequence we also deduce $\mathrm{Br}(X)_{\mathrm{tors}} \cong \mathrm{Br}(X_{\mathrm{red}})_{\mathrm{tors}}$. As we shall see later (Theorem 3.3.2), this translates as an isomorphism $\mathrm{Br}(X)_{\mathrm{Az}} \cong \mathrm{Br}(X_{\mathrm{red}})_{\mathrm{Az}}$.

In characteristic $p > 0$, the kernel and the cokernel of $\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathrm{red}})$ are p-torsion groups.

Remark 7.1.1 The study of the kernel of $\mathrm{Br}(Y \times_k A) \to \mathrm{Br}(Y)$, where A is a local artinian k-algebra, led Artin and Mazur to define the formal Brauer group of the k-variety Y, see [AM77, Ch. II, §4]. The group $\mathrm{H}^2(Y, \mathcal{O}_Y)$ is the tangent space to the formal Brauer group of Y (when the latter exists). This group is of importance in studying varieties over fields of positive characteristic. It is of particular interest in the case of K3 surfaces (e.g. smooth quartics in $\mathbb{P}^3_k$) over a finite field.

A reduced, reducible scheme

Here is another type of example of non-torsion elements in the Brauer group, which works over fields of arbitrary characteristic.

Lemma 7.1.2 *Let k be a field. Let U be a non-empty open subset of a smooth projective curve C of genus at least 1 over k. For any integer r there exists a field K finitely generated over k such that the dimension of the $\mathbb{Q}$-vector space $\mathrm{Pic}(U_K) \otimes_{\mathbb{Z}} \mathbb{Q}$ is at least r. There exists a field extension L/k such that $\mathrm{Pic}(U_L) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an infinite-dimensional $\mathbb{Q}$-vector space.*

Proof. One may assume $C(k) \neq \emptyset$. It is enough to prove that if A is an abelian variety over k, then $\dim_{\mathbb{Q}}(A(K) \otimes_{\mathbb{Z}} \mathbb{Q})$, where K is finitely generated over k, can be made arbitrarily large, while $\dim_{\mathbb{Q}}(A(L) \otimes_{\mathbb{Z}} \mathbb{Q})$ can be made infinite for even larger field extension L/k. Indeed, the generic point of A is a point of $A(k(A))$ no multiple of which belongs to $A(k)$. Now extend the ground field from k to $k(A)$ and iterate the process. □

Let k be a prime field. Let $S \subset \mathbb{P}^3_k$ be a smooth cubic surface. Up to replacing k by a finite extension, we can find a plane $H \subset \mathbb{P}^3_k$ which intersects S transversally along a smooth cubic E with a rational point. Write $Y = S \cup H \subset \mathbb{P}^3_k$ and $X = S \sqcup H$. Let $p : X \to Y$ be the natural morphism and let $i : E \hookrightarrow Y$ be the natural inclusion. Both these morphisms are finite, thus i_* and π_* are exact functors for the étale topology [Mil80, Cor. II.3.6]. Hence $R^j p_* = 0$ and $R^j i_* = 0$ for any $j > 0$. We have an exact sequence of sheaves for the étale topology on Y

$$1 \longrightarrow \mathbb{G}_{m,Y} \longrightarrow p_*\mathbb{G}_{m,X} \longrightarrow i_*\mathbb{G}_{m,E} \longrightarrow 1.$$

The associated long exact cohomology sequence gives an exact sequence

$$\mathrm{Pic}(S) \oplus \mathrm{Pic}(H) \longrightarrow \mathrm{Pic}(E) \longrightarrow \mathrm{Br}(Y)$$

Now $\mathrm{Pic}(S) \oplus \mathrm{Pic}(H)$ is a finitely generated free abelian group of rank at most 8. The group $\mathrm{Pic}(E)$ contains $E(k)$ as a subgroup. The same statements hold after replacing k by any field extension K. Using Lemma 7.1.2 one finds a field K finitely generated over its prime subfield k such that $\mathrm{Br}(Y_K)$ contains non-torsion elements and $\dim_{\mathbb{Q}}(\mathrm{Br}(Y_K) \otimes_{\mathbb{Z}} \mathbb{Q})$ is arbitrarily large. One can also find a field extension L/k such that $\dim_{\mathbb{Q}}(\mathrm{Br}(Y_L) \otimes_{\mathbb{Z}} \mathbb{Q}) = \infty$.

One may replace H and S by any two smooth surfaces in $\mathbb{P}^3$ transversally intersecting in a smooth curve of genus at least 1. The same argument also works for the Zariski topology, thus producing examples with non-torsion groups $\mathrm{H}^2_{\mathrm{zar}}(X_L, \mathbb{G}_m)$.

Replacing $S, H, E \subset \mathbb{P}^3_k$ by their respective intersections with any Zariski open set $W \subset \mathbb{P}^3_k$ such that $W \cap E \neq \emptyset$ produces examples where Y is affine and $\mathrm{Br}(Y_K)$ is non-torsion of rank as big as one wishes.

The above example implies the existence of an affine variety X over a finite field such that $\mathrm{Br}(X)$ is not a torsion group. Indeed, let us start with a field L of positive characteristic p and an affine variety Y over L with a non-torsion element $\beta \in \mathrm{Br}(Y)$. The field L is a filtered union of $\mathbb{F}_p$-algebras of finite type A_i, $i \in I$. There exists an $i \in I$ such that Y comes from an affine A_i-scheme of finite type Y_i, and β is the image of some $\beta_i \in \mathrm{Br}(Y_i)$. The element β_i in the Brauer group of the affine $\mathbb{F}_p$-variety Y_i is not torsion.

7.2 Schemes of dimension 1

The following proposition clarifies some points in [Gro68, II, §1].

Proposition 7.2.1 *Let X be a reduced, noetherian, 1-dimensional scheme. The Brauer group* $\mathrm{Br}(X)$ *is a torsion group. If $\alpha \in \mathrm{Br}(X)$ vanishes when evaluated at each generic point of X and also at each singular point of X, then $\alpha = 0$.*

Proof. Let us write $x = \mathrm{Spec}(k(x))$ for a closed point of X, and $y = \mathrm{Spec}(k(y))$ for any of the finitely many points of X of dimension 1. Let $i_x : x \to X$ and $i_y : y \to X$ be the natural morphisms. Then we have an exact sequence of étale sheaves

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow \prod_y i_{y*}\mathbb{G}_{m,k(y)} \longrightarrow \bigoplus_x i_{x*}F_x \longrightarrow 0, \tag{7.1}$$

where F_x is an étale sheaf on x which is the constant sheaf $\mathbb{Z}$, except possibly when x is one of the finitely many singular points of X. Using Hilbert's Theorem 90 for the fields $k(y)$, we deduce from (7.1) an exact sequence

$$0 \longrightarrow \bigoplus_x \mathrm{H}^1(k(x), F_x) \longrightarrow \mathrm{Br}(X) \longrightarrow \prod_y \mathrm{Br}(k(y)). \tag{7.2}$$

Note that $\mathrm{H}^1(x, F_x) = 0$ if x is a regular point, since $\mathrm{H}^1(k(x), \mathbb{Z}) = 0$. From this exact sequence we conclude that $\mathrm{Br}(X)$ is a torsion group.

Let $X_x = \mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{X,x})$ be the henselisation of X at a singular point x. Then we have a similar exact sequence

$$0 \longrightarrow \mathrm{H}^1(k(x), F_x) \longrightarrow \mathrm{Br}(X_x) \longrightarrow \prod_{y_x} \mathrm{Br}(k(y_x)),$$

where the product is over the generic points y_x of X_x. The two sequences are compatible via the maps induced by the natural morphism $X_x \to X$.

If $\alpha \in \mathrm{Br}(X)$ vanishes at each generic point of X, then α is the image of a well-defined element $\{\zeta_x\} \in \oplus_x \mathrm{H}^1(k(x), F_x)$, where the sum is over the singular points of X. By Theorem 3.4.2 the evaluation map $\mathrm{Br}(X_x) \to \mathrm{Br}(k(x))$ is an isomorphism. Thus if α also vanishes when evaluated at the closed point x, then the image of α in $\mathrm{Br}(X_x)$ is zero, hence $\zeta_x = 0$. This proves the proposition. □.

Remark 7.2.2 If the 1-dimensional scheme X is affine, one may give a proof of Proposition 7.2.1 in terms of Azumaya algebras, using conductors and patching diagrams [CTOP02, Prop. 1.12]. See also [Chi74] and [KO74a]. For X arbitrary, the result then follows from the fact that the set of singular points of X is contained in an affine open subset and from Theorem 3.5.5.

Lemma 7.2.3 *Let X be a noetherian separated scheme of dimension* 1*. Then X has an ample invertible sheaf.*

Proof. See [Stacks, Prop. 09NZ]. □

Proposition 7.2.4 *Let X be a noetherian separated scheme of dimension ≤ 1. The natural inclusion $\mathrm{Br}_{\mathrm{Az}}(X) \hookrightarrow \mathrm{Br}(X)$ is an equality.*

Proof. By Theorem 7.2.1, $\mathrm{Br}(X)$ is a torsion group. Now Lemma 7.2.3 and Gabber's theorem 3.3.2 give the result. □

Remark 7.2.5 As we saw in Section 7.1, there exists 2-dimensional schemes X such that $\mathrm{Br}(X)$ is not a torsion group.

Let X be as in Proposition 7.2.1, and let $\tilde{X}\to X$ be the normalisation of X. If one lets $\tilde{x}$ run through the closed points of $\tilde{X}$ above the singular points $x \in X$, one obtains an obvious complex

$$\mathrm{Br}(X) \longrightarrow \mathrm{Br}(\tilde{X}) \oplus \bigoplus_x \mathrm{Br}(k(x)) \longrightarrow \bigoplus_{\tilde{x}} \mathrm{Br}(k(\tilde{x})),$$

where x runs through the closed points of X. The proposition implies that the first map here is injective. One may wonder whether the complex is exact. This has been studied from the Azumaya point of view in [Chi74] and [KO74a]. In the case of a curve over a field k of characteristic zero, this will be established in Section 7.3. The proof there relies on a closer knowledge of the sheaves F_x.

7.3 Singular curves and their desingularisation

Let k be a field of characteristic 0 with an algebraic closure $\bar{k}$ and Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$. In this section we give a complement to Proposition 7.2.1.

Let C be a *reduced*, separated curve over k. We define the *normalisation* $\widetilde{C}$ as the disjoint union of normalisations of the irreducible components of C. The normalisation morphism $\nu : \widetilde{C}\to C$ factors as

$$\widetilde{C} \xrightarrow{\nu'} C' \xrightarrow{\nu''} C,$$

where C' is a maximal intermediate curve universally homeomorphic to C, see [BLR90, Section 9.2, p. 247] or [Liu10, Section 7.5, p. 308]. The curve C' is obtained from $\widetilde{C}$ by identifying the points which have the same image in C. In particular, there is a canonical bijection $\nu'' : C'(K)\tilde{\longrightarrow} C(K)$ for any field extension K/k. The curve C' has relatively mild singularities: for each singular point $s \in C'(\bar{k})$ the branches of $\overline{C}'$ through s intersect like n coordinate axes at $0 \in \mathbb{A}^n_{\bar{k}}$.

We define three reduced 0-dimensional schemes naturally arising in this situation. Let Λ be the k-scheme of geometric irreducible components of C (or the geometric connected components of $\widetilde{C}$); it is the disjoint union of finite integral k-schemes $\lambda = \mathrm{Spec}(k(\lambda))$ such that $k(\lambda)$ is the algebraic closure of k in the function field of the corresponding irreducible component $k(C_\lambda) = k(\widetilde{C}_\lambda)$. Let

$$\Pi = C_{\mathrm{sing}}, \quad \Psi = \big(\Pi \times_C \widetilde{C}\big)_{\mathrm{red}}. \tag{7.3}$$

Thus Ψ is the union of fibres of $\nu : \widetilde{C}\to C$ over the singular points of C with their reduced subscheme structure. The morphism ν'' induces an isomorphism $\big(\Pi\times_C C'\big)_{\rm red}\tilde{\longrightarrow}\Pi$, so we can identify these schemes. Let $i:\Pi\to C$, $i':\Pi\to C'$ and $j:\Psi\to\widetilde{C}$ be the natural closed immersions. We have a commutative diagram

$$\begin{array}{ccccc} \widetilde{C} & \xrightarrow{\nu'} & C' & \xrightarrow{\nu''} & C \\ {\scriptstyle j}\uparrow & & {\scriptstyle i'}\uparrow & \nearrow{\scriptstyle i} & \\ \Psi & \xrightarrow{\nu'} & \Pi & & \end{array}$$

The restriction of ν to the smooth locus of C induces isomorphisms

$$\widetilde{C}\setminus j(\Psi)\tilde{\longrightarrow}C'\setminus i'(\Pi)\tilde{\longrightarrow}C\setminus i(\Pi).$$

An algebraic group over Π is a product $G=\prod_\pi i_{\pi *}(G_\pi)$, where π ranges over the irreducible components of Π, $i_\pi:\mathrm{Spec}(k(\pi))\to\Pi$ is the natural closed immersion, and G_π is an algebraic group over the field $k(\pi)$.

Lemma 7.3.1 (i) *The canonical maps* $\mathbb{G}_{m,C'}\to\nu'_*\mathbb{G}_{m,\widetilde{C}}$ *and* $\mathbb{G}_{m,C'}\to i'_*\mathbb{G}_{m,\Pi}$ *give rise to the exact sequence of étale sheaves on* C'

$$0\longrightarrow\mathbb{G}_{m,C'}\longrightarrow\nu'_*\mathbb{G}_{m,\widetilde{C}}\oplus i'_*\mathbb{G}_{m,\Pi}\longrightarrow i'_*\nu'_*\mathbb{G}_{m,\Psi}\longrightarrow 0, \tag{7.4}$$

where $\nu'_*\mathbb{G}_{m,\Psi}$ *is an algebraic torus over* Π.

(ii) *The canonical map* $\mathbb{G}_{m,C}\to\nu''_*\mathbb{G}_{m,C'}$ *gives rise to the exact sequence of étale sheaves on* C*:*

$$0\longrightarrow\mathbb{G}_{m,C}\longrightarrow\nu''_*\mathbb{G}_{m,C'}\longrightarrow i_*\mathbb{U}\longrightarrow 0, \tag{7.5}$$

where $\mathbb{U}$ *is a commutative unipotent group over* Π.

Proof. See [BLR90], the proofs of Propositions 9.2.9 and 9.2.10, or [Liu10, Lemma 7.5.12]. By [Mil80, Thm. II.2.15 (b), (c)] it is enough to check the exactness of (7.4) at each geometric point $\bar{x}$ of C'. If $\bar{x}\notin i'(\Pi)$, this is obvious since locally at $\bar{x}$ the morphism ν' is an isomorphism, and the stalks $(i'_*\mathbb{G}_{m,\Pi})_{\bar{x}}$ and $(i'_*\nu'_*\mathbb{G}_{m,\Psi})_{\bar{x}}$ are zero. Now let $\bar{x}\in i'(\Pi)$, and let $\mathcal{O}^{\rm sh}_{C',\bar{x}}$ be the strict henselisation of the local ring of $\bar{x}$ in C'. Each geometric point $\bar{y}$ of $\widetilde{C}$ belongs to exactly one geometric connected component of $\widetilde{C}$. Let $\mathcal{O}^{\rm sh}_{\widetilde{C},\bar{y}}$ be the strict henselisation of the local ring of $\bar{y}$ in its geometric connected component. By the construction of C' we have an exact sequence

$$0\longrightarrow\mathcal{O}^{\rm sh}_{C',\bar{x}}\longrightarrow k(\bar{x})\times\prod_{\nu'(\bar{y})=\bar{x}}\mathcal{O}^{\rm sh}_{\widetilde{C},\bar{y}}\longrightarrow\prod_{\nu'(\bar{y})=\bar{x}}k(\bar{y})\longrightarrow 0,$$

where $\mathcal{O}^{\rm sh}_{\widetilde{C},\bar{y}}\to k(\bar{y})$ is the reduction modulo the maximal ideal of $\mathcal{O}^{\rm sh}_{\widetilde{C},\bar{y}}$, and $k(\bar{x})\to k(\bar{y})$ is the multiplication by -1. We obtain an exact sequence of abelian groups

$$1\longrightarrow(\mathcal{O}^{\rm sh}_{C',\bar{x}})^*\longrightarrow k(\bar{x})^*\times\prod_{\nu'(\bar{y})=\bar{x}}(\mathcal{O}^{\rm sh}_{\widetilde{C},\bar{y}})^*\longrightarrow\prod_{\nu'(\bar{y})=\bar{x}}k(\bar{y})^*\longrightarrow 1.$$

Using [Mil80, Cor. II.3.5 (a), (c)] one sees that this is the sequence of stalks of (7.4) at $\bar{x}$, so that (i) is proved.

To prove (ii) consider the exact sequence

$$0 \longrightarrow \mathbb{G}_{m,C} \longrightarrow \nu''_* \mathbb{G}_{m,C'} \longrightarrow \nu''_* \mathbb{G}_{m,C'}/\mathbb{G}_{m,C} \longrightarrow 0.$$

The morphism ν'' is an isomorphism away from $i(\Pi)$, so the restriction of the sheaf $\nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C}$ to $C \smallsetminus i(\Pi)$ is zero. Hence $\nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C} = i_*\mathbb{U}$ for some sheaf $\mathbb{U}$ on Π. To see that $\mathbb{U}$ is a unipotent group scheme it is enough to check the stalks at geometric points. Let $\bar{x}$ be a geometric point of $i(\Pi)$, and let $\bar{y}$ be the unique geometric point of C' such that $\nu''(\bar{y}) = \bar{x}$. Let $\mathcal{O}^{\rm sh}_{C,\bar{x}}$ and $\mathcal{O}^{\rm sh}_{C',\bar{y}}$ be the corresponding strictly henselian local rings. The stalk $(\nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C})_{\bar{x}}$ is $(\mathcal{O}^{\rm sh}_{C',\bar{y}})^*/(\mathcal{O}^{\rm sh}_{C,\bar{x}})^*$, and by [Liu10, Lemma 7.5.12 (c)], this is a unipotent group over the field $k(\bar{x})$. □

For fields $k_1, \ldots, k_n$, we have $\mathrm{Br}\big(\coprod_{i=1}^n \mathrm{Spec}(k_i)\big) = \oplus_{i=1}^n \mathrm{Br}(k_i)$.

Proposition 7.3.2 *Let k be a field of characteristic 0. Let C be a reduced curve over k, and let Λ, Π and Ψ be the schemes defined in (7.3). Let $\Lambda = \coprod_\lambda \mathrm{Spec}(k(\lambda))$ be the decomposition into the disjoint union of connected components, so that $\widetilde{C} = \coprod_\lambda \widetilde{C}_\lambda$, where $\widetilde{C}_\lambda$ is a smooth geometrically integral curve over the field $k(\lambda)$. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(C) \longrightarrow \mathrm{Br}(\Pi) \oplus \bigoplus_{\lambda\in\Lambda} \mathrm{Br}(\widetilde{C}_\lambda) \longrightarrow \mathrm{Br}(\Psi), \tag{7.6}$$

where the maps are the composition of canonical maps

$$\mathrm{Br}(\widetilde{C}_\lambda) \longrightarrow \mathrm{Br}(\widetilde{C}_\lambda \cap \Psi) \longrightarrow \mathrm{Br}(\Psi),$$

and the opposite of the restriction map $\mathrm{Br}(\Pi) \to \mathrm{Br}(\Psi)$.

Proof. Let π range over the irreducible components of Π, so that $\mathbb{U} = \prod_\pi i_{\pi*}(U_\pi)$, where U_π is a commutative unipotent group over the field $k(\pi)$. Since i_* is an exact functor [Mil80, Cor. II.3.6], we have $\mathrm{H}^n(C, i_*\mathbb{U}) = \mathrm{H}^n(\Pi, \mathbb{U}) = \prod_\pi \mathrm{H}^n(k(\pi), U_\pi)$. The field k has characteristic 0, and it is well known that this implies that any commutative unipotent group has zero cohomology in degree $n > 0$. (Such a group has a composition series with factors $\mathbb{G}_a$, and $\mathrm{H}^n(k, \mathbb{G}_a) = 0$ for any $n > 0$, see [SerCL, Ch. X, Prop. 1].) Thus the long exact sequence of cohomology groups associated to (7.5) gives rise to an isomorphism $\mathrm{Br}(C) = \mathrm{H}^2(C, \mathbb{G}_{m,C}) \xrightarrow{\sim} \mathrm{H}^2(C, \nu''_*\mathbb{G}_{m,C'})$. Since ν'' is finite, the functor ν''_* is exact [Mil80, Cor. II.3.6], so we obtain an isomorphism $\mathrm{Br}(C) \xrightarrow{\sim} \mathrm{Br}(C')$. We now apply similar arguments to (7.4). Hilbert's theorem 90 gives $\mathrm{H}^1(\Pi, \nu'_*\mathbb{G}_{m,\Psi}) = \mathrm{H}^1(\Psi, \mathbb{G}_{m,\Psi}) = 0$, so we obtain the exact sequence (7.6). □

7.4 Isolated singularities

This section elaborates on [Gro68, Ch. II, §1, Rem. 11 (b)] and on further literature [D68, D72, Oja74], [Chi76, Thm. 1.1], [DF92, Ber05, Kol16].

Let X be a normal integral noetherian scheme with function field K. Assume that the singular locus $X_{\rm sing}$ is the union of finitely many closed points $P_1, \ldots, P_n$. Let k_i denote the residue field at P_i, let $k_{i,\rm s}$ be a separable closure of k_i and let $G_i = {\rm Gal}(k_{i,\rm s}/k_i)$, for $i = 1, \ldots, n$. We write R_i for the local ring $\mathcal{O}_{X,P_i}$ and $R_i^{\rm sh}$ for the strict henselisation of R_i. Let ${\rm Cl}(X)$ be the *class group* of X, defined as the cokernel of the divisor map

$$\mathrm{div} : K^* \longrightarrow \bigoplus_{x \in X^{(1)}} \mathbb{Z}.$$

We define the étale sheaf $\mathcal{D}iv_X$ by the condition that the following sequence is exact:

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow j_*\mathbb{G}_{m,K} \longrightarrow \mathcal{D}iv_X \longrightarrow 0. \tag{7.7}$$

Taking étale cohomology of (7.7) and using Lemma 2.4.1, we get an isomorphism

$$\mathrm{H}^1_{\text{ét}}(X, \mathcal{D}iv_X) \tilde{\longrightarrow} \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(K)]. \tag{7.8}$$

Sending a Cartier divisor to the associated Weil divisor defines a natural injective map $\mathcal{D}iv_X \to \oplus_{x \in X^{(1)}} i_{x*}\mathbb{Z}_{k(x)}$. This is an isomorphism when X is regular. Let us define $\mathcal{P}_X$ as the cokernel of this map. This gives an exact sequence

$$0 \longrightarrow \mathcal{D}iv_X \longrightarrow \bigoplus_{x \in X^{(1)}} i_{x*}\mathbb{Z}_{k(x)} \longrightarrow \mathcal{P}_X \longrightarrow 0. \tag{7.9}$$

It is clear that $\mathcal{P}_X$ is supported on $X_{\rm sing}$, hence $\mathcal{P}_X = \oplus_{i=1}^n \mathcal{P}_X(R_i^{\rm sh})$. Looking at the stalks of the terms of (7.7) and (7.9) at the points P_i we see that $\mathcal{P}_X(R_i^{\rm sh}) = {\rm Cl}(R_i^{\rm sh})$ for each i. Thus

$$\mathcal{P}_X = \bigoplus_{i=1}^n i_{P_i*}({\rm Cl}(R_i^{\rm sh})).$$

Taking étale cohomology of (7.9) and using Lemma 2.4.1 together with (7.8), we then get an exact sequence

$$0 \to \mathrm{H}^0(X, \mathcal{D}iv_X) \to \bigoplus_{x \in X^{(1)}} \mathbb{Z} \to \bigoplus_{i=1}^n {\rm Cl}(R_i^{\rm sh})^{G_i} \to \mathrm{Br}(X) \to \mathrm{Br}(K).$$

Using the definition of ${\rm Cl}(X)$, we deduce the exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X) \longrightarrow \bigoplus_{i=1}^n \mathrm{Cl}(R_i^{\rm sh})^{G_i} \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(K).$$

If X is the spectrum of a semi-local ring, one obtains the exact sequence

$$0 \longrightarrow \mathrm{Cl}(X) \longrightarrow \bigoplus_{i=1}^{n} \mathrm{Cl}(R_i^{\mathrm{sh}})^{G_i} \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(K).$$

If $X = \mathrm{Spec}(R)$ is the spectrum of a local ring R with field of fractions K. In this case the exact sequence takes the form

$$0 \longrightarrow \mathrm{Cl}(R) \longrightarrow \mathrm{Cl}(R^{\mathrm{sh}})^{G} \longrightarrow \mathrm{Br}(R) \longrightarrow \mathrm{Br}(K).$$

Let $U = X \smallsetminus \{x\}$ and $U^{\mathrm{sh}} = \mathrm{Spec}(R^{\mathrm{sh}}) \smallsetminus \{\overline{x}\}$. Since U is regular, we have $\mathrm{Pic}(U) = \mathrm{Cl}(R)$ and $\mathrm{Pic}(U^{\mathrm{sh}}) = \mathrm{Cl}(R^{\mathrm{sh}})$. The last displayed sequence then becomes the formula (7) in [Gro68, Ch. II, §1].

Remark 7.4.1 [Ber05] Grothendieck claims that for any normal scheme X with isolated singular points $\{P_i\}$ the above computations give the general formula [Gro68, §1, (7)]:

$$\mathrm{H}^1_{\text{ét}}(X, \mathcal{D}iv_X) \cong \bigoplus_{i=1}^{n} [\mathrm{Pic}(\mathrm{Spec}(R_i^{\mathrm{sh}}) \smallsetminus \overline{P_i})^{G_i} / \mathrm{Im}\big(\mathrm{Pic}(\mathrm{Spec}(R_i) \smallsetminus P_i)\big)]. \quad (7.10)$$

This is not correct. Given the above computations, this would imply

$$\bigoplus_{i=1}^{n} [\mathrm{Cl}(R_i^{\mathrm{sh}})^{G_i} / \mathrm{Im}(\mathrm{Cl}(R_i))] \xrightarrow{\sim} \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(K)].$$

There is a natural surjective map

$$[\bigoplus_{i=1}^{n} \mathrm{Cl}(R_i^{\mathrm{sh}})^{G_i}] / \mathrm{Im}(\mathrm{Cl}(X)) \longrightarrow \bigoplus_{i=1}^{n} [\mathrm{Cl}(R_i^{\mathrm{sh}})^{G_i} / \mathrm{Im}(\mathrm{Cl}(R_i))].$$

Formula (7.10) holds if and only if the map $\mathrm{Cl}(X) \to \oplus_{i=1}^{n} \mathrm{Cl}(R_i)$ is surjective. Ojanguren's Example (4) in Section 7.7 is precisely built on an example where this map is not surjective [Oja74, §2, p. 511].

Example 7.4.2 Let R be the local ring of the vertex of the cone over a smooth projective plane curve $X \subset \mathbb{P}^2_{\mathbb{C}}$ of degree d. This is a 2-dimensional local normal domain. As explained in Childs [Chi76, Thm. 6.1], work of Danilov [D68, D72] gives that $\mathrm{Cl}(R^{\mathrm{h}})/\mathrm{Cl}(R) = \mathrm{Cl}(\hat{R})/\mathrm{Cl}(R)$. Moreover, this quotient is the finite dimensional complex vector space $\oplus_{i \geq 1} \mathrm{H}^1(X, \mathcal{O}_X(i))$, which has positive dimension if $d \geq 4$. Hence for these values of d the kernel of the map $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is a non-zero vector space over $\mathbb{C}$. In particular, there are non-torsion elements in this kernel. Note that this implies that the kernel of $\mathrm{Br}_{\mathrm{Az}}(R) \to \mathrm{Br}(K)$ is zero, because $\mathrm{Br}_{\mathrm{Az}}(R)$ is always a torsion group.

Let $U_i = \mathrm{Spec}(R_i)$ be the affine Zariski open neighbourhoods of the vertex of the cone. We have $R = \varinjlim R_i$, in fact, R is the union of the rings R_i. By Section 2.2.4 we have $\mathrm{Br}(R) = \varinjlim \mathrm{Br}(U_i)$. Let $\alpha \in \mathrm{Br}(R)$ be a non-torsion element in the kernel of the map $\mathrm{Br}(R) \to \mathrm{Br}(K)$. There exist an i and $\alpha_i \in \mathrm{Br}(R_i)$ such that the image of α_i in R is α. Thus α_i is a non-torsion element in the kernel of $\mathrm{Br}(R_i) \to \mathrm{Br}(K)$. Now $Y = \mathrm{Spec}(R_i)$ is an affine normal integral surface such that $\mathrm{Br}(Y)$ contains an element of infinite order which goes to zero in $\mathrm{Br}(K)$.

7.5 Intersections of hypersurfaces

Proposition 7.5.1 *Let k be an algebraically closed field of characteristic $p \geq 0$. Let $X \subset \mathbb{P}^N_k$ be a closed subscheme.*

(i) *If X is defined by the vanishing of at most $N-3$ homogeneous forms, then $\mathrm{Br}(X)$ has no prime-to-p torsion. This holds for any hypersurface $X \subset \mathbb{P}^N_k$ where $N \geq 4$.*

(ii) *If X is defined by the vanishing of at most $N-4$ homogeneous forms, then $\mathrm{Br}(X)$ is uniquely ℓ-divisible for any prime $\ell \neq p$. This holds for any hypersurface $X \subset \mathbb{P}^N_k$ where $N \geq 5$.*

Proof. Let $\ell \neq p$ be a prime. The more general result [Kat04, Cor. B.6] gives that the restriction map $\mathrm{H}^2(\mathbb{P}^N_k, \mathbb{Z}/\ell) \to \mathrm{H}^2(X, \mathbb{Z}/\ell)$ is an isomorphism under hypothesis (i) and that $\mathrm{H}^3(\mathbb{P}^N_k, \mathbb{Z}/\ell) \to \mathrm{H}^3(X, \mathbb{Z}/\ell)$ is an isomorphism under hypothesis (iii). The Kummer sequence then gives that $\mathrm{Br}(\mathbb{P}^N_k)[\ell] \to \mathrm{Br}(X)[\ell]$ is surjective. Since $\mathrm{Br}(\mathbb{P}^N_k) = 0$, this concludes the proof in case (i). In case (i) from $\mathrm{H}^3(\mathbb{P}^N_k, \mathbb{Z}/\ell) = 0$ we deduce $\mathrm{H}^3(X, \mathbb{Z}/\ell) = 0$, and the Kummer sequence gives $\mathrm{Br}(X)/\ell \hookrightarrow \mathrm{H}^3(X, \mathbb{Z}/\ell) = 0$. □

Purity on some singular varieties

Corollary 4.4.5 can be extended to some singular complete intersections. K. Česnavičius showed us that the following theorem is essentially a special case of results of Michèle Raynaud [MR62], a text which contains many more purity theorems in a possibly singular context. Recent work of Česnavičius and Scholze vastly extend these results.

Theorem 7.5.2 *Let k be a separably closed field of characteristic zero. Let $X \subset \mathbb{P}^N_k$ be a complete intersection of dimension $d \geq 3$. Assume that the codimension in X of the singular locus X_{sing} is at least 4. Let $U = X \setminus X_{\mathrm{sing}}$. Then $\mathrm{Br}(U) = 0$, hence $\mathrm{Br}_{\mathrm{nr}}(k(X)) = 0$.*

Proof. The assumption on X and on the codimension of the singular locus implies [SGA2, XI, Cor. 3.14] that X is geometrically locally factorial. Theorem 3.5.4 then gives that the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is injective. The restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ is surjective since X is locally factorial and is injective since the codimension of X_{sing} in X is at least 2, so it is an isomorphism. Quite generally, for any complete intersection $X \subset \mathbb{P}^N_k$ of dimension d and any $i < d$, the restriction map $\mathrm{H}^i(\mathbb{P}^N_k, \mu_n) \to \mathrm{H}^i(X, \mu_n)$ is an isomorphism, see [Kat04]. In particular, $\mathbb{Z}/n = \mathrm{H}^2(\mathbb{P}^N_k, \mu_n) = \mathrm{H}^2(X, \mu_n)$. Now, from the Kummer sequence, we obtain a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
 & & \mathrm{Pic}(\mathbb{P}^N_k)/n & \xrightarrow{\cong} & \mathrm{H}^2(\mathbb{P}^N_k,\mu_n) & & & & \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & & & \\
0 & \longrightarrow & \mathrm{Pic}(X)/n & \xrightarrow{\cong} & \mathrm{H}^2(X,\mu_n) & \longrightarrow & \mathrm{Br}(X)[n] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}(U)/n & \longrightarrow & \mathrm{H}^2(U,\mu_n) & \longrightarrow & \mathrm{Br}(U)[n] & \longrightarrow & 0
\end{array}
$$

To complete the proof it is enough to show that the restriction map

$$\mathrm{H}^2(X,\mu_n) \longrightarrow \mathrm{H}^2(U,\mu_n)$$

is an isomorphism.

Let us describe the relevant results from [MR62]. Let X be a noetherian scheme, let $Y \subset X$ be a closed subscheme, and let $U = X \smallsetminus Y$. The *étale depth* $\mathrm{depth}_Y(X)(\mathbb{Z}/\ell)$ of X along Y is defined in [MR62, Déf. 1.2], which refers to [MR62, Prop. 1.1 (iii)]. If $n = \mathrm{depth}_Y(X)(\mathbb{Z}/\ell)$, then for any X' étale over X, the restriction map $\mathrm{H}^i(X',\mathbb{Z}/\ell)\to\mathrm{H}^i(X'\times_X U,\mathbb{Z}/\ell)$ is an isomorphism for $i < n-1$ and an injection for $i = n-1$.

One defines a similar notion locally at any point x of X, as follows. Let $\overline{X}_{\bar{x}} = \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{\bar{x}})$ be the strict henselisation of X at a geometric point $\bar{x}$ above x. Define $\mathrm{depth}_x(X)(\mathbb{Z}/\ell) = \mathrm{depth}_{\bar{x}}(\overline{X}_{\bar{x}})(\mathbb{Z}/\ell)$, that is, the étale depth of the local scheme $\overline{X}_{\bar{x}}$ at its closed point $\bar{x}$. By [MR62, Thm. 1.8], $\mathrm{depth}_Y(X)(\mathbb{Z}/\ell)$ can be computed locally:

$$\mathrm{depth}_Y(X)(\mathbb{Z}/\ell) = \inf_{y\in Y}\ \mathrm{depth}_y(X)(\mathbb{Z}/\ell),$$

where y ranges through the points of the scheme Y.

The *geometric depth* of an excellent local ring A is defined in [MR62, Déf. 5.3]. If A is a complete intersection, then the geometric depth of A coincides with the dimension of A [MR62, Prop. 5.4]. For an excellent local ring A of characteristic zero, the étale depth is greater than or equal to the geometric depth [MR62, Thm. 5.6].

We now resume the proof of the theorem. So let X be as in the statement of the theorem, let $Y = X_{\mathrm{sing}}$, and let $U = X \smallsetminus Y$. Since X is a complete intersection, so is $\overline{X}_{\bar{y}}$, where y is a point of Y and $\bar{y}$ is a geometric point over y. Since $\mathrm{codim}_X(Y) \geq 4$, we have $\dim(\overline{X}_{\bar{y}}) \geq 4$. We conclude that the étale depth at the local ring of X at y is at least 4. Thus $\mathrm{depth}_Y(X)(\mathbb{Z}/\ell) \geq 4$ for any prime number ℓ, hence the restriction map $\mathrm{H}^2(X,\mathbb{Z}/\ell)\to\mathrm{H}^2(U,\mathbb{Z}/\ell)$ is an isomorphism. Thus $\mathrm{H}^2(X,\mu_n) \cong \mathrm{H}^2(U,\mu_n)$ for any $n > 0$. □

Corollary 7.5.3 *Let k be a field of characteristic zero. Let $X \subset \mathbb{P}^N_k$ be a complete intersection of dimension at least* 3. *Assume that the singular locus of X is of codimension at least* 4 *in X. Then the natural map $\mathrm{Br}(k)\to\mathrm{Br}_{\mathrm{nr}}(k(X)/k)$ is an isomorphism.*

Proof. Let $\bar{k}$ be an algebraic closure of k. Let $U = X \setminus X_{\text{sing}}$. We have $\bar{k}^* = \bar{k}[U]^* = \bar{k}[X]^*$. The assumptions on X and on $\operatorname{codim}_X(X_{\text{sing}})$ imply [SGA2, XI, Cor. 3.14] that X is geometrically locally factorial. It follows that the restriction map $\operatorname{Pic}(\overline{X}) \to \operatorname{Pic}(\overline{U})$ is an isomorphism. By a generalisation of the Lefschetz theorem, for a complete intersection X of dimension at least 3 in $\mathbb{P}^N_{\bar{k}}$, the restriction map $\mathbb{Z} = \operatorname{Pic}(\mathbb{P}^N_{\bar{k}}) \to \operatorname{Pic}(\overline{X})$ is an isomorphism [SGA2, XII, Cor. 3.7], and both groups are generated by the hyperplane section class, which is defined over k. By the above theorem $\operatorname{Br}(\overline{X}) = \operatorname{Br}(\overline{U}) = 0$. From the exact sequence (4.9) we then get isomorphisms

$$\operatorname{Br}(k) = \operatorname{Br}(X) = \operatorname{Br}(U).$$

Let $f : Z \to X$ be a proper desingularisation of X which induces an isomorphism $V = f^{-1}(U) \cong U$. The composition $\operatorname{Br}(k) \to \operatorname{Br}(Z) \to \operatorname{Br}(U)$ is an isomorphism, and $\operatorname{Br}(Z) \to \operatorname{Br}(U)$ is injective since Z is smooth. Thus $\operatorname{Br}(k) \to \operatorname{Br}(Z)$ is an isomorphism, hence $\operatorname{Br}(k) \to \operatorname{Br}_{\text{nr}}(k(X)/k)$ is an isomorphism. □

7.6 Projective cones

Proposition 7.6.1 *Let k be a field of characteristic zero. Let $Y \subset \mathbb{P}^n_k$, $n \geq 2$, be an integral closed subvariety. Let $X \subset \mathbb{P}^{n+1}_k$ be the projective cone over Y. Write $U = X_{\text{smooth}}$.*

(i) *The restriction map $\operatorname{Br}(X) \to \operatorname{Br}(U)$ is the composition of the by evaluation at P map $\operatorname{Br}(X) \to \operatorname{Br}(k)$ and the map $\operatorname{Br}(k) \to \operatorname{Br}(U)$ induced by the structure morphism $U \to \operatorname{Spec}(k)$.*

(ii) *If Y is smooth, then U is the complement to the vertex of the cone X and $\operatorname{Br}(U) \cong \operatorname{Br}_{\text{nr}}(k(X)/k)$.*

Proof. Let $\alpha \in \operatorname{Br}(X)$. Let $K = k(X)$ be the function field of X. The K-variety $X_K = X \times_k K$ has two obvious K-points: the point P_K given by the vertex $P \in X(k)$ and the point given by the generic point $\eta \in X$. Any point $M \in X_K(K)$ distinct from P_K lies on the projective line $\mathbb{P}^1_K \subset X_K$ through M and P_K. Since $\operatorname{Br}(K) \to \operatorname{Br}(\mathbb{P}^1_K)$ is an isomorphism (Theorem 4.5.1 (vii)) we have

$$\alpha(\eta) = \alpha(P_K) = \operatorname{res}_{K/k}(\alpha(P)) \in \operatorname{Br}(K).$$

But $\alpha(\eta)$ is just the image of α under the restriction map $\operatorname{Br}(X) \to \operatorname{Br}(k(X))$. The latter map is the composition $\operatorname{Br}(X) \to \operatorname{Br}(U) \to \operatorname{Br}(k(X))$, where the map $\operatorname{Br}(U) \to \operatorname{Br}(k(X))$ is injective since U is smooth over k (Theorem 3.5.4). Hence $\operatorname{Br}(X) \to \operatorname{Br}(U)$ factors as

$$\operatorname{Br}(X) \longrightarrow \operatorname{Br}(k) \longrightarrow \operatorname{Br}(U),$$

where the first arrow is evaluation at P and the second arrow is induced by the structure map $U \to \operatorname{Spec}(k)$.

Assume that Y is smooth. Then $U = X \setminus \{P\}$ is a smooth integral variety. The projection map $p : U \to Y$ makes U an $\mathbb{A}^1$-bundle over Y, thus the induced map $p^* : \mathrm{Br}(Y) \to \mathrm{Br}(U)$ is an isomorphism. Note that X is birationally equivalent to $Y \times_k \mathbb{P}^1_k$, hence $\mathrm{Br}_{\mathrm{nr}}(k(X)/k) \cong \mathrm{Br}_{\mathrm{nr}}(k(Y)/k)$. Since Y is smooth, $\mathrm{Br}(Y) = \mathrm{Br}_{nr}(k(Y)/k) \subset \mathrm{Br}(k(Y))$ (Proposition 5.2.2). □

Let us discuss the case where $Y \subset \mathbb{P}^{N-1}_{\mathbb{C}}$, $N \geq 3$, is a smooth projective hypersurface. Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ be the projective cone over Y. The vertex is the only singularity of X; it has codimension $N-1$ in X. Let $U \subset X$ be the complement to the vertex of X. By Proposition 7.6.1, the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is zero. On the other hand, Proposition 7.5.1 says that $\mathrm{Br}(X)$ is torsion-free for $N \geq 4$ and $\mathrm{Br}(X)$ is uniquely divisible for $N \geq 5$.

For $N \geq 5$, we actually have $\mathrm{Br}(X) = \mathrm{Br}(U) = 0$. Indeed, X is geometrically locally factorial, hence $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is injective. As U is an $\mathbb{A}^1$-bundle over a smooth hypersurface $Y \subset \mathbb{P}^{N-1}$ with $N-1 \geq 4$, we have $\mathrm{Br}(Y) = 0$ and $\mathrm{Br}(U) = 0$.

It remains to investigate the case where Y is a smooth curve in $\mathbb{P}^2_{\mathbb{C}}$ or a smooth surface in $\mathbb{P}^3_{\mathbb{C}}$. In the first case $\mathrm{Br}(U) = \mathrm{Br}(Y) = 0$. In the second case we know that $\mathrm{Br}(X)$ is torsion-free. We also know that if the surface is of degree at least 4 then $\mathrm{Br}(U) \cong \mathrm{Br}(Y) \neq 0$.

Example 7.6.2 Let us show that the condition on the codimension of the singular locus in Theorem 7.5.2 and Corollary 7.5.3 is necessary. Let $k = \mathbb{C}$ and let $Y \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth surface of degree $d \geq 4$. Then $\mathrm{NS}(Y)$ is torsion-free and we have $b_2 > \rho$ since $\mathrm{H}^2(Y, \mathcal{O}_Y) \neq 0$, by Hodge theory. Proposition 4.2.6 implies that $\mathrm{Br}(Y) = (\mathbb{Q}/\mathbb{Z})^{b_2-\rho} \neq 0$. Thus in the above notation we have $\mathrm{Br}(U) = \mathrm{Br}(Y) \neq 0$, while the map $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is zero. This gives an example of a hypersurface of dimension 3 with an isolated singularity of codimension 3 for which the map $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is not an isomorphism. Note that $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(X)) = \mathrm{Br}_{\mathrm{nr}}(Y) = \mathrm{Br}_{\mathrm{nr}}(U) \neq 0$ in this case.

7.7 Some examples

(1) Let k be a field of characteristic different from 2 with $a, b \in k^*$ such that the quaternion algebra class $(a, b) \in \mathrm{Br}(k)$ is non-zero. (For example, $k = \mathbb{R}$ and $a = b = -1$.) Consider the singular affine curve over k defined by the equation

$$y^2 = x^2(x+b).$$

Let X be the open set given by $x \neq -b$. Consider the quaternion algebra

$$A = (a, x+b) \in \mathrm{Br}_{\mathrm{Az}}(X).$$

Over the function field $k(X)$ of X, we have

$$(a, x+b) = (a, (y/x)^2) = 0 \in \mathrm{Br}(k(X)).$$

But the evaluation of A at the singular point $(x, y) = (0, 0)$ is the non-zero element $(a, b) \in \mathrm{Br}(k)$, thus $A \neq 0$ lies in the kernel of $\mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}(k(X))$. Compare with Proposition 7.2.1.

(2) Let k and $a, b \in k^*$ be the same as in (1). Consider the normal affine surface over k defined by the equation

$$y^2 - az^2 = x^2(x+b).$$

Let X be the open set given by $x \neq -b$. Consider the quaternion algebra

$$A = (a, x+b) \in \mathrm{Br}_{\mathrm{Az}}(X).$$

Over the function field $k(X)$ of X, we have

$$(a, x+b) = (a, (y^2 - az^2)/x^2) = 0 \in \mathrm{Br}(k(X)).$$

The evaluation of A at the singular point $(x, y, z) = (0, 0, 0)$ is the non-zero element $(a, b) \in \mathrm{Br}(k)$. Thus $A \neq 0$ lies in the kernel of $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$.

(3) Let k and $a, b \in k^*$ be the same as in (1). Consider the quadratic cone $X \subset \mathbb{A}^4_{\mathbb{R}}$ defined by

$$x^2 - ay^2 - bz^2 + abt^2 = 0.$$

Its singular locus is the point $P = (0, 0, 0, 0)$, which has codimension 3 in X. The class $(a, b) \in \mathrm{Br}(k)$ gives rise to $\alpha = (a, b)_X \in \mathrm{Br}_{\mathrm{Az}}(X)$. This class is non-zero, because its evaluation at P is $(a, b) \in \mathrm{Br}(k)$. But the image of α in $\mathrm{Br}(k(X))$ is zero, since

$$(a, b)_{k(X)} = (a, (x^2 - ay^2)/(z^2 - at^2)) = 0 \in \mathrm{Br}(k(X)).$$

This example shows that in Theorem 3.5.4 of Auslander and Buchsbaum one cannot remove the assumption that the codimension of the singular locus is at least 4.

(4) If X is a noetherian integral scheme with an isolated singularity $P \in X$, and R_P is the local ring of X at P, then the restriction map

$$\mathrm{Br}(X) \longrightarrow \mathrm{Br}(R_P)$$

is injective. Indeed one may write $X = U \cup V$ where U is regular and V contains P. By Theorem 3.5.5 this implies that the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(V)$ is injective. Passing over to the limit over all V containing P gives the result.

The affine surface X over $\mathbb{C}$ given by $z^3 = (1 - x - y)xy$ is normal with exactly three singular points P_i, $i = 1, 2, 3$. Let R_i be the local ring of X at P_i. Ojanguren shows in [Oja74] that the natural map

$$\mathrm{Br}_{\mathrm{Az}}(X) \longrightarrow \prod_{i=1}^{3} \mathrm{Br}_{\mathrm{Az}}(R_i)$$

has a non-trivial kernel.

Chapter 8

Varieties with a group action

One often needs to study the Brauer group of a variety equipped with an action of an algebraic group. The Brauer groups of connected algebraic groups themselves as well as the Brauer groups of their homogeneous spaces can be explicitly computed in many cases. In Section 8.1 we deal with tori and in Section 8.2 with simply connected semisimple groups. We then turn our attention to the unramified Brauer group of homogeneous spaces; the challenge here is to compute these groups without having to construct an explicit smooth projective model. In Section 8.3 we discuss Bogomolov's theorems which compute the unramified Brauer group of the invariant field of a linear action of a finite group over an algebraically closed field, and a related theorem of Saltman. Finally, in Section 8.4 we give an overview of the unramified Brauer groups of homogeneous spaces over an arbitrary field (mostly without proofs).

8.1 Tori

The étale cohomology of split tori has been studied by many authors, e.g. [Mag78, GiPi08, GiSe14].

Lemma 8.1.1 *Let X be a smooth, geometrically integral variety over a field k of characteristic 0. Let $\Gamma = \mathrm{Gal}(k_s/k)$. There are split exact sequences*

$$0 \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^1(\mathbb{G}_{m,X}, \mathbb{Q}/\mathbb{Z}) \longrightarrow (\mathbb{Q}/\mathbb{Z}(-1))^{\Gamma} \longrightarrow 0,$$

where $\mathbb{Q}/\mathbb{Z}(-1)$ is the direct limit of $\mathbb{Z}/n(-1)$ for $n\to\infty$, and

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(\mathbb{G}_{m,X}) \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

Proof. Let Y be the closed subset of $\mathbb{A}^1_X$ which is the zero section of the structure morphism $\mathbb{A}^1_X\to X$. Then $X \cong Y$. The open subset $\mathbb{A}^1_X \smallsetminus Y$ is isomorphic to

$\mathbb{G}_{m,X}$. The unit section of the structure morphism $\mathbb{G}_{m,X}\to X$ is an embedding $X\hookrightarrow\mathbb{G}_{m,X}$ such that the composition $X\hookrightarrow\mathbb{G}_{m,X}\to\mathbb{A}^1_X\to X$ is an isomorphism.

For any integer $n>0$ we have the Gysin exact sequence (2.15)

$$\ldots\to\mathrm{H}^i_{\text{ét}}(\mathbb{A}^1_X,\mathbb{Z}/n)\to\mathrm{H}^i_{\text{ét}}(\mathbb{G}_{m,X},\mathbb{Z}/n)\to\mathrm{H}^{i-1}_{\text{ét}}(X,\mathbb{Z}/n(-1))\to\mathrm{H}^{i+1}_{\text{ét}}(\mathbb{A}^1_X,\mathbb{Z}/n)\to\ldots$$

As $n>0$ is invertible in X, the natural maps $\mathrm{H}^i_{\text{ét}}(X,\mathbb{Z}/n)\to\mathrm{H}^i_{\text{ét}}(\mathbb{A}^1_X,\mathbb{Z}/n)$ are isomorphisms. Specialisation at the unit section of $\mathbb{G}_{m,X}\to X$ shows that all maps $\mathrm{H}^i_{\text{ét}}(\mathbb{A}^1_X,\mathbb{Z}/n)\to\mathrm{H}^i_{\text{ét}}(\mathbb{G}_{m,X},\mathbb{Z}/n)$ are split injective. Putting everything together, we get split short exact sequences

$$0\longrightarrow\mathrm{H}^i_{\text{ét}}(X,\mathbb{Z}/n)\longrightarrow\mathrm{H}^i_{\text{ét}}(\mathbb{G}_{m,X},\mathbb{Z}/n)\longrightarrow\mathrm{H}^{i-1}_{\text{ét}}(X,\mathbb{Z}/n(-1))\longrightarrow 0.$$

For $i=1$, this gives the first exact sequence. For $i=2$, this gives the exact sequence

$$0\longrightarrow\mathrm{H}^2_{\text{ét}}(X,\mu_n)\longrightarrow\mathrm{H}^2_{\text{ét}}(\mathbb{G}_{m,X},\mu_n)\longrightarrow\mathrm{H}^1_{\text{ét}}(X,\mathbb{Z}/n)\longrightarrow 0.$$

One then uses the compatible exact sequences

$$0\longrightarrow\mathrm{Pic}(X)/n\longrightarrow\mathrm{H}^2_{\text{ét}}(X,\mu_n)\longrightarrow\mathrm{Br}(X)[n]\longrightarrow 0$$

and

$$0\longrightarrow\mathrm{Pic}(\mathbb{G}_{m,X})/n\longrightarrow\mathrm{H}^2_{\text{ét}}(\mathbb{G}_{m,X},\mu_n)\longrightarrow\mathrm{Br}(\mathbb{G}_{m,X})[n]\longrightarrow 0$$

given by the Kummer sequence. The map $\mathrm{Pic}(X)\to\mathrm{Pic}(\mathbb{G}_{m,X})$ is the composition $\mathrm{Pic}(X)\to\mathrm{Pic}(\mathbb{A}^1_X)\to\mathrm{Pic}(\mathbb{G}_{m,X})$. The first map is an isomorphism since X is regular and the second map is surjective since $\mathbb{A}^1_X$ is regular. Since $\mathbb{G}_{m,X}/X$ has the unit section, we conclude that the map $\mathrm{Pic}(X)\to\mathrm{Pic}(\mathbb{G}_{m,X})$ is an isomorphism. We now get the exact sequence

$$0\longrightarrow\mathrm{Br}(X)[n]\longrightarrow\mathrm{Br}(\mathbb{G}_{m,X})[n]\longrightarrow\mathrm{H}^1_{\text{ét}}(X,\mathbb{Z}/n)\longrightarrow 0.$$

Since X and $\mathbb{G}_{m,X}$ are regular, both $\mathrm{Br}(X)$ and $\mathrm{Br}(\mathbb{G}_{m,X})$ are torsion groups, so we obtain the second exact sequence of the lemma. □

Let k be a field with separable closure k_{s}. Let T be an algebraic torus. Then $T^{\mathrm{s}}=T\times_k k_{\mathrm{s}}\cong\mathbb{G}^d_{m,k_{\mathrm{s}}}$ for some positive integer d. By an easy case of Rosenlicht's lemma, the group $k_{\mathrm{s}}[T]^*$ of invertible functions on T^{s} is the direct sum of k_{s}^* and the character group $\hat{T}=\mathrm{Hom}_{k_{\mathrm{s}}-\mathrm{groups}}(T^{\mathrm{s}},\mathbb{G}_{m,k_{\mathrm{s}}})$. In particular, for any integer n invertible in k, there is a natural isomorphism $\mathrm{H}^0(T^{\mathrm{s}},\mathbb{G}_{m,k_{\mathrm{s}}})/n=\hat{T}/n$.

Proposition 8.1.2 *Let k be a field of characteristic 0. Let T be an algebraic torus of dimension $d\geq 1$ over k with character group $\hat{T}$.*

(a) *There is a Γ-equivariant isomorphism*

$$\mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}},\mathbb{Q}/\mathbb{Z})\xrightarrow{\sim}\hat{T}\otimes\mathbb{Q}/\mathbb{Z}(-1)$$

and a non-canonical isomorphism $\mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}},\mathbb{Q}/\mathbb{Z})\simeq(\mathbb{Q}/\mathbb{Z})^d$.

(b) *There is a* Γ*-equivariant isomorphism*

$$\wedge^2(\hat{T}) \otimes \mathbb{Q}/\mathbb{Z}(-1) \xrightarrow{\sim} \mathrm{Br}(T^{\mathrm{s}})$$

and a non-canonical isomorphism $\mathrm{Br}(T^{\mathrm{s}}) \simeq (\mathbb{Q}/\mathbb{Z})^{d(d-1)/2}$. *If* k *is algebraically closed,* $T = \mathrm{Spec}(k[x_1, x_1^{-1}, \ldots, x_d, x_d^{-1}])$ *and* ζ *is a primitive* n*-th root of unity, the composite map*

$$\wedge^2(\hat{T}) \otimes \mathbb{Z}/n \xrightarrow{\sim} \mathrm{Br}(T)[n] \otimes \mu_n \longrightarrow \mathrm{Br}(k(T))[n] \otimes \mu_n$$

sends $x_i \wedge x_j$ *to* $(x_i, x_j)_\zeta \otimes \zeta$, *where* $(x_i, x_j)_\zeta$ *is defined at the end of Section 1.3.4.*

(c) *There is a split exact sequence of abelian groups*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}_1(T) \longrightarrow \mathrm{H}^2(k, \hat{T}) \longrightarrow 0.$$

Proof. (a) Since $\mathrm{Pic}(T^{\mathrm{s}}) = 0$, for any integer n, the Kummer sequence gives a natural isomorphism

$$\mathrm{H}^0_{\text{ét}}(T^{\mathrm{s}}, \mathbb{G}_m)/n \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}}, \mu_n),$$

hence $\hat{T}/n \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}}, \mu_n)$. We thus obtain an isomorphism

$$\mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \hat{T} \otimes \mathbb{Q}/\mathbb{Z}(-1).$$

(b) Using this isomorphism and the second (split) exact sequence of Lemma 8.1.1 for $X = \mathbb{G}_m^{d-1}$, we obtain by induction a non-canonical isomorphism $\mathrm{Br}(T^{\mathrm{s}}) \simeq (\mathbb{Q}/\mathbb{Z})^{d(d-1)/2}$. In particular, for each $n \geq 1$, the order of $\mathrm{Br}(T^{\mathrm{s}})[n]$ is $n^{d(d-1)/2}$.

Consider the cup-product pairing of étale cohomology groups

$$\mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}}, \mu_n) \times \mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}}, \mu_n) \longrightarrow \mathrm{H}^2_{\text{ét}}(T^{\mathrm{s}}, \mu_n^{\otimes 2}) = \mathrm{Br}(T^{\mathrm{s}})[n] \otimes \mu_n, \tag{8.1}$$

where the last equality follows from the Kummer sequence and the vanishing of $\mathrm{Pic}(T^{\mathrm{s}})$. This pairing is compatible with the cup-product pairing of Galois cohomology groups

$$\mathrm{H}^1(k_{\mathrm{s}}(T), \mu_n) \times \mathrm{H}^1(k_{\mathrm{s}}(T), \mu_n) \longrightarrow \mathrm{H}^2(k_{\mathrm{s}}(T), \mu_n^{\otimes 2}) \tag{8.2}$$

via the injective map $\mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}}, \mu_n) \hookrightarrow \mathrm{H}^1(k_{\mathrm{s}}(T), \mu_n)$ induced by the inclusion of the generic point $\mathrm{Spec}(k_{\mathrm{s}}(T)) \to T^{\mathrm{s}}$. Since $\mathrm{char}(k) = 0$, the field k_{s} is algebraically closed. Thus $(a, a) = (a, -a) = 0$ for any $a \in \mathrm{H}^1(k_{\mathrm{s}}(T), \mu_n)$, so the pairings (8.2) and (8.1), are alternating. We thus have a Galois equivariant map

$$\xi : \wedge^2(\hat{T}) \otimes \mathbb{Z}/n \longrightarrow \mathrm{Br}(T^{\mathrm{s}})[n] \otimes \mu_n.$$

It is enough to prove that ξ is an isomorphism of abelian groups. We already know that the two groups have the same cardinality, so it remains to show that ξ is injective.

Let us fix an isomorphism of k_{s}-tori

$$T^{\mathrm{s}} \simeq \mathbb{G}^d_{m,k_{\mathrm{s}}} = \mathrm{Spec}(k_{\mathrm{s}}[x_1, x_1^{-1}, \ldots, x_d, x_d^{-1}]) \subset \mathbb{A}^d_{m,k_s} = \mathrm{Spec}(k_s[x_1, \ldots, x_d]).$$

The free $\mathbb{Z}/n$-module $\wedge^2(\hat{T})\otimes\mathbb{Z}/n$ is generated by the elements $x_i\wedge x_j$ for $1\leq i<j\leq d$. Let $\alpha=\sum_{i<j}a_{i,j}x_i\wedge x_j$ be a non-zero element of $\wedge^2(\hat{T})\otimes\mathbb{Z}/n$, where each $a_{i,j}$ is a non-negative integer less than n. Write β for the image of α in $\mathrm{Br}(k_{\mathrm{s}}(T))[n]\otimes\mu_n$. Let r be the smallest value such that $a_{r,t}\neq 0$ for some t. Let K_r be the field $k_{\mathrm{s}}(x_1,\ldots,x_{r-1},x_{r+1},\ldots,x_d)$. The residue of β at the divisor $x_r=0$ of $\mathbb{A}^d_{m,k_s}$ is the class $\prod_{t>r}x_t^{a_{r,t}}$ in K_r^*/K_r^{*n}. This class is not trivial, hence $\beta\neq 0$. This shows that the composition of ξ with the natural map $\mathrm{Br}(T^{\mathrm{s}})[n]\otimes\mu_n\to\mathrm{Br}(k_{\mathrm{s}}(T))[n]\otimes\mu_n$ is injective, so ξ is injective. This proves (b).

(c) In view of $\mathrm{Pic}(T^s)=0$, the spectral sequence

$$E_2^{p,q}=\mathrm{H}^p(k,\mathrm{H}^q_{\text{ét}}(T^{\mathrm{s}},\mathbb{G}_m))\Longrightarrow\mathrm{H}^{p+q}_{\text{ét}}(T,\mathbb{G}_m)$$

gives rise to an isomorphism $\mathrm{H}^2(k,k_{\mathrm{s}}[T]^*)\tilde{\longrightarrow}\mathrm{Br}_1(T)$, hence to an isomorphism $\mathrm{H}^2(k,k_{\mathrm{s}}^*)\oplus\mathrm{H}^2(k,\hat{T})\tilde{\longrightarrow}\mathrm{Br}_1(T)$ which gives (c). □

Proposition 8.1.3 *Let k be a perfect field, let $n\geq 1$ be an integer and let $T=\mathbb{G}^n_{m,k}$ be a split torus. The natural map $\mathrm{Br}(T)\to\mathrm{Br}(T^{\mathrm{s}})^\Gamma$ is surjective.*

Proof. For any k-variety X we have the spectral sequence

$$E_2^{pq}=\mathrm{H}^p(k,\mathrm{H}^q(\overline{X},\mathbb{G}_m))\Rightarrow\mathrm{H}^{p+q}(X,\mathbb{G}_m).$$

It is functorial contravariant in the k-variety X.

If $X=T$ is a k-torus, then $\mathrm{H}^0(X^{\mathrm{s}},\mathbb{G}_m)=k_{\mathrm{s}}^*\oplus\hat{T}$ and $\mathrm{H}^1(X^{\mathrm{s}},\mathbb{G}_m)=\mathrm{Pic}(T^s)=0$. The spectral sequence thus gives rise to an exact sequence

$$0\to\mathrm{H}^2(\Gamma,k_{\mathrm{s}}^*\oplus\hat{T})\to\mathrm{Br}(T)\to\mathrm{Br}(T^{\mathrm{s}})^\Gamma\to\mathrm{H}^3(\Gamma,k_{\mathrm{s}}^*\oplus\hat{T})\to\mathrm{H}^3(T,\mathbb{G}_m).$$

Write

$$\mathrm{Br}_e(T)=\mathrm{Ker}[\mathrm{Br}(T)\to\mathrm{Br}(k)],\quad \mathrm{H}^3_e(T,\mathbb{G}_m)=\mathrm{Ker}[\mathrm{H}^3(T,\mathbb{G}_m)\to\mathrm{H}^3(k,\mathbb{G}_m)]$$

for the kernels of the evaluation maps at the neutral element $e\in T(k)$. Then we get an exact sequence

$$0\longrightarrow\mathrm{H}^2(\Gamma,\hat{T})\longrightarrow\mathrm{Br}_e(T)\longrightarrow\mathrm{Br}(T^{\mathrm{s}})^\Gamma\longrightarrow\mathrm{H}^3(\Gamma,\hat{T})\longrightarrow\mathrm{H}^3_e(T,\mathbb{G}_m).$$

Since the spectral sequence is functorial in X, for any k-homomorphism of tori $R\to T$, we get a commutative diagram of exact sequences

$$\begin{array}{ccccc}\mathrm{Br}_e(T) & \longrightarrow & \mathrm{Br}(T^{\mathrm{s}})^\Gamma & \longrightarrow & \mathrm{H}^3(\Gamma,\hat{T})\\ \downarrow & & \downarrow & & \downarrow\\ \mathrm{Br}_e(R) & \longrightarrow & \mathrm{Br}(R^{\mathrm{s}})^\Gamma & \longrightarrow & \mathrm{H}^3(\Gamma,\hat{R})\end{array}$$

If R is of dimension 1, then $\mathrm{Br}(R^{\mathrm{s}})=0$ (here we use the hypothesis that k is perfect; for k arbitrary, we would only get a result up to the characteristic of k). This implies that the composition of maps

$$\mathrm{Br}(T^{\mathrm{s}})^\Gamma\longrightarrow\mathrm{H}^3(\Gamma,\hat{T})\longrightarrow\mathrm{H}^3(\Gamma,\hat{R})$$

is zero.

If T is split, then $T = \mathbb{G}_{m,k}^n$, $\hat{T} = \mathbb{Z}^n$, and we have $\mathrm{H}^3(\Gamma, \mathbb{Z}^n) = \mathrm{H}^3(\Gamma, \mathbb{Z})^{\oplus n}$. Thus the map $\mathrm{Br}(T^{\mathrm{s}})^\Gamma \to \mathrm{H}^3(\Gamma, \hat{T})$ is zero, hence the map $\mathrm{Br}(T) \to \mathrm{Br}(T^{\mathrm{s}})^\Gamma$ is surjective. $\square$

In the case when the field k is algebraically closed, Harari and Skorobogatov [HS03, Thm. 1.6] computed the Brauer group of a torsor for a k-torus.

Theorem 8.1.4 *Let k be an algebraically closed field of characteristic 0. Let X be an integral smooth variety over k such that $k^* = k[X]^*$ and $\mathrm{Pic}(X)$ is a finitely generated free abelian group. Let $f : Y \to X$ be an X-torsor for a torus such that $k^* = k[Y]^*$ and $\mathrm{Pic}(Y)$ is a finitely generated free abelian group. Then the map $f^* : \mathrm{Br}(X) \to \mathrm{Br}(Y)$ is an isomorphism.*

8.2 Simply connected semisimple groups

Proposition 8.2.1 *Let k be a field of characteristic 0. Let G be a simply connected semisimple group over k. Let E be a k-torsor for G and let X be a smooth, projective, geometrically integral variety over k birationally equivalent to E. Then the following natural maps are isomorphisms:*

(i) $\mathrm{Br}(k) \tilde{\longrightarrow} \mathrm{Br}(E)$;
(ii) $\mathrm{Br}(k) \tilde{\longrightarrow} \mathrm{Br}(X)$.

Proof. For a semisimple and simply connected group G we have $k_{\mathrm{s}}^* = k_{\mathrm{s}}[E]^*$, $\mathrm{Pic}(E^{\mathrm{s}}) = 0$, and $\mathrm{Br}(k) \tilde{\longrightarrow} \mathrm{Br}(G)$, see [San81, §6], [Gi09].

The exact sequence (4.8) then gives an isomorphism $\mathrm{Br}(k) \tilde{\longrightarrow} \mathrm{Br}(E)$ in (i). For X as in the proposition, there exists a non-empty open set $U \subset E$ and a birational morphism $U \to X$. Since X is projective and E is smooth, we may assume that U contains all codimension 1 points of E. By purity for the Brauer group, the restriction map $\mathrm{Br}(E) \to \mathrm{Br}(U)$ is an isomorphism. Since X is smooth, the map $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is injective. Now we obtain (ii) from (i). $\square$

If G is not simply connected, then $\mathrm{Br}(k) \to \mathrm{Br}(G)$ is not necessarily an isomorphism even when k is algebraically closed of characteristic zero, see [Ive76].

Let us state an important theorem of Bruhat and Tits, see [BT87].

Theorem 8.2.2 (Bruhat–Tits) *Let K be a complete local field with perfect residue field of cohomological dimension 1. Let X be a K-torsor for a simply connected semisimple group over K. Then X has a K-point.*

One application of this theorem is Theorem 10.1.10 below, which says the following. If $f : X \to Y$ is a dominant morphism of smooth, projective, geometrically integral varieties over a field k of characteristic zero such that the generic fibre is birationally equivalent to a $k(Y)$-torsor for a simply connected semisimple group over $k(Y)$, then the induced map $f^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is an isomorphism. This result has the following corollary.

Corollary 8.2.3 *Let k be a field of characteristic 0. Let $H \hookrightarrow \mathrm{GL}_n$ be an arbitrary linear group over k. Let $H \hookrightarrow G$ be an embedding into a simply connected semisimple group G. Then $\mathrm{Br}_{\mathrm{nr}}(k(\mathrm{GL}_n/H)) \cong \mathrm{Br}_{\mathrm{nr}}(k(G/H))$.*

Proof. (cf. [LA15, Prop. 26]) Let $P = G \times_k \mathrm{GL}_n$. Consider the quotient P/H with respect to the diagonal action of H on the right. The projection of $P \to G$ induces a morphism $P/H \to G/H$ which is a left GL_n-torsor. Similarly, the morphism $P/H \to \mathrm{GL}_n/H$ induced by the projection $P \to \mathrm{GL}_n$ is a left G-torsor. Any GL_n-torsor is locally trivial for the Zariski topology, thus P/H is birationally equivalent to $G/H \times_k \mathrm{GL}_k$, hence P/H and G/H have isomorphic unramified Brauer groups (Corollary 5.2.5). Since G is simply connected and semisimple, Theorem 10.1.10 implies that the map $\mathrm{Br}_{\mathrm{nr}}(k(\mathrm{GL}_n/H)) \to \mathrm{Br}_{\mathrm{nr}}(k(P/H))$ is an isomorphism. □

8.3 Theorems of Bogomolov and Saltman

In this section we discuss theorems of Bogomolov and Saltman. We refer to [CTS07, §6] and to [GS17, Ch. 6, §6] for most proofs and for history of the subject. An abelian group generated by at most two elements will be called *bicyclic*.

Theorem 8.3.1 [CTS07, Thm. 6.1] *Let L be a field finitely generated over an algebraically closed field k of characteristic zero. Let G be a finite group of automorphisms of L over k, and let $\mathcal{B}_G$ be the set of bicyclic subgroups of G. Then*

$$\mathrm{Br}_{\mathrm{nr}}(L^G) = \{\alpha \in \mathrm{Br}(L^G) \mid \alpha_H \in \mathrm{Br}_{\mathrm{nr}}(L^H) \text{ for all } H \in \mathcal{B}_G\},$$

where α_H is the restriction of $\alpha \in \mathrm{Br}(L^G)$ to $\mathrm{Br}(L^H)$.

Proof. [CTS07, *loc.cit.*] Let $K = L^G$ and let $\alpha \in \mathrm{Br}(K)$ be such that $\partial_A(\alpha) \neq 0$ for some discrete valuation ring $A \subset K$ with fraction field K. We must show that there exists a subgroup $H \in \mathcal{B}_G$ such that

$$\alpha_H \notin \mathrm{Br}_{\mathrm{nr}}(L^H).$$

The following facts can be found in [SerCL, I, §7]. Let $\mathfrak{p}$ be a prime ideal in the semi-local Dedekind ring $\tilde{A}$ which is the integral closure of A in L, let $D \subset G$ be the associated decomposition group, and let $I \subset D$ be the inertia group, which is a normal subgroup of G. The localisation $B = \tilde{A}_{\mathfrak{p}} \subset L$ is a discrete valuation ring. There is a tower of fields: $K \subset L^D \subset L^I \subset L$ and a corresponding tower of discrete valuation rings obtained by taking the traces $A = B^G \subset B^D \subset B^I$ of B on the subfields. The corresponding residue field extensions are $F = F \subset E = E$, and we have $D/I = \mathrm{Gal}(E/F) = \mathrm{Gal}(L^I/L^D)$. The Galois extension L^I/K is unramified, i.e. a uniformising parameter of A is still a uniformising parameter in B^I.

Moreover, since the residue characteristic is zero, the inertia group I can be identified with a cyclic group, namely, the group μ of roots of unity in F [SerCL, IV, §2, Cor. 1 et 2]. Furthermore, the conjugacy action of D on the normal subgroup I is then trivial, since this action can be identified with the action of $D/I = \mathrm{Gal}(E/F)$ on $\mu \subset F$, and all the roots of unity are in $k \subset F$. Thus I is central in D.

If $\alpha_I \notin \mathrm{Br}_{\mathrm{nr}}(L^I)$, we are done, since I is a cyclic subgroup of G. We may thus assume that $\alpha_I \in \mathrm{Br}_{\mathrm{nr}}(L^I)$. Since B^D/A is an unramified extension of discrete valuation rings which induces an isomorphism on the residue fields, the assumption $\partial_A(\alpha) \neq 0$ implies $\partial_{B^D}(\alpha) \neq 0 \in \mathrm{H}^1(F, \mathbb{Q}/\mathbb{Z})$. On the other hand, $\partial_{B^I}(a) = 0 \in \mathrm{H}^1(E, \mathbb{Q}/\mathbb{Z})$. Since B^I/B^D is unramified, the commutative diagram:

$$\begin{array}{ccc} \mathrm{Br}(K^I) & \xrightarrow{\partial_B} & \mathrm{H}^1(E, \mathbb{Q}/\mathbb{Z}) \\ \uparrow & & \uparrow \mathrm{Res}_{F/E} \\ \mathrm{Br}(K^D) & \xrightarrow{\partial_{B^D}} & \mathrm{H}^1(F, \mathbb{Q}/\mathbb{Z}) \end{array}$$

implies that $\partial_{B^D}(\alpha)$ may be identified with a non-trivial character of $D/I = \mathrm{Gal}(E/F)$. Let $g \in D$ be an element of D whose class $\bar{g}$ in D/I satisfies $\partial_{B^D}(\alpha)(\bar{g}) \neq 0 \in \mathbb{Q}/\mathbb{Z}$, let $H = \langle I, g\rangle \subset D$ be the subgroup spanned by I and g, and let F_1 be the residue class field of B^H. Inserting $\mathrm{Br}(K^H) \to \mathrm{H}^1(F_1, \mathbb{Q}/\mathbb{Z})$ in the above diagram, one immediately sees that $\partial(\alpha_H) \neq 0$, since $\partial(\alpha_H)$ may be identified with a character of $\mathrm{Gal}(E/F_1) = D/H$ which does not vanish on $\bar{g}$. This is enough to conclude, since H is an extension of the cyclic group $\langle \bar{g} \rangle$ by the central cyclic subgroup I (see above), hence is an abelian group spanned by two elements. □

Let G be a finite group. Consider a faithful representation $G \to \mathrm{GL}(V)$, where V is a finite dimensional complex vector space. Write $\mathbb{C}(V)$ for the purely transcendental extension of $\mathbb{C}$, which is the field of rational functions on V considered as an affine space over $\mathbb{C}$. Then the subfield of invariants $\mathbb{C}(V)^G$ is the function field of the quotient V/G. Speiser's lemma (see, e.g. [CTS07, Thm. 3.3]) states that the stably birational equivalence class of V/G does not depend on the choice of a faithful representation $G \to \mathrm{GL}(V)$. By Corollary 5.2.5, this implies that $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G)$ does not depend on the choice of V. In particular, considering the left action of $\mathrm{GL}(V)$ on $\mathrm{End}(V)$ gives a faithful representation of G in $\mathrm{End}(V)$, so we get an isomorphism $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G) = \mathrm{Br}_{\mathrm{nr}}(\mathrm{GL}(V)/G)$.

If G is a finite abelian group, it is a consequence of a theorem of Fischer that the field of invariants $\mathbb{C}(V)^G$ is purely transcendental, hence $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G) = 0$. Combining this with Theorem 8.3.1, one gets the following result.

Theorem 8.3.2 (Bogomolov) *Let $G \subset \mathrm{GL}(V)$ be a finite group. Then the unramified Brauer group of the field $\mathbb{C}(V)^G$ is given by the formula*

$$\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G) = \mathrm{Ker}[\mathrm{H}^2(G, \mathbb{C}^*) \longrightarrow \prod_{A \in \mathcal{B}} \mathrm{H}^2(A, \mathbb{C}^*)],$$

where $\mathcal{B}$ is the set of bicyclic subgroups $A \subset G$ and $\mathrm{H}^2(G, \mathbb{C}^*) \to \mathrm{H}^2(A, \mathbb{C}^*)$ is the restriction map.

See [Bog87], [CTS07, Thm. 7.1], [GS17, Thm. 6.6.12]. Fischer's theorem implies that the set $\mathcal{B}$ of bicyclic subgroups can be replaced by the larger set of all abelian subgroups. One may also write $\mathrm{H}^2(G, \mathbb{C}^*) \cong \mathrm{H}^3(G, \mathbb{Z})$ and similarly for each A. The same formula gives the value of $\mathrm{Br}_{\mathrm{nr}}(H/G)$, where G is a finite subgroup of $H = \mathrm{SL}_{n,\mathbb{C}}$ or any simply connected semisimple group over $\mathbb{C}$ (see [CT12b] and [LA17]).

This theorem has led to numerous examples of finite p-groups G such that the quotient $\mathrm{GL}_{n,\mathbb{C}}/G$ is not rational (E. Noether's problem). D. Saltman (1984) was the first to use the unramified Brauer group to disprove the rationality of $\mathrm{GL}_{n,\mathbb{C}}/G$ for some finite groups G. Bogomolov [Bog87] developed a technique for computing $\mathrm{Br}_{\mathrm{nr}}(\mathrm{GL}_{n,\mathbb{C}}/G)$ when G is a central extension of abelian groups. See [CTS07, §7] and the references therein. Since [CTS07] was written, many papers have been devoted to the computation of the group $\mathrm{Br}_{\mathrm{nr}}(\mathrm{GL}_{n,\mathbb{C}}/G)$ in Theorem 8.3.2, which often goes under the name of 'Bogomolov multiplier'. (Recall that $\mathrm{H}^2(G, \mathbb{C}^*) \cong \mathrm{H}^3(G, \mathbb{Z})$ is the Schur multiplier of the finite group G.) Kunyavskiĭ [Ku10] proved that the Bomogolov multiplier vanishes for all simple groups.

Theorem 8.3.3 (Saltman) [Sal90] *Let G be a finite group and let M be a faithful G-lattice. Let $\mathbb{C}(M)$ be the field of fractions of the group algebra $\mathbb{C}[M]$. Then*

$$\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M)^G) = \mathrm{Ker}[\mathrm{H}^2(G, \mathbb{C}^* \oplus M) \longrightarrow \prod_{A \in \mathcal{B}} \mathrm{H}^2(A, \mathbb{C}^* \oplus M)],$$

where $\mathcal{B}$ is the set of bicyclic subgroups $A \subset G$.

In other words, $\mathbb{C}(M)$ is the field of functions $\mathbb{C}(T)$ of a complex torus T equipped with an action of a group G. Then $\mathbb{C}(M)^G$ is the field of functions $\mathbb{C}(T/G)$ of the quotient T/G.

Further work along these lines has been done by D. Saltman, E. Peyre [P08], and in joint work of B. Kahn and Nguyen Thi Kim Ngan [KN16].

There is an extension of Theorem 8.3.1 to almost free actions of reductive groups, see [Bog89, Thm. 2.1] and [CTS07, Thm. 6.4].

Theorem 8.3.4 (Bogomolov) *Let k be an algebraically closed field of characteristic zero, let G be a reductive group over k, and let X be an integral affine variety over k with an action of G such that all stabilisers are trivial. Write $\mathcal{B}_G$ for the set of finite bicyclic subgroups of $G(k)$. Then*

$$\mathrm{Br}_{\mathrm{nr}}(k(X)^G) = \{\alpha \in \mathrm{Br}(k(X)^G) \mid \alpha_A \in \mathrm{Br}_{\mathrm{nr}}(k(X)^A) \text{ for all } A \in \mathcal{B}_G\},$$

where α_A is the restriction of $\alpha \in \mathrm{Br}(k(X)^G)$ to $\mathrm{Br}(k(X)^A)$.

The following theorem was proved in several instalments.

Theorem 8.3.5 *Let k be an algebraically closed field of characteristic* 0*. Let G be a connected linear algebraic group over k and let $H \subset G$ be a connected algebraic subgroup. Let X_c be a smooth compactification of $X = G/H$. Then* $\mathrm{Br}(X_c) = 0$*.*

The case $H = \mathrm{PGL}_n \subset G = \mathrm{GL}_N$ is due to Saltman [Sal85].

For G simply connected, the result is a theorem of Bogomolov [Bog89, Thm. 2.4]. For a detailed account of his proof see [CTS07, §9]. The proof given there builds upon Theorem 8.3.4.

The result in the general case was obtained by Borovoi, Demarche and Harari in [BDH13]. Their proof uses a long arithmetic detour. A direct reduction to the case G semisimple simply connected was then given by Borovoi [Bor13].

In the special case when $G = \mathrm{GL}_n$ and H is a connected semisimple group, a proof in arbitrary characteristic is given by Blinstein and Merkurjev in [BM13, Thm. 5.10].

Over a separably closed field of characteristic $p > 0$, assuming that the connected groups G and H are smooth and reductive, Borovoi, Demarche and Harari [BDH13] prove that $\mathrm{Br}(X_c)$ is a p-primary torsion group.

Remark 8.3.6 1. Let $k = \mathbb{C}$. There exists a subgroup $A \subset \mathrm{SL}_n$, where A is an extension of a finite abelian group by a torus, such that $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(\mathrm{SL}_n/A)) \neq 0$. Such examples can be constructed by a method suggested by C. Demarche. Suppose that a group H' is a central extension of a finite abelian group A by a finite abelian group Z. Let us embed Z into a torus T and define $H = (T \times H')/Z$. Then H is a central extension of A by T. Suppose we are given an embedding $H \hookrightarrow G = \mathrm{SL}_n$. Since T commutes with H, there is a right action of T on G/H'. But H is generated by T and H', hence the natural morphism $G/H' \to G/H$ is a right torsor under the quotient torus T/Z. This torus is split, hence G/H' is stably birationally equivalent to G/H. Thus the natural map $\mathrm{Br}_{\mathrm{nr}}(G/H) \to \mathrm{Br}_{\mathrm{nr}}(G/H')$ is an isomorphism. Using Theorem 8.3.2, Bogomolov [Bog87] has constructed examples with $\mathrm{Br}_{\mathrm{nr}}(G/H') \neq 0$. (See also [CTS07].)

2. Let $k = \mathbb{C}$. For a subgroup $A \subset G$, where G is semisimple and simply connected and A is an extension of a group of multiplicative type by a semisimple simply connected group, we have $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G/A)) = 0$. This follows by combining Theorem 8.3.5 with [LA15, Prop. 26] (itself an elaboration on Corollary 8.2.3).

8.4 Homogeneous spaces over an arbitrary field

For $g \in \Gamma$ we denote by $\langle g \rangle$ the *closed* subgroup of Γ generated by g. For a continuous discrete Galois module M and $i \geq 0$ we define

$$Ш^i_\omega(\Gamma, M) = \mathrm{Ker}[\mathrm{H}^i(\Gamma, M) \longrightarrow \prod_{g \in \Gamma} \mathrm{H}^i(\langle g \rangle, M)].$$

Using hypercohomology one extends this definition to bounded complexes of Galois modules. The following statements are proved using standard properties of Galois cohomology.

(1) If $K \subset k_{\mathrm{s}}$ is a Galois extension of k such that $\mathrm{Gal}(k_{\mathrm{s}}/K)$ acts trivially on M, then the inflation map $\mathrm{H}^1(\mathrm{Gal}(K/k), M^{\mathrm{Gal}(k_{\mathrm{s}}/K)}) \to \mathrm{H}^1(\Gamma, M)$ induces an isomorphism

$$\mathrm{Ker}[\mathrm{H}^1(\mathrm{Gal}(K/k), M) \longrightarrow \prod_{g\in \mathrm{Gal}(K/k)} \mathrm{H}^1(\langle g\rangle, M)] = Ш^1_\omega(\Gamma, M).$$

(2) If, in addition, the abelian group M is finitely generated and free, then the inflation map $\mathrm{H}^2(\mathrm{Gal}(K/k), M^{\mathrm{Gal}(k_{\mathrm{s}}/K)}) \to \mathrm{H}^2(\Gamma, M)$ induces an isomorphism

$$\mathrm{Ker}[\mathrm{H}^2(\mathrm{Gal}(K/k), M) \longrightarrow \prod_{g\in \mathrm{Gal}(K/k)} \mathrm{H}^2(\langle g\rangle, M)] = Ш^2_\omega(\Gamma, M).$$

Work of many authors [Vos98, CTS77, San81, Bog89, CTK98, BK00, BKG04, CTK06, CTS07, CT08, Bor13, BM13] has led to the following results.

Theorem 8.4.1 *Let k be a field of characteristic 0 with an algebraic closure $\bar{k}$ and $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Let X be a homogeneous space of a connected linear algebraic group such that the stabilisers of geometric points are connected. Let X_c be a smooth compactification of X. Then the following properties hold.*

(i) $\mathrm{Br}(\overline{X}_c) = 0$*, hence* $\mathrm{Br}(X_c) = \mathrm{Br}_1(X_c)$.

(ii) *The Γ-lattice $\mathrm{Pic}(\overline{X}_c)$ is a flasque Γ-module, that is, for every closed subgroup $C \subset \Gamma$ we have $\mathrm{Ext}^1_C(\mathrm{Pic}(\overline{X}_c), \mathbb{Z}) = 0$.*

(iii) *For any procyclic subgroup $C \subset \Gamma$ we have $\mathrm{H}^1(C, \mathrm{Pic}(\overline{X}_c)) = 0$.*

(iv) *There is an exact sequence*

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}(X_c) \longrightarrow Ш^1_\omega(\Gamma, \mathrm{Pic}(\overline{X}_c)) \longrightarrow \mathrm{H}^3(k, \bar{k}^*).$$

(v) *If $X(k) \neq \emptyset$, then there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X_c) \longrightarrow Ш^1_\omega(\Gamma, \mathrm{Pic}(\overline{X}_c)) \longrightarrow 0.$$

Let us for simplicity assume $X(k) \neq \emptyset$. Then $X = G/H$, where G and H are connected linear algebraic groups. Once $\mathrm{Br}(\overline{X}_c) = 0$ has been established (Theorem 8.3.5), one has $\mathrm{Br}(X_c)/\mathrm{Br}(k) = \mathrm{H}^1(\Gamma, \mathrm{Pic}(\overline{X}_c))$. Statement (ii) [CTK06, Thm. 5.1] implies (iii) for purely algebraic reasons (the duality for Tate cohomology of a finite group with values in a lattice and the periodicity of cohomology of a finite cyclic group). From (iii) we immediately get (iv) which implies (v).

Corollary 8.4.2 *Let k be a field of characteristic 0. Let X be a smooth, projective, geometrically integral variety over k with a k-point. Assume that X is stably k-birational to a homogeneous space of a connected linear algebraic group such that the stabilisers of geometric points are connected. If there exists a finite cyclic extension K/k such that $\mathrm{Pic}(X_K) = \mathrm{Pic}(\overline{X})$, then the map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is an isomorphism and the Γ-module $\mathrm{Pic}(\overline{X})$ is a direct summand of a permutation Γ-module.*

This is a consequence of Proposition 5.2.13, Theorem 8.4.1, and the following general lemma.

Lemma 8.4.3 *Let k be field of characteristic zero and let W be a smooth projective variety over k. Assume that $\mathrm{Pic}(\overline{W})$ is a finitely generated torsion-free abelian group. Assume that $\mathrm{H}^1(C,\mathrm{Pic}(\overline{W}))=0$ for all procyclic subgroups $C\subset\Gamma$. If there exists a cyclic finite field exension K/k such that $\mathrm{Pic}(W_K)=\mathrm{Pic}(\overline{W})$, then $\mathrm{H}^1(k,\mathrm{Pic}(\overline{W}))=0$ and the Γ-module $\mathrm{Pic}(\overline{W})$ is a direct summand of a permutation Γ-module.*

Proof. Let $K\subset\bar{k}$ be a Galois extension of k such that $G=\mathrm{Gal}(K/k)$ is cyclic. Let $M=\mathrm{Pic}(\overline{W})$. The group $\mathrm{Gal}(\bar{k}/K)$ acts trivially on the finitely generated torsion-free abelian group M, hence $\mathrm{H}^1(K,M)=0$. The restriction-inflation sequence gives $\mathrm{H}^1(G,M)=\mathrm{H}^1(K/k,M)=\mathrm{H}^1(k,M)$. The map $\Gamma=\mathrm{Gal}(\bar{k}/k)\to G$ is surjective, so we can find a $g\in\Gamma$ whose image generates G. Let $E=\bar{k}^g$ be the fixed field of g. The field extensions K/k and E/k are linearly disjoint. In particular, $\mathrm{H}^1(K/k,M)\cong\mathrm{H}^1(KE/E,M)$. We have $\mathrm{H}^1(KE,M)=0$, so the restriction-inflation sequence gives $\mathrm{H}^1(KE/E,M)\cong\mathrm{H}^1(E,M)$, and the latter group is trivial by assumption. We thus get $\mathrm{H}^1(K/k,M)=0$ and then $\mathrm{H}^1(k,M)=0$. This remains true if k is replaced by a finite field extension. The last part of the statement is then a consequence of a theorem of Endo and Miyata (cf. [CTS77, Prop. 2, p. 184]): if G is a finite cyclic group acting on a finitely generated torsion-free abelian group M such that $\mathrm{H}^1(H,M)=0$ for all subgroups $H\subset G$, then M is a direct summand of a permutation G-module. □

Example 8.4.4 A Châtelet surface Y given by the affine equation

$$y^2-az^2=(x-e_1)(x-e_2)(x-e_3),$$

where $a\in k\smallsetminus k^{*2}$ and $e_i\neq e_j$ for $i\neq j$, admits a smooth compactification Y_c such that $\mathrm{Pic}(Y_{c,K})=\mathrm{Pic}(\overline{Y}_c)$, where $K=k(\sqrt{a})$. However, $\mathrm{Br}(Y_c)/\mathrm{Br}(k)=(\mathbb{Z}/2)^2$ (see Exercise 10.2.6). Corollary 8.4.2 then shows that such a Châtelet surface is not stably k-birational to any homogeneous space of a connected linear group with connected geometric stabilisers.

One would like to have a formula for $Ш^1_\omega(\Gamma,\mathrm{Pic}(\overline{X}_c))$ in terms of the homogeneous space X and not in terms of a smooth compactification. Let G be a connected linear algebraic group over a field k of characteristic 0. Let X be a homogeneous space of G defined over k. Let $\overline{H}\subset\overline{G}$ be the stabiliser of a $\bar{k}$-point of X. *Assume that $\overline{H}$ is an extension of a group of multiplicative type $\overline{S}$ by a connected linear algebraic group with trivial group of characters.* Then there is a natural group k-scheme S of multiplicative type such that $\overline{S}=S\times_k\bar{k}$. Let T be a torus over k which is the maximal toric quotient of G. Then there is an induced homomorphism $S\to T$ defined over k. Let $[\hat{T}\to\hat{S}]$ be the dual map of respective groups of characters, viewed as a complex of Galois modules in degrees -1 and 0.

Theorem 8.4.5 [BDH13, Thm. 8.1, Cor. 8.3] *With notation as above assume* $\mathrm{Pic}(\overline{G})=0$*. Let* X_c *be a smooth compactification of* X*. Then there is an exact sequence*

$$0\to\mathrm{Br}_1(X_c)/\mathrm{Br}_0(X_c)\to\text{Ш}^1_\omega(k,[\hat{T}\to\hat{S}])\to\mathrm{Ker}[\mathrm{H}^3(k,\bar{k}^*)\to\mathrm{H}^3_{\text{ét}}(X_c,\mathbb{G}_m)].$$

If $\overline{H}$ *is connected, then* S *is a torus and we have the same sequence with* $\mathrm{Br}_1(X_c)/\mathrm{Br}_0(X_c)$ *replaced by* $\mathrm{Br}(X_c)/\mathrm{Br}_0(X_c)$*.*

Let us mention some special cases, some of which are used in the proof of the general result.

- $G=T$ *is a torus and* $\overline{H}=1$. Here $S=1$, and

  $$\text{Ш}^1_\omega([\hat{T}\to\hat{S}])=\text{Ш}^1_\omega([\hat{T}\to 0])=\text{Ш}^2_\omega(k,\hat{T}).$$

  Under the assumption $X(k)\neq\emptyset$, i.e. $X=T$, the result in this case appeared in [CTS87b]. The proof uses the theorem of Endo and Miyata mentioned above: for any finite cyclic group G any H^1-trivial G-lattice is a direct summand of a permutation G-lattice (cf. [CTS77, Prop. 2 p. 184]).

- G *is a simply connected semisimple group,* $\mu\subset G$ *is a finite central subgroup and* $X=G/\mu$. Here $T=1$, $S=\mu$, so

  $$\text{Ш}^1_\omega([\hat{T}\to\hat{S}])=\text{Ш}^1_\omega([0\to\hat{\mu}])=\text{Ш}^1_\omega(k,\hat{\mu}),$$

  where $\hat{\mu}=\mathrm{Hom}_{k\text{-groups}}(\mu,\mathbb{G}_{m,k})$. The result in this case was obtained in [CTK98]. The proof relies on a reduction to the case of a finite ground field k together with the above mentioned theorem on tori.

- G *is a simply connected semisimple group and* $\overline{H}$ *is connected.* Here $T=1$ and we have

  $$\text{Ш}^1_\omega([\hat{T}\to\hat{S}])=\text{Ш}^1_\omega([0\to\hat{S}])=\text{Ш}^1_\omega(k,\hat{S}).$$

  Under the assumption $X(k)\neq\emptyset$, the result in this case appeared in [CTK06] where Theorem 8.2.2 was used.

- $G=\mathrm{GL}_{n,k}$ *and* $H\subset G$ *is semisimple.* In this case

  $$\text{Ш}^1_\omega([\hat{T}\to\hat{S}])=\text{Ш}^1_\omega([\mathbb{Z}\to 0])=\text{Ш}^2_\omega(k,\mathbb{Z})=0.$$

The proof of $\mathrm{Br}_1(X_c)/\mathrm{Br}_0(X_c)=\text{Ш}^2_\omega(k,\hat{T})$, where X_c is a smooth compactification of a torus T, is done directly at the level of the field k. The proofs of most other computations of $\mathrm{Br}_1(X_c)/\mathrm{Br}_0(X_c)=\text{Ш}^1_\omega(\Gamma,\mathrm{Pic}(\overline{X}_c))$ rely on various reductions involving change of the ground field k. Let us mention some of them, without going into details.

If X is a homogeneous space of a semisimple group G, it is helpful to reduce to the case when G is quasi-split, that is, G contains a Borel subgroup B. Indeed, in this case the maximal torus of B is a quasi-trivial torus, that is, a product of tori of the form $R_{k'/k}(\mathbb{G}_{m,k'})$, where k' is finite separable extension of k. This implies that G is a rational variety over k. To reduce to this situation one extends the ground field k to the function field K of the variety of Borel subgroups of G. One then uses the fact that the map $\mathrm{Pic}(X_c\times_k k_\mathrm{s})\to\mathrm{Pic}(X_c\times_k K_\mathrm{s})$ is an isomorphism, see Proposition 5.2.14.

Another way to reduce to the case when G is quasi-split is first to reduce to the case when k is the fraction field of a finitely generated $\mathbb{Z}$-algebra over which G, X, X_c can be extended, and then use Chebotarev's density theorem to reduce the whole situation to the case of a finite field where the Galois action is preserved. See [CTK98] for details.

One also uses algebraic and arithmetic results from the theory of connected linear algebraic groups: a semisimple algebraic group over a finite field is quasi-split; a quasi-split semisimple group over a field k is birationally equivalent to the product of an affine space and a torus. One also uses Theorem 8.2.2.

The above theorems do not cover the case of quotients $\mathrm{GL}_{n,k}/G$ where G is a non-commutative finite subgroup subscheme of $\mathrm{GL}_{n,k}$. Such an extension of Theorem 8.3.2 to more general ground fields is given in [CT12a] for constant G and in [LA17] for more general G. The case when G is constant and $k=\mathbb{Q}$ is of interest in connection with the inverse Galois problem [Ha07a, Dem10, HW]. For further work on unramified Brauer groups of quotients, see [Dem10] and [LA14, LA15, LA17].

Exercise 8.4.6 [CTS77, Prop. 7] Let K/k be a finite Galois extension of fields. Let $T=R^1_{K/k}(\mathbb{G}_{m,K})$ be the kernel of the norm map $R_{K/k}(\mathbb{G}_{m,K})\to\mathbb{G}_{m,k}$. Show that $\mathrm{Br}_\mathrm{nr}(k(T)/k)\cong \mathrm{H}^3(\mathrm{Gal}(K/k),\mathbb{Z})$. If $\mathrm{Gal}(K/k)\cong(\mathbb{Z}/p)^2$, where p is a prime number, show that $\mathrm{Br}_\mathrm{nr}(k(T)/k)\cong\mathbb{Z}/p$. Thus T is not k-rational. This example of a non-k-rational linear algebraic group was first given by C. Chevalley (with a different proof).

Chapter 9

Schemes over local rings and fields

The object of study in this chapter is a scheme over the spectrum of a local ring. A separately standing Section 9.1 is devoted to the concepts of a split variety and of a split fibre of a morphism of varieties; for arithmetic applications and for the calculation of the Brauer group, split fibres should be considered as 'good' or 'non-degenerate'. In Section 9.2 we look at the classical case of quadrics over a discrete valuation ring.

In the ensuing sections the local ring is henselian or complete. In Section 9.3 we consider regular integral proper schemes of relative dimenson 1 over a henselian discrete valuation ring. The study of the Brauer group of such schemes goes back to Artin and Grothendieck [Gro68, III, §3]. We also discuss the parallel situation of proper regular desingularisations of a 2-dimensional henselian local ring, already considered in [Art87]. This leads to local-global theorems for the Brauer group of the function field. It also leads to comparison of index and exponent of a central simple algebra of the function field of such schemes under suitable assumptions on the residue field of the local ring, as initiated by Artin and by Saltman. In Section 9.4 we analyse the Brauer group of the generic fibre of a smooth proper scheme over a henselian discrete valuation ring. In Section 9.5 we discuss various properties of the Brauer group of a variety over a local field with respect to evaluation at rational and closed points.

9.1 Split varieties and split fibres

Split varieties

Recall our standard convention that a variety over k is a separated scheme of finite type over k. For an irreducible variety X over k we write k_X for the algebraic closure of k in the field of functions $k(X)$, which is the residue field $k(\eta)$ at the generic point $\eta \in X$.

Recall that we write $k_{\rm s}$ for a separable closure of k and $\bar{k}$ for an algebraic closure of k. We write $\overline{X} = X \times_k \bar{k}$.

Let us recall birational criteria for an integral scheme to be geometrically reduced or geometrically irreducible. Following Bourbaki [BouV, §15, no. 2, Déf. 1], a commutative k-algebra A is called *separable* if the ring $A \otimes_k L$ is reduced (i.e. has no nilpotents) for any field extension L/k. By [BouV, §15, no. 2, Prop. 3] A is a separable k-algebra if and only if $A \otimes_k \bar{k}$ is a separable $\bar{k}$-algebra, which is equivalent to $A \otimes_k \bar{k}$ being reduced [BouV, §15, no. 5, Thm. 3 (c)].

Let X be an integral scheme over k. Then X is geometrically reduced if and only if $k(X)$ is a separable k-algebra [EGA $\rm IV_2$, Prop. 4.6.1]. Next, X is geometrically irreducible if and only if k is separably closed in $k(X)$, that is, the only separable algebraic field extension of k in $k(X)$ is k itself [EGA $\rm IV_2$, Prop. 4.5.9]. See also [Po18, Section 2.2].

Definition 9.1.1 *Let X be an irreducible variety over a field k. The* **multiplicity** *of X is the length of the (artinian) local ring of X at the generic point η of X. The* **geometric multiplicity** *of X is the length of the (artinian) local ring of $\overline{X}$ at a point $\bar{\eta}$ of $\overline{X}$ over η.*

The definition of geometric multiplicity does not depend on the choice of $\bar{\eta}$ because such points are conjugate under the action of $\mathrm{Aut}(\bar{k}/k)$.

The multiplicity of X is 1 if and only if X contains a non-empty open reduced subscheme. The geometric multiplicity of X is 1 if and only if X contains a non-empty open geometrically reduced subscheme. By the birational criterion, this is equivalent to $k(X)$ being separable over k. Equivalently, X contains a dense open smooth subscheme, cf. [Stacks, Lemma 056V]. The multiplicity divides the geometric multiplicity; the ratio is the geometric multiplicity of the reduced subscheme $X_{\rm red}$ [BLR90, §9.1, Lemma 4 (a)]. It is a power of the characteristic exponent of k [BLR90, §9.1, Lemma 4 (c)].

Lemma 9.1.2 *Let $X\to Y$ be a morphism of integral schemes over a field k. Suppose that Y is normal. Then there is a natural embedding $k_Y \subset k_X$.*

Proof. Let $y \in Y$ be a point and let $\mathcal{O}_{Y,y}$ be the local ring of Y at y. Since Y is normal, $\mathcal{O}_{Y,y}$ is integrally closed in the function field $k(Y)$. Thus the inclusions $k \subset k_Y \subset k(Y)$ induce inclusions $k \subset k_Y \subset \mathcal{O}_{Y,y}$. It follows that k_Y is contained in $\mathrm{H}^0(Y, \mathcal{O}_Y)$, so that the structure morphism $Y\to\mathrm{Spec}(k)$ factors through $\mathrm{Spec}(k_Y)$. Thus the structure morphism $X\to\mathrm{Spec}(k)$ also factors through $\mathrm{Spec}(k_Y)$, hence $k_Y \subset k(X)$. □

The following definition was introduced in [Sko96].

Definition 9.1.3 *A variety over a field k is* **split** *if it contains a non-empty open geometrically integral subscheme.*

Proposition 9.1.4 *Let X be a variety over a field k. The following properties are equivalent.*

(i) *X is split;*

(ii) *X contains a non-empty open integral subscheme U such that $k_U = k$ which is geometrically reduced;*

(iii) *X contains a non-empty open integral subscheme of geometric multiplicity 1 which is geometrically irreducible;*

(iv) *X contains a non-empty open integral subscheme which is smooth and geometrically irreducible.*

Proof. Let us show that (i) implies (ii). Let $U \subset X$ be a non-empty open geometrically integral subscheme. By the birational criterion, k is separably closed in $k(U)$ and $k(U)$ is separable over k, hence k_U is separable over k so that $k_U = k$.

Conversely, $k_U = k$ implies that k is separably closed in $k(X)$, so X is geometrically irreducible. Thus (ii) implies (i).

A non-empty open integral subscheme $U \subset X$ has geometric multiplicity 1 if and only if it contains a dense open subscheme which is geometrically reduced, so (i) and (iii) are equivalent. This happens precisely when U contains a dense open smooth subscheme, so (iii) and (iv) are equivalent. □

Lemma 9.1.5 *A variety X over k which contains a smooth k-point is split.*

Proof. Let P be a smooth k-point of X. Then there exists a smooth irreducible Zariski open set $U \subset X$ which contains P. In particular, U is geometrically reduced. Lemma 9.1.2 gives $k_U = k$, so U is geometrically irreducible. □

Split fibres

Proposition 9.1.6 *Let R be a regular local ring with residue field k, maximal ideal $\mathfrak{m}$ and field of fractions K. Let $f : X \to \mathrm{Spec}(R)$ be an R-scheme of finite type such that X is regular and the generic fibre X_K is a smooth K-scheme. Let $i : R \hookrightarrow R'$ be an extension of local rings such that $\mathfrak{m}$ generates the maximal ideal $\mathfrak{m}' \subset R'$ and the residue field k' of R' is a separable extension of k (not necessarily algebraic). Then any morphism $\sigma : \mathrm{Spec}(R') \to X$ such that $f\sigma = i^*$ factors through the smooth locus of $f : X \to \mathrm{Spec}(R)$.*

Proof. It is enough to show that $P = \sigma(\mathrm{Spec}(k'))$ is a smooth point of the closed fibre X_k. Let A be the local ring of X at P with maximal ideal $\mathfrak{m}_A \subset A$ and residue field $k(P) = A/\mathfrak{m}_A$. We have homomorphisms of local rings $f^* : R \to A$ and $\sigma^* : A \to R'$ such that $\sigma^* f^* = i$. They induce embeddings of residue fields $k \subset k(P) \subset k'$. The induced maps $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}'/\mathfrak{m}'^2$ are linear maps of k-vector spaces such that the composition $i^* : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}'/\mathfrak{m}'^2$ is induced by $i : R \hookrightarrow R'$. We claim that the k'-vector space $\mathfrak{m}'/\mathfrak{m}'^2$ is obtained from the k-vector space $\mathfrak{m}/\mathfrak{m}^2$ by extending scalars from k to k'. Indeed, tensoring the exact sequence of R-modules

$$0 \longrightarrow \mathfrak{m}^2 \longrightarrow \mathfrak{m} \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow 0$$

with R', using $R'\mathfrak{m} = \mathfrak{m}'$ which implies $R'\mathfrak{m}^2 = \mathfrak{m}'^2$, we obtain an isomorphism $\mathfrak{m}'/\mathfrak{m}'^2 \xrightarrow{\sim} (\mathfrak{m}/\mathfrak{m}^2) \otimes_k k'$ whose composition with $i^* : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}'/\mathfrak{m}'^2$ is the natural map $\mathfrak{m}/\mathfrak{m}^2 \to (\mathfrak{m}/\mathfrak{m}^2) \otimes_k k'$. It follows that $f^* : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is injective.

Thus if $\{s_1, \ldots, s_m\} \subset \mathfrak{m}$, where $m = \dim(R)$, is a regular system of parameters of R, then $f^*(s_1), \ldots, f^*(s_m)$ can be completed to a regular system of parameters of A, that is, there exist $t_{m+1}, \ldots, t_n \in \mathfrak{m}_A$ such that $f^*(s_1), \ldots, f^*(s_m), t_{m+1}, \ldots, t_n$ is a regular system in A. Indeed, it is enough to choose $t_{m+1}, \ldots, t_n \in \mathfrak{m}_A$ such that the classes of $f^*(s_1), \ldots, f^*(s_m), t_{m+1}, \ldots, t_n$ form a basis of the $k(P)$-vector space $\mathfrak{m}_A/\mathfrak{m}_A^2$. Since X is regular, A is a regular local ring, so $\dim(A) = n$.

The quotient $B = A/(\mathfrak{m} \otimes_R A)$ is the local ring of X_k at P; its maximal ideal is $\mathfrak{m}_B = \mathfrak{m}_A/(\mathfrak{m} \otimes_R A)$ and its residue field is $B/\mathfrak{m}_B = k(P)$. The $k(P)$-vector space $\mathfrak{m}_B/\mathfrak{m}_B^2$ has a basis consisting of the images of $t_{m+1}, \ldots, t_n$, hence $\dim(\mathfrak{m}_B/\mathfrak{m}_B^2) = n - m = \dim(A) - \dim(R) \leq \dim(B)$, see [Liu10, Thm. 4.3.12] for the last inequality. But $\dim(B) \leq \dim(\mathfrak{m}_B/\mathfrak{m}_B^2)$ for any local ring B, so $\dim(B) = \dim(\mathfrak{m}_B/\mathfrak{m}_B^2)$ so that B is a regular local ring. Finally, the residue field of B is $k(P) \subset k'$, which is separable over k since k'/k is separable, so P is smooth in X_k. □

Corollary 9.1.7 *Let R be a regular local ring with residue field k. Let X be a regular scheme which is an R-scheme of finite type. If the morphism $X \to \mathrm{Spec}(R)$ has a section, then this section meets the closed fibre X_k in a smooth k-point. Hence X_k is a split k-variety.*

Proof. Taking $R' = R$ in Proposition 9.1.6 we obtain a smooth k-point P in the closed fibre X_k. The last statement now follows from Lemma 9.1.5. □

A variety Z over a field k is *geometrically split* if the k_{s}-scheme $Z^{\mathrm{s}} = Z \times_k k_{\mathrm{s}}$ is split. Equivalently, Z contains a non-empty smooth open subscheme. In particular, a variety over k is geometrically split if and only if it contains a smooth closed point.

Corollary 9.1.8 *Let $f : X \to Y$ be a dominant, proper and flat morphism of regular varieties over a field k. Let P be a point of Y. The fibre X_P is geometrically split if and only if f has a section locally at P for the étale topology, i.e. the morphism $X \times_Y \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ has a section, where R is the strict henselisation of the local ring of Y at P.*

Proof. Let $X' = X \times_Y R$ and let X'_0 be the closed fibre of X'/R. It is enough to show that X'_0 is split if and only if X'/R has a section.

The Y-scheme $\mathrm{Spec}(R)$ is a direct limit of étale schemes V/Y, thus X' is a limit of $V \times_Y X$. But $V \times_Y X$ is étale over a regular scheme X, hence X' is regular. Now R is a regular local ring and X' is regular, so if X'/R has a section, then X'_0 is split by Corollary 9.1.7.

Conversely, since X'_0 is split over a separably closed field, Proposition 9.1.4 (iv) implies that X'_0 has a smooth rational point P. By assumption the morphism $X \to Y$ is flat, so X' is a flat R-scheme. Hence the morphism $X' \to \mathrm{Spec}(R)$

is smooth in a neighbourhood of P. Since R is henselian, P can be lifted to a section of X'/R. □

In the case of a regular integral scheme over a discrete valuation ring, the multiplicity of an irreducible component of the closed fibre has a clear geometric meaning.

Lemma 9.1.9 *Let R be a discrete valuation ring with maximal ideal $\mathfrak{m} = (\pi)$ and residue field $k = R/\mathfrak{m}$. Let X be a regular integral scheme with a faithfully flat morphism $f : X \to \mathrm{Spec}(R)$. Then the (non-empty) closed fibre X_t is the principal divisor*

$$(\pi) = \sum_{i=1}^{n} m_i C_i \ \in \ \mathrm{Div}(X),$$

where $C_1, \ldots, C_n$ are the (reduced) irreducible components of X_k, and m_i is the multiplicity of C_i, for $i = 1, \ldots, n$.

Proof. Since f is faithfully flat, X_k is non-empty, and each C_i is a divisor on X. Since X is regular, each C_i is a Cartier divisor and the local ring $\mathcal{O}_{X,C_i}$ of X at the generic point of C_i is a discrete valuation ring. The local ring of X_k at the generic point of C_i is $\mathcal{O}_{X,C_i}/\pi\mathcal{O}_{X,C_i}$, which by assumption is a local Artinian ring of length m_i. Hence the valuation of π is m_i. Thus the Cartier divisors $X_k = (\pi)$ and $\sum_{i=1}^{n} m_i C_i$ coincide at codimension 1 points of X; this implies that they coincide as Cartier divisors on X. □

Proposition 9.1.10 *Let R be a discrete valuation ring with field of fractions K, maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Let Y and Y' be regular, integral and flat R-schemes of finite type, with smooth generic fibres Y_K and Y'_K. Assume that Y' is a proper R-scheme. If there is a rational map from Y_K to Y'_K, then for any irreducible component $C \subset Y_k$ of geometric multiplicity 1 there exists an irreducible component $C' \subset Y'_k$ of geometric multiplicity 1 such that $k_{C'} \subset k_C$. In particular, if Y_k is split, then Y'_k is split too.*

Proof. Write $F = K(Y_K)$. Let $\mathcal{O}_C$ be the local ring of Y at the generic point of C. Since Y is integral, the field of fractions of $\mathcal{O}_C$ is F; the residue field of $\mathcal{O}_C$ is $k(C)$. Since Y is regular, $\mathcal{O}_C$ is a discrete valuation ring. Since Y/R is flat, $\mathcal{O}_C$ is a flat, hence torsion-free R-module, so the natural homomorphism $R \to \mathcal{O}_C$ is injective. The multiplicity of C is 1, so Lemma 9.1.9 shows that the maximal ideal of $\mathcal{O}_C$ is $\mathfrak{m} \otimes_R \mathcal{O}_C = \mathfrak{m}\mathcal{O}_C$. Moreover, the geometric multiplicity of C is 1, and as was noted in the discussion following Definition 9.1.1, this implies that the smooth locus C_{smooth} is a dense open subscheme of C.

Let $X \subset Y$ be the open subscheme obtained by removing from the closed fibre Y_k all the irreducible components other than C, and then removing the closed subset $C \setminus C_{\mathrm{smooth}}$. The natural morphism $X \to \mathrm{Spec}(R)$ is smooth. Indeed, X is flat over R, with smooth generic fibre $X_K = Y_K$ and smooth closed fibre C_{smooth}. The local ring of the closed fibre of X/R is $\mathcal{O}_C$.

As X is smooth over R, the projection $Y' \times_R X \to Y'$ is smooth. But Y' is regular, so $Y' \times_R X$ is regular too. Hence $Y' \times_R \mathcal{O}_C$ is regular. Since the maximal ideal of $\mathcal{O}_C$ is $\mathfrak{m}\mathcal{O}_C$, the closed fibre of $Y' \times_R \mathcal{O}_C \to \mathcal{O}_C$ is $Y'_k \times_k k(C)$.

A rational map from Y_K to Y'_K can be thought of as an F-point of Y'_K. Recall that $F = K(Y_K)$ is the field of fractions of $\mathcal{O}_C$. The morphism $Y' \times_R \mathcal{O}_C \to \mathcal{O}_C$ is proper, so by the valuative criterion of properness any F-point of its generic fibre extends to a section of the morphism. A section of $Y' \times_R \mathcal{O}_C \to \mathcal{O}_C$ gives rise to a $k(C)$-point P of the closed fibre $Y'_k \times_k k(C)$.

Since $Y' \times_R \mathcal{O}_C$ is regular and of finite type over $\mathcal{O}_C$, any section meets the closed fibre at a smooth point (Corollary 9.1.7), therefore P is a smooth point of $Y'_k \times_k k(C)$. This defines a morphism $\mathrm{Spec}(k(C)) \to Y'_k$ whose image is in $Y'_{k,\mathrm{smooth}}$. Let U be the connected component of $Y'_{k,\mathrm{smooth}}$ containing the image of P. The Zariski closure of U in Y'_k is an irreducible component $C' \subset Y'_k$ of geometric multiplicity 1. The morphism $U \to \mathrm{Spec}(k)$ factors through $U \to \mathrm{Spec}(k_{C'})$. The composition $\mathrm{Spec}(k(C)) \to P \to U \to \mathrm{Spec}(k_{C'})$ gives rise to an embedding $k_{C'} \subset k(C)$, hence $k_{C'} \subset k_C$, as required.

Finally, since Y_k is split if and only if Y_k contains an irreducible component C of geometric multiplicity 1 such that $k_C = k$ (Proposition 9.1.4), we see that C' has the same properties, hence Y'_k is split. □

Corollary 9.1.11 *Let R be a discrete valuation ring with field of fractions K and residue field k. Let X be a regular, integral, proper and flat R-scheme of finite type, with smooth generic fibre. Let Σ_X be the (possibly, empty) partially ordered set of irreducible components of geometric multiplicity 1 of X_k, where C dominates D if there exists an embedding of k_D into k_C. The set of finite separable field extensions $k \subset k_C$, where C is a minimal element of Σ_X, is a birational invariant of the generic fibre X_K as a smooth, integral, proper variety over K. In particular, the property of the closed fibre X_k to be split is a birational invariant of X_K.*

Proof. Suppose that X and Y are regular, integral, proper and flat R-schemes, with smooth generic fibres, such that $K(X_K) \cong K(Y_K)$. Define the partially ordered set Σ_Y in the same way as Σ_X. Let C be a minimal element of Σ_X. By Proposition 9.1.10 there exists a $C' \in \Sigma_Y$ such that $k_{C'}$ can be embedded into k_C. By the same proposition, there is a $C'' \in \Sigma_X$ such that $k_{C''}$ can be embedded into $k_{C'}$. By minimality of C we have $k_{C''} \simeq k_C$, hence $k_C \simeq k_{C'}$. Since C is minimal in Σ_X, then, by Proposition 9.1.10, C' is minimal in Σ_Y. The last statement then follows from the fact that X_k is split if and only if there is a $C \in \Sigma_X$ such that $k_C = k$, see Proposition 9.1.4. □

In some concrete cases, for example when the generic fibre is a quadric and the residue field if of characteristic different from 2, it is not difficult to determine this set of finite separable extensions.

One can give a criterion for the closed fibre to be split in terms of the generic fibre [Sko96, Lemma 1.1].

Theorem 9.1.12 *Let R be a discrete valuation ring with field of fractions K and residue field k. Let X be a regular, integral, proper and flat R-scheme of finite type, with smooth generic fibre. Then the closed fibre X_k is split if and only if there exists a flat local homomorphism of discrete valuation rings $i : R \to R'$ satisfying the following properties, where k' is the residue field of R' and K' is the fraction field of R':*

(a) *k' is a separable extension of k, and k is algebraically closed in k';*
(b) *the maximal ideal of R generates the maximal ideal of R';*
(c) *the generic fibre X_K has a K'-point.*

Following Bourbaki, an extension k'/k satisfying the conditions in (a) is called *regular* [BouV, §17, no. 4, Déf. 2].

Proof of Theorem 9.1.12. Assume that X_k is split, so that X_k contains a non-empty open geometrically integral subscheme U. Let R' be the local ring of X at the generic point of U. Since X is flat over R, the Zariski closure of U has codimension 1 in X. Then since X is regular, R' is a discrete valuation ring. It is clear that the residue field of R' is $k(U)$ and the fraction field is $k(X)$. Since U is geometrically integral over k, the field $k(U)$ is a separable extension of k in which k is algebraically closed, so (a) is satisfied. The multiplicity of U is 1, so Lemma 9.1.9 shows that the maximal ideal of R' is generated by the maximal ideal of R, which is (b). Finally, the generic point of X_K is a K'-point, so (c) holds as well.

To prove the converse, let $i : R \to R'$ be as in the statement of the theorem. By the valuative criterion of properness, the given K'-point of X_K extends to an R-morphism $\phi : \mathrm{Spec}(R') \to X$. Since the field extension $k \subset k'$ is separable, by Proposition 9.1.6 the morphism ϕ factors through the smooth locus X_{smooth} of X/R. Let $P = \phi(\mathrm{Spec}(k'))$ be the image of the closed point of $\mathrm{Spec}(R')$ in $X_{\mathrm{smooth}} \cap X_k$. It follows that X_k contains an open irreducible smooth subscheme U such that $P \in U$.

Let us show that U is geometrically integral. Since U is smooth over k, it is geometrically reduced. By Lemma 9.1.2 applied to the morphism of k-schemes $\phi : \mathrm{Spec}(k') \to U$, the field k_U is a subfield of the algebraic closure of k in k'. But k is algebraically closed in k' by assumption, hence $k_U = k$, so U is geometrically irreducible. □

Under the additional assumption that R' is finitely generated as an R-algebra, this statement follows from Proposition 9.1.10. However, the case when R' is not a finitely generated R-algebra (or a localisation of such an algebra) is of greater interest, e.g., the case when R' contains the completion of R, because it is usually easier to find a K'-point in X_K when R' is complete.

As an example of application of this theorem let us prove the following

Proposition 9.1.13 *Let k be a field of characteristic 0. Let $f : X \to Y$ be a proper dominant morphism of smooth and geometrically integral varieties over k. Assume that the generic fibre X_η is birationally equivalent to a $k(Y)$-torsor for a simply connected semisimple group over $k(Y)$. Then for any point $y \in Y$ of codimension 1, the fibre X_y is split.*

Proof. Write $\kappa = k(y)$. The completion of the local ring $\mathcal{O}_y$ of Y at y is isomorphic to $\kappa[[t]]$. Let $\kappa \subset L$ be a field extension as in Lemma 5.2.7. As $\mathrm{cd}(L) \leq 1$, by Theorem 8.2.2 any torsor for a simply connected semisimple group over $L((t))$ has an $L((t))$-point. The generic fibre X_η of the morphism $f: X \to Y$ is a proper variety over $k(\eta) = k(Y)$ birationally equivalent to such a homogeneous space. By the lemma of Lang and Nishimura, X_η has an $L((t))$-point. The local extension of discrete valuation rings $\mathcal{O}_y \subset L[[t]]$ satisfies the conditions of Theorem 9.1.12, so by this theorem the fibre X_y is split. □

Let us give an example (taken from [LS18]) when one can determine if the closed fibre is split using only the information about the birational equivalence class of the generic fibre without constructing an explicit model.

Proposition 9.1.14 *Let k be a field of characteristic 0. Let $k_1, \ldots, k_n$ be finite field extensions of k, and let $m_1, \ldots, m_n$ be positive integers such that*

$$\text{g.c.d.}\big(m_1, \ldots, m_n\big) = 1.$$

Let m be an integer and let X be the affine $k((t))$-variety with equation

$$\prod_{i=1}^{n} \mathrm{N}_{k_i/k}(x_i)^{m_i} = t^m, \tag{9.1}$$

where x_i is a k_i-variable. Let $\mathcal{X}$ be a regular scheme equipped with a proper morphism $\mathcal{X} \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth, geometrically integral, and contains X as an open subscheme. Then the closed fibre $\mathcal{X}_k$ is split if and only if $r|m$, where

$$r = \text{g.c.d.}\big(m_1[k_1 : k], \ldots, m_n[k_n : k]\big).$$

Proof. Equation (9.1) with right hand side replaced by 1 defines a k-torus. Hence X is a $k((t))$-torsor for this torus; in particular, it is geometrically integral.

If $r|m$ we can write $m = s_1 m_1[k_1 : k] + \ldots + s_n m_n[k_n : k]$ for some $s_i \in \mathbb{Z}$. Then $x_i = t^{s_i}$, for $i = 1, \ldots, n$, is a $k((t))$-point of X. By the valuative criterion of properness, it gives rise to a section of $\mathcal{X} \to \mathrm{Spec}(k[[t]])$. By Corollary 9.1.7 the closed fibre $\mathcal{X}_k$ is split.

Conversely, assume that $\mathcal{X}_k$ is split, so $\mathcal{X}_k$ has a geometrically irreducible component C of multiplicity 1. Let $\mathcal{O}_C$ be the local ring of C in $\mathcal{X}$. This a discrete valuation ring with field of fractions $k((t))(X)$ and residue field $k(C)$. Let $A = \widehat{\mathcal{O}}_C$ be the completion of $\mathcal{O}_C$. This is also a discrete valuation ring with residue field $k(C)$. Let K be the field of fractions of A and let $v : K^* \to \mathbb{Z}$ be the valuation. Then $k[[t]] \subset A$ is an unramified extension of complete discrete valuation rings, so $v(t) = 1$. In fact, A is isomorphic to $k(C)[[t]]$. Since C is geometrically irreducible, k is algebraically closed in $k(C)$, hence also in K.

The generic fibre X has a canonical $k((t))(X)$-point Q defined by the generic point of X. This point is contained in the affine open subset given by (9.1). Since $k((t))(X) \subset K$, we can think of Q as a K-point of X. Suppose that Q has coordinates (x_i), where $x_i \in K \otimes_k k_i$ for $i = 1, \ldots, n$. Since k is algebraically closed in K, the k-algebra $K_i = K \otimes_k k_i$ is a field, hence K_i is a complete local field which is an unramified extension of K of degree $[k_i : k]$. This implies that $v(\mathrm{N}_{K_i/K}(x_i)) = s_i[k_i : k]$ for some $s_i \in \mathbb{Z}$. But then (9.1) gives that $m = s_1 m_1[k_1 : k] + \ldots + s_n m_n[k_n : k]$, so we are done. □

We refer to [CT11] for further discussion and applications of the type of results discussed here.

9.2 Quadrics over a discrete valuation ring

In this section R is a discrete valuation ring with fraction field K. Let $\mathfrak{m} \subset R$ be the maximal ideal and let $k = R/\mathfrak{m}$ be the residue field. We assume that $\mathrm{char}(k) = 0$. For $a \in R$ we denote by $\bar{a} \in k$ the reduction of a modulo $\mathfrak{m}$.

Conics over a discrete valuation ring

Let X be a smooth conic over K. It has a regular model $\mathcal{X} \subset \mathbb{P}^2_R$ given either by an equation

$$x^2 - ay^2 - bz^2 = 0$$

with $a, b \in R^*$ (which we refer to as case (I)), or by an equation

$$x^2 - ay^2 - \pi z^2 = 0,$$

where $a \in R^*$ and π is a uniformizing parameter (which we refer to as case (II)). In fact, if $\bar{a}$ is a square in k, then the conic X also has a model of type (I).

Proposition 9.2.1 *Let $W \to \mathrm{Spec}(R)$ be a proper flat morphism such that W is regular and connected, and the generic fibre of $W \to \mathrm{Spec}(R)$ is a smooth conic over K. Then the natural map $\mathrm{Br}(R) \to \mathrm{Br}(W)$ is surjective.*

Proof. Let X be the generic fibre of $W \to \mathrm{Spec}(R)$ and let $\mathcal{X} \to \mathrm{Spec}(R)$ be the integral model of X given above. By a special case of Proposition 3.7.9 that only involves purity for regular 2-dimensional schemes (which has been known for some time, see [Gro68, II, Prop. 2.3]), there is an isomorphism $\mathrm{Br}(W) \simeq \mathrm{Br}(\mathcal{X})$ compatible with the maps $\mathrm{Br}(R) \to \mathrm{Br}(W)$ and $\mathrm{Br}(R) \to \mathrm{Br}(\mathcal{X})$.

Thus we can assume that $W = \mathcal{X}$ as above. The conic X over K is a Severi–Brauer variety of dimension 1. The exact sequence (6.1) shows that the map $\mathrm{Br}(K) \to \mathrm{Br}(X)$ is surjective. Since $\mathrm{char}(K) \neq 2$, its kernel is spanned by the class of the quaternion algebra $(a, b)_K$ in case (I) and $(a, \pi)_K$ in case (II).

Pick any $\beta \in \mathrm{Br}(\mathcal{X})$. Let β_K be the image of β under the injective map $\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(X)$. Let $\alpha \in \mathrm{Br}(K)$ be any element mapping to β_K. Consider the exact sequence

$$0 \longrightarrow \mathrm{Br}(R) \longrightarrow \mathrm{Br}(K) \longrightarrow \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$$

from Proposition 3.6.1 (i). Comparing residues on $\mathrm{Spec}(R)$ and on $\mathcal{X}$ using Theorem 3.7.4 one shows that the residue $\delta_R(\alpha)$ is either 0 or is equal to the non-trivial class in $\mathrm{H}^1(k(\sqrt{\bar{a}})/k, \mathbb{Z}/2)$, and this last case may happen only in case (II). In the first case we have $\alpha \in \mathrm{Br}(R)$, hence the images of α and β in $\mathrm{Br}(X)$ coincide, thus they also coincide in $\mathrm{Br}(\mathcal{X})$ since X is regular. In the second case we have

$$\delta_R(\alpha) = \delta_R((a, \pi))$$

hence $\alpha = (a, \pi) + \gamma$ with $\gamma \in \mathrm{Br}(R)$. We then get

$$\beta = (a, \pi)_{K(X)} + \gamma_{K(X)} \in \mathrm{Br}(K(X)).$$

But $(a, \pi)_{K(X)} = 0$. Thus $\beta - \gamma_{\mathcal{X}} \in \mathrm{Br}(\mathcal{X}) \subset \mathrm{Br}(K(X))$ vanishes, hence $\beta = \gamma_{\mathcal{X}} \in \mathrm{Br}(\mathcal{X})$. The map $\mathrm{Br}(R) \to \mathrm{Br}(\mathcal{X})$ is thus surjective. This proves the statement for $\mathcal{X}$, and hence also for W. □

Corollary 9.2.2 *In the notation of Proposition 9.2.1 let $Y \subset W$ be a divisor which is an irreducible component of multiplicity* 1 *of the closed fibre of $W \to \mathrm{Spec}(R)$. Then the image of the restriction map $\mathrm{Br}(W) \to \mathrm{Br}(Y)$ is contained in the image of $\mathrm{Br}(k) \to \mathrm{Br}(Y)$.*

Quadric surfaces over a discrete valuation ring

The references for this section are [Sko90], [CTS93, §3], [CTS94, Thm. 2.3.1], and [Pir18, Thm. 3.17].

Let $X \subset \mathbb{P}^3_K$ be a smooth quadric, defined by a quadratic form q of rank 4 over K. By a linear change of variables and multiplication of q by an element of K^* we can reduce q to one of the following forms.

(I) $q = \langle 1, -a, -b, abd \rangle$, where $a, b, d \in R^*$.
(II) $q = \langle 1, -a, -b, \pi \rangle$, where $a, b \in R^*$ and $\pi \in R$ is a uniformiser.
(III) $q = \langle 1, -a, -\pi, \pi b \rangle$, where $a, b \in R^*$ and $\pi \in R$ is a uniformiser.

In case (III) the discriminant of q is the class of ab in R^*/R^{*2}. Its image $\overline{a} \cdot \overline{b} \in k^*$ is a square if and only if ab is a square in the completion of K with respect to the valuation of R.

Let $\mathcal{X} \subset \mathbb{P}^3_R$ be the subscheme $q = 0$. Let Y/k be the closed fibre of $\mathcal{X}/R$.

In case (I) the morphism $\mathcal{X} \to \mathrm{Spec}(R)$ is smooth.

In case (II) the scheme $\mathcal{X}$ is regular and Y is a cone over a smooth conic.

In case (III) the closed fibre Y is given by the equation $x^2 - \bar{a}y^2 = 0$ in $\mathbb{P}^3_k$. If $\bar{a}$ is a square, this is the union of two planes intersecting along the line $x = y = 0$. If $\bar{a}$ is not a square, this is an integral scheme which splits up over $k(\sqrt{a})$ as the union of two planes. In each case the scheme $\mathcal{X}$ is singular at the points $x = y = 0$, $z^2 - \bar{b}t^2 = 0$. (See [Sko90, §2].)

Proposition 9.2.3 *In case* (III) *let W be a projective, regular, integral scheme over R such that there is a birational R-morphism $W \to \mathcal{X}$. Then we have the following statements.*

In case (I) *the map $\mathrm{Br}(R) \to \mathrm{Br}(\mathcal{X})$ is surjective. If $d \in R$ is not a square, this map is an isomorphism. If d is a square, the kernel is spanned by the class $(a, b) \in \mathrm{Br}(R)$.*

In case (II) *the map* $\mathrm{Br}(R)\to\mathrm{Br}(\mathcal{X})$ *is an isomorphism.*

In case (III)*, if either* $\bar{a}$ *or* $\bar{b}$ *is a square in* k*, or* $\bar{a}\cdot\bar{b}$ *is not a square in* k*, then* $\mathrm{Br}(R)\to\mathrm{Br}(W)$ *is surjective. Any element of* $\mathrm{Br}(K)$ *whose image in* $\mathrm{Br}(X)$ *lies in* $\mathrm{Br}(W)$ *belongs to* $\mathrm{Br}(R)$*.*

In case (III)*, if* $\bar{a}\cdot\bar{b}$ *is a square in* k*, then the image of* $(a,\pi)\in\mathrm{Br}(K)$ *in* $\mathrm{Br}(X)$ *belongs to* $\mathrm{Br}(W)$ *and spans the cokernel of the map* $\mathrm{Br}(R)\to\mathrm{Br}(W)$*. If, moreover,* $\bar{a}$ *is not a square in* k*, then this cokernel is non-zero.*

Proof. To make our notation uniform, in cases (I) and (II) we set $W=\mathcal{X}$. By [Har77, Prop. III.9.7], the morphism $f: W\to\mathrm{Spec}(R)$ is flat; since f is projective, it is also surjective. Thus each fibre of f has dimension 2 at every point. Let $Y=W\times_{\mathrm{Spec}(k)}\mathrm{Spec}(R)$ be the closed fibre of f. Each irreducible component x of Y of multiplicity e gives rise to a commutative diagram

$$\begin{array}{ccccccc} & & & & \mathrm{Br}(X) & \xrightarrow{\partial_x} & \mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z}) \\ & & & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathrm{Br}(R) & \longrightarrow & \mathrm{Br}(K) & \xrightarrow{\partial} & \mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z}) \end{array}$$

Here the bottom exact sequence is given by Proposition 3.6.1 (ii) and the right hand vertical arrow is the restriction map followed by multiplication by e (by the functoriality of residues, see Theorem 3.7.4). Since X is a smooth quadric over K, the middle vertical map is surjective by Proposition 6.2.3 (a). Since W is regular, the group $\mathrm{Br}(W)$ is the intersection of the kernels $\mathrm{Ker}(\partial_x)$, for all irreducible components x of Y.

In cases (I) and (II), the closed fibre Y is geometrically integral over k, so $x=Y$ and $e=1$, hence the map $\mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z})$ is injective. This is enough to prove the claim in these cases.

Let us consider case (III). Let $\alpha\in\mathrm{Br}(K)$ be such that the image of α in $\mathrm{Br}(X)$ belongs to $\mathrm{Br}(W)$. If $\bar{a}$ is a square, then the closed fibre of $\mathcal{X}\to\mathrm{Spec}(R)$ contains a geometrically integral component of multiplicity 1 which is one of the two components of $x^2-\bar{a}y^2=0$. It gives rise to a geometrically integral component x of Y of multiplicity 1. The above diagram then implies that $\partial(\alpha)=0$, so $\alpha\in\mathrm{Br}(R)$. If $\bar{b}$ is a square, we consider the quadratic form $q'=\langle 1,-b,-\pi,\pi a\rangle$. Since $X\subset\mathbb{P}^3_K$ can be also given by $q'=0$, we can apply the same argument.

Now assume that neither $\bar{a}$ nor $\bar{b}$ is a square. The closed fibre of $\mathcal{X}\to\mathrm{Spec}(R)$ is the integral subscheme of $\mathbb{P}^3_k$ given by $x^2-\bar{a}y^2=0$. It gives rise to an integral component x of Y of multiplicity 1 such that the integral closure of k in $k(x)$ is $k(\sqrt{\bar{a}})$. From the diagram it follows that $\partial(\alpha)$ belongs to

$$\mathrm{Ker}[\mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z})]=\mathrm{Ker}[\mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^1(k(\sqrt{\bar{a}}),\mathbb{Q}/\mathbb{Z})],$$

which is the $\mathbb{Z}/2$-module generated by the class of $\bar{a}$ in k^*/k^{*2}. Applying this argument to the model given by $q'=0$ we obtain that $\partial(\alpha)$ belongs to the $\mathbb{Z}/2$-module spanned by the class of $\bar{b}$ in k^*/k^{*2}. If $\bar{a}\cdot\bar{b}$ is not a square, we conclude that $\partial(\alpha)=0$, proving the statement.

Finally, let $\bar{a}\cdot\bar{b}$ be a square, whereas neither $\bar{a}$ nor $\bar{b}$ is a square. If $\partial(\alpha)\neq 0$, then $\partial(\alpha)=\bar{a}=\partial((a,\pi))$. We now show that (a,π) has trivial residues on W. We actually prove the triviality of residues of (a,π) with respect to any rank one discrete valuation v of the function field $K(X)$ of X. It is enough to consider only those v which extend the valuation of K defined by R. In $K(X)$ we have

$$x^2-ay^2=\pi(z^2-b),$$

where both sides are non-zero. Thus in $\mathrm{Br}(K(X))$ we have the equality

$$(a,\pi)=(a,x^2-ay^2)+(a,z^2-b)=(a,z^2-b),$$

since $(a,x^2-ay^2)=0$ by Proposition 1.1.7. To compute residues, we can go over to the field extension $\widehat{K}\subset\widehat{K(X)}$, where $\widehat{K}$ is the completion of K and $\widehat{K(X)}$ is the completion of $K(X)$ defined by v. We have $ab\in\hat{K}^{*2}$, hence $ab\in\widehat{K(X)}^{*2}$. But then in $\mathrm{Br}(\widehat{K(X)})$ we have $(a,z^2-b)=(b,z^2-b)=0$. Hence the residue of (a,π) at v is zero. □

The following statement is a stronger version of Corollary 1.4.8 in the situation considered here.

Corollary 9.2.4 *In the notation of Proposition 9.2.3 let $D\subset W$ be an integral divisor contained in the closed fibre of $W\to\mathrm{Spec}(R)$. Then the image of the restriction map $\mathrm{Br}(W)\to\mathrm{Br}(k(D))$ is contained in the image of $\mathrm{Br}(k)\to\mathrm{Br}(k(D))$.*

Proof. This is clear when the map $\mathrm{Br}(R)\to\mathrm{Br}(W)$ is surjective. It remains to consider case (III) when $\bar{a}$ and $\bar{b}$ are not squares, but $\bar{a}\cdot\bar{b}$ is. To prove the result, we may assume that R is henselian. Then ab is a square in R. By Proposition 9.2.3, the group $\mathrm{Br}(W)$ is generated by the image of $\mathrm{Br}(R)$ and the image of the class (a,π). The equation of the quadric X can be written as

$$X^2-aY^2-\pi Z^2+a\pi T^2=0.$$

Proposition 1.1.7 implies that the image of (a,π) in $\mathrm{Br}(X)$ is zero, hence the image of (a,π) in $\mathrm{Br}(W)\subset\mathrm{Br}(X)$ is zero. □

Remark 9.2.5 If R is a *henselian* discrete valuation ring, then the proof of Corollary 9.2.4 also shows that the map $\mathrm{Br}(R)\to\mathrm{Br}(W)$ is surjective in all cases.

9.3 Two-dimensional schemes over a henselian local ring

Let D be a central simple algebra over a field F. Let $\mathrm{ind}(D)$ be the *index* of D, that is, the square root of the dimension of the division algebra representing the class $[D]\in\mathrm{Br}(F)$. The index $\mathrm{ind}(D)$ can be also characterised as the smallest degree of a field extension of F that splits D. Let $\exp(D)$ be the *exponent* of D, that is, the order of $[D]$ in $\mathrm{Br}(F)$. The following facts were established in the 1930s by Brauer, Albert and others, see [Alb31].

- $\exp(D)$ divides $\mathrm{ind}(D)$; moreover, the primes which divide $\exp(D)$ are the same as the primes which divide $\mathrm{ind}(D)$.
- Let F be a number field or a p-adic field. Every central division algebra D over F of exponent $\exp(D)=n$ is cyclic of degree n, hence is split by a cyclic extension of F of degree n. In particular, $\mathrm{ind}(D)=\exp(D)$.
- If F is a number field and D splits over each completion of F, then D splits over F (the Albert–Brauer–Hasse–Noether theorem).
- Every central simple algebra over the function field of a curve over $\mathbb{C}$ is split (Tsen's theorem).

Such properties have applications to quadratic forms over F: the local-to-global principle for a quadratic form to be isotropic (i.e. to have the zero value on some non-zero vector) and the determination of the u-invariant of F (the maximum dimension of an anisotropic quadratic form over F).

One may wonder whether similar properties hold for other 'arithmetic fields'. Among the first examples one can think of are field extensions of $\mathbb{C}$ of transcendence degree 2. In this case the equality of index and exponent was established relatively recently by de Jong [deJ04]. One may also consider more local situations, such as function fields in one variable over $\mathbb{C}((t))$ or the purely local situation, that is, finite extensions of $\mathbb{C}((x,y))$. Further up the cohomological dimension there are function fields of curves over a p-adic field. As early as 1970, Lichtenbaum [Lic69], using Tate's duality theorems for abelian varieties over a p-adic field, established a local-to-global principle in this context. Later, Saltman [Sal97] showed that over such a field the index divides the square of the exponent.

We shall explain some of these results. Our starting point is the following theorem which is a more general version of a theorem of Artin about families of curves over a henselian discrete valuation ring (written up by Grothendieck [Gro68, III, Thm. (3.1)]).

Theorem 9.3.1 *Let R be a henselian local ring with residue field k. Let X be a regular scheme of dimension 2 equipped with a proper morphism $X\to\mathrm{Spec}(R)$ whose closed fibre X_0 has dimension 1. Then we have the following statements.*

(i) *The natural map $\mathrm{Br}(X)\to\mathrm{Br}(X_0)$ is an isomorphism.*

(ii) *If k is separably closed or finite, then $\mathrm{Br}(X)=0$.*

Proof. For part (i) see [CTOP02, Thm. 1.8, Remark 1.8.1]. Part (ii) then follows from Theorem 4.5.1 (iv) and (v). □

Remark 9.3.2 (a) For ℓ invertible in k, the ℓ-primary part of this theorem is relatively easy to prove using the Kummer exact sequence [CTOP02, Thm. 1.3].

(b) When R is a discrete valuation ring, Theorem 9.3.1 removes the excellence hypothesis in Artin's theorem [Gro68, III, Thm. 3.1].

(c) The following two situations are of particular interest.

- The "semi-global" case: R is a henselian discrete valuation ring, X is integral, and the generic fibre of $X \to \mathrm{Spec}(R)$ is smooth and geometrically integral. This is the case considered in [Gro68, III, §3], with the additional hypothesis that the discrete valuation ring is excellent (used only to handle the p-torsion part of the theorem, where $p = \mathrm{char}(k)$).

- The "local" case: R is a 2-dimensional henselian local domain and the morphism $X \to \mathrm{Spec}(R)$ is birational. Then X is a resolution of singularities of $\mathrm{Spec}(R)$. If R is excellent, such a desingularisation always exists (Hironaka, Abhyankar, Lipman).

In the "semi-global" case, we have the following theorem.

Theorem 9.3.3 *Let R be an excellent henselian discrete valuation ring with residue field k and fraction field K. Let F be the function field of a smooth, projective, connected curve over K. Let D be a central division algebra over F.*

(i) *If k is algebraically closed of characteristic* 0, *then* $\mathrm{ind}(D) = \exp(D)$. *Moreover, D is cyclic and split by a field extension $F(\sqrt[n]{f})$ for some $f \in F^*$.*

(ii) *If k is a finite field, then* $\mathrm{ind}(D) | \exp(D)^2$.

Proof. Let us prove (i). This is a very slight variation on the proof of [CTOP02, Thm. 2.1] which is Theorem 9.3.5 below.

There exists a regular, projective, integral model $X \to \mathrm{Spec}(R)$ of the smooth, projective curve over K with function field F. The purity theorem gives an exact sequence (3.13):

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(F) \longrightarrow \oplus_{x \in X^{(1)}} \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}).$$

By Theorem 9.3.1 (ii), we have $\mathrm{Br}(X) = 0$. Thus the total residue map on $\mathrm{Br}(F)$ is an injection

$$\mathrm{Br}(F) \hookrightarrow \oplus_{x \in X^{(1)}} \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}).$$

Let $n = \exp(D)$ and let $\xi \in \mathrm{Br}(F)[n]$ be the class of D. Let R be the sum of the closures of codimension 1 points of X where ξ has a non-zero residue. By blowing up X we can assume that R is a strict normal crossing divisor. Since $\dim(X) = 2$, the singular locus R_{sing} is a union of closed points; these are the points where any two of the components meet. Replace X by its blow up in R_{sing}, write C for the strict transform of R and write E for the exceptional divisor. Thus D is unramified over $X \setminus (C + E)$ and both C and E are (not necessarily connected) regular curves in X such that $C+E$ has normal crossings. If $C + E = 0$, i.e., if ξ is unramified on X, then $\xi = 0$ and the theorem is clear. We thus assume that $C + E \neq 0$.

Let S be a finite set of closed points of X including all points of intersection of C and E and at least one point of each component of $C + E$. Since X is projective over $\mathrm{Spec}(R)$, there exists an affine open subset $U \subset X$ containing S. The semi-localisation of U at S is a semi-local regular domain, hence a unique factorisation domain. Thus there exists an $f \in F^*$ such that the divisor of f on

X is of the form $\mathrm{div}_X(f) = C+E+G$, where the support of G does not contain any point of S, hence in particular has no common component with $C+E$. Let L/k be the splitting field of the polynomial $T^n - f$. At the generic point of each component of $C+E$, the extension L/F is totally ramified of degree n. In particular, L/F is a field extension of degree n. Let ξ_L be the image of ξ under the restriction map $\mathrm{Br}(F)\to\mathrm{Br}(L)$. To prove (i) it suffices to show that $\xi_L = 0$.

Let X' be the normalisation of X in L and let $\pi : Y\to X'$ be a projective birational morphism such that Y is regular and integral. Let B be the integral closure of R in L. The ring B is a henselian discrete valuation ring with the same residue field k as R. By the universal property of normalisation, the composition $X'\to X\to\mathrm{Spec}(R)$ factors though a projective morphism $X'\to\mathrm{Spec}(B)$, hence induces a projective morphism $Y\to\mathrm{Spec}(B)$. By Theorem 9.3.1 (ii), we have $\mathrm{Br}(Y) = 0$. Just as above, the total residue map on Y defines an injection

$$\mathrm{Br}(L) \hookrightarrow \oplus_{y\in Y^{(1)}}\mathrm{H}^1(\kappa(y), \mathbb{Q}/\mathbb{Z}).$$

It is thus enough to show that ξ_L is unramified on Y. Let $y \in Y$ be a codimension one point. We show that $\partial_y(\xi_L) = 0$. Let $x \in X$ be the image of y under the map $Y\to X'\to X$.

Suppose first that $\mathrm{codim}(x) = 1$. If $\overline{\{x\}}$ is not a component of $C + E$, then $\partial_x(\xi) = 0$, hence, by functoriality of residues, $\partial_y(\xi_L) = 0$. Suppose that $D = \overline{\{x\}}$ is a component of $C + E$. Then f is a uniformising parameter of the discrete valuation ring $\mathcal{O}_{X,x}$. The extension L/F is totally ramified at x. The restriction map $\mathrm{Br}(F)\to\mathrm{Br}(L)$ induces multiplication by the ramification index on the character groups of the residue fields (Proposition 1.4.6). Hence ξ_L is unramified at y.

Suppose now that $\mathrm{codim}(x) = 2$. Note that x is in the closed fibre, hence the residue field $\kappa(x) = k$, which is algebraically closed. If $x \notin C + E$, then $\xi \in \mathrm{Br}(\mathcal{O}_{X,x})$, hence ξ_L is unramified at y. If x is a regular point of $C + E$, then without loss of generality we can assume that x belongs to C but not to E. Let C_0 be the irreducible component of C that contains x, and let $V \subset C_0$ be the complement to the intersection of C_0 with the union of all the other components of C. Then $x \in V$. Let $\pi \in \mathcal{O}_{X,x}$ be a local equation of C at x. (This is also a local equation of V at x.) By the exact sequence (3.11) the residue $\partial_\pi(\xi) \in \kappa(C_0)^*/\kappa(C_0)^{*n}$ comes from an element of $\mathrm{H}^1(V, \mathbb{Z}/n)$. Since C is regular we can choose a $\delta \in \mathcal{O}_{X,x}$ such that (π, δ) is a regular system of parameters of $\mathcal{O}_{X,x}$. As $\partial_\pi(\xi)$ comes from an element of $\mathrm{H}^1(V, \mathbb{Z}/n)$, it goes to zero under the map $\kappa(C_0)^*/\kappa(C_0)^{*n}\to\mathbb{Z}/n$ induced by the valuation defined by x on the field $\kappa(C_0)$, which is the fraction field of the discrete valuation ring $\mathcal{O}_{X,x}/(\pi)$. Thus $\partial_\pi(\xi)$ is the class of a unit of $\mathcal{O}_{X,x}/(\pi)$, and such a unit lifts to a unit μ of $\mathcal{O}_{X,x}$. Now the residues of $\xi-(\mu, \pi)$ at all points of codimension one of $\mathcal{O}_{X,x}$ are trivial. Since $\mathcal{O}_{X,x}$ is a regular two-dimensional ring, this implies that $\xi - (\mu, \pi)$ is the class of an element $\eta \in \mathrm{Br}(\mathcal{O}_{X,x})$. Now

$$\partial_y(\xi_L) = \partial_L((\mu, \pi)) = \overline{\mu}^{v_y(\pi)} \bmod \kappa(y)^{*n}\ ,$$

where $\kappa(y)$ is the residue field of y and $\overline{\mu}$ is the class of μ in $\kappa(y)$. This class comes from $\kappa(x) = k$, which is algebraically closed, therefore $\overline{\mu}$ is an n-th power and $\partial_y(\xi_L) = 0$.

Suppose now that x belongs to $C \cap E$. There exists a regular system of parameters (π, δ) defining (C, E) such that $f = u\pi\delta$, where $u \in \mathcal{O}^*_{X,x}$. Since the ramification of ξ on $\mathrm{Spec}(\mathcal{O}_{X,x})$ is only along π and δ, it can be shown that

$$\xi = \eta + (\pi, \mu_1) + (\delta, \mu_2) + r(\pi, \delta) \ ,$$

for some $\eta \in \mathrm{Br}(\mathcal{O}_{X,x})$, where $\mu_1, \mu_2 \in \mathcal{O}^*_{X,x}$ and $r \in \mathbb{Z}$. (This uses a Bloch–Ogus argument similar to the one used in the proof of Theorem 10.5.1, see [CTOP02] for details.) Since $f = u\pi\delta$, we get

$$(\pi, \delta) = (\pi, fu^{-1}\pi^{-1}) = (\pi, f) + (\pi, -u) \ .$$

The symbol (π, f) vanishes over L and the other symbols become unramified at y.

For the proof of (ii) in the case when the index is coprime to the residual characteristic we refer to [Sal97] (see also the review Zentralblatt Zbl. 0902.16021). This restriction was recently lifted by Parimala and Suresh [PS14]. □.

Remark 9.3.4 The technique used in the proof is essentially that of [FS89] and [Sal97]. For function fields of curves over the field of fractions K of a complete discrete valuation ring R with arbitrary residue field k, Harbater, Hartmann and Krashen introduced a new, patching technique which among other things gives bounds [HHK09, Thm. 5.5] for the index in terms of similar bounds for the field K and for the function fields of curves over the residue field k.

Here is a "local" analogue of Theorem 9.3.3.

Theorem 9.3.5 *Let R be a 2-dimensional henselian local, normal, excellent domain with fraction field F and residue field k. Let D be a central division algebra over F.*

(i) *If k is separably closed and $\exp(D)$ is prime to $\mathrm{char}(k)$, then $\mathrm{ind}(D) = \exp(D)$. Moreover, D is cyclic.*

(ii) *If k is a finite field and $\exp(D)$ is coprime to $\mathrm{char}(k)$, then $\mathrm{ind}(D)|\exp(D)^2$.*

Proof. For (i) see [FS89, Thm. 1.6], [CTOP02, Thm. 2.1]. For (ii) see [Hu13, Thm. 3.4]. □

Lemma 9.3.6 *Let R be a discrete valuation ring with fraction field K. Let $\widehat{R}$ be the completion of R and let $\widehat{K}$ be the fraction field of $\widehat{R}$. If the image of $\alpha \in \mathrm{Br}(K)$ in $\mathrm{Br}(\widehat{K})$ lies in $\mathrm{Br}(\widehat{R}) \subset \mathrm{Br}(\widehat{K})$, then α belongs to $\mathrm{Br}(R) \subset \mathrm{Br}(K)$.*

Proof. There exists an integer n, and elements $x_1 \in \mathrm{H}^1(K, \mathrm{PGL}_n)$ and $x_2 \in \mathrm{H}^1(\widehat{R}, \mathrm{PGL}_n)$, with the same image in $\mathrm{H}^1(\widehat{K}, \mathrm{PGL}_n)$, such that the injective map $\mathrm{H}^1(K, \mathrm{PGL}_n) \to \mathrm{Br}(K)$ sends x_1 to α. There is an embedding of reductive group R-schemes $\mathrm{PGL}_{n,R} \hookrightarrow \mathrm{GL}_{N,R}$ for some N. Then $E = \mathrm{GL}_{N,R}/\mathrm{PGL}_{n,R}$ is a smooth R-scheme. We have an exact sequence of pointed sets [SerCG, Ch. 1, §5, Prop. 36]

$$E(K) \longrightarrow \mathrm{H}^1(K, \mathrm{PGL}_n) \longrightarrow \mathrm{H}^1(K, \mathrm{GL}_N),$$

and a similar sequence for $\widehat{R}$ in place of K. By Hilbert's Theorem 90 we have $\mathrm{H}^1(K, \mathrm{GL}_N) = 0$ (Theorem 1.3.1), so we can lift x_1 to a point $y_1 \in E(K)$. It is known that $\mathrm{H}^1(R, \mathrm{GL}_{N,R}) = 0$ for any local ring R, cf. [Mil80, Ch. III, Lemma 4.10], hence we can lift x_2 to a point $y_2 \in E(\widehat{R})$. There exists an element $g \in \mathrm{GL}_N(\widehat{K})$ such that $gy_1 = y_2$. As $\mathrm{GL}_{N,K}$ is an open subset of an affine space, any element $g \in \mathrm{GL}_N(\widehat{K})$ can be written as a product $g_2 g_1$ where $g_1 \in \mathrm{GL}_N(K)$ and $g_2 \in \mathrm{GL}_N(\widehat{R})$. Then $g_1 y_1 = g_2^{-1} y_2$ is an element of $E(\widehat{K})$ contained in $E(K) \cap E(\widehat{R}) = E(R)$. This implies that $\alpha \in \mathrm{Br}(R)$. □

For a more general statement, see [CTPS12, Lemma 4.1].

Theorem 9.3.7 *Let R be a henselian local domain with residue field k. Let X be an integral regular scheme of dimension 2 equipped with a proper morphism $X \to \mathrm{Spec}(R)$ whose closed fibre X_0 is of dimension 1. Let F be the function field of X. Let Ω_X be the set of rank 1 valuations v on F associated to codimension 1 points on X. Let F_v denote the completion of F with respect to v. Then the natural restriction map $\mathrm{Br}(F) \to \prod_{v \in \Omega_X} \mathrm{Br}(F_v)$ is injective.*

Proof. Since $\alpha \in \mathrm{Br}(F)$ is trivial in each $\mathrm{Br}(F_v)$ for v attached to the points of codimension 1 of the regular scheme X, by Lemma 9.3.6 (via a patching argument) α can be represented by an Azumaya algebra over an open set $U \subset X$ which contains all codimension 1 points of X. Since X is regular and 2-dimensional, by a theorem of Auslander, Goldman and Grothendieck [Gro68, II, §2, Thm. 2.1] there exists an Azumaya algebra over X whose class in $\mathrm{Br}(F)$ is α. We thus have $\alpha \in \mathrm{Br}(X) \subset \mathrm{Br}(F)$.

An irreducible component C of the curve X_0 defines a valuation $v \in \Omega_X$. The image of α in $\mathrm{Br}(F_v)$ belongs to the subgroup $\mathrm{Br}(\mathcal{O}_v) \subset \mathrm{Br}(F_v)$, where $\mathcal{O}_v$ is the ring of integers of the complete field F_v. By assumption, this image is zero. Thus the image of α in the Brauer group of the function field of C is zero.

Now let P be a closed point of X_0. Since X is regular, there exists a closed integral curve $D \subset X$ through P which is regular at P. Arguing as above, we see that the value of α at the generic point of D is zero. This implies that the restriction of α to the local ring of P on D is zero, hence $\alpha(P) = 0$. We now apply Proposition 4.5.1 (i) to conclude that the image of α in $\mathrm{Br}(X_0)$ is zero. Now Theorem 9.3.1 implies that $\alpha = 0$. □

Remark 9.3.8 1. The above proof is essentially given by Y. Hu in [Hu12, §3]. It extends proofs in [CTOP02].

2. For R complete, in the semi-global case, a different proof of Theorem 9.3.7 is given in [CTPS12, Theorem 4.3]. This proof relies on the work of Harbater, Hartmann and Krashen [HHK09].

3. There exist examples of R, X and F as above such that the map

$$\mathrm{H}^1(F,\mathbb{Q}/\mathbb{Z}) \longrightarrow \prod_{v\in\Omega_X} \mathrm{H}^1(F_v,\mathbb{Q}/\mathbb{Z})$$

has a non-trivial kernel. See [CTPS12, §6].

4. Let D be a central simple algebra over a field F. The relation between $\mathrm{ind}(D)$ and $\exp(D)$ over specific fields F has been the object of much study. Suppose $F = \mathbb{C}(X)$ is the function field of an integral algebraic variety X of dimension d over $\mathbb{C}$. It would be interesting to know if $\mathrm{ind}(D)|\exp(D)^{d-1}$ for any D over F, which is the best possible bound [CT02]. The case $d = 1$ is Tsen's theorem. The case $d = 2$ is a theorem of de Jong [deJ04], [CT06]. For more work on the comparison of index and exponent over various fields of geometric or arithmetic origin, see [Lie08, Lie11, Lie15], [KL08], [HHK09] and [AAI$^+$].

The following theorem combines [CTPS16, Prop. 2.10] and work of Izquierdo [Izq19].

Theorem 9.3.9 *Let R be a 2-dimensional, local, normal, excellent, henselian domain with algebraically closed residue field k of characteristic 0. Let K be the fraction field of R. Let $X\to\mathrm{Spec}(R)$ be a resolution of singularities such that the reduced divisor associated to the closed fibre Y/k is a divisor on X with strict normal crossings. For each place v of K, let K_v be the completion of K at v. Then we have the following statements.*

(i) *There is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{v\in X^{(1)}} \mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z}) \longrightarrow \bigoplus_{x\in X^{(2)}} \mathbb{Q}/\mathbb{Z}(-1) \longrightarrow 0.$$

(ii) *For each $v \in R^{(1)}$ there are isomorphisms*

$$\mathrm{Br}(K_v)\xrightarrow{\sim}\mathrm{H}^1(k(v),\mathbb{Q}/\mathbb{Z})\xrightarrow{\sim}\mathbb{Q}/\mathbb{Z}(-1).$$

(iii) *The sum of these maps for all $v \in R^{(1)}$ fits into an exact sequence*

$$\mathrm{Br}(K) \longrightarrow \oplus_{v\in R^{(1)}}\mathrm{Br}(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z}(-1) \longrightarrow 0,$$

(iv) *If R is regular, the map $\mathrm{Br}(K)\to\oplus_{v\in R^{(1)}}\mathrm{Br}(K_v)$ is injective. Assume Y is a curve. Let Γ be the graph associated to the reduced divisor Y whose vertices correspond to the irreducible components of Y_{red} and the edges correspond to the intersection points of components. This graph is connected. Let $c = n_e - n_v + 1$ be Betti number of Γ. Let $m_Y = c + 2\sum_{y\in Y^{(1)}} g_y$, where g_y is the genus of the smooth, irreducible, projective curve defined by y. Then*

$$\mathrm{Ker}[\mathrm{Br}(K)\to \oplus_{v\in R^{(1)}} \mathrm{Br}(K_v)] \cong (\mathbb{Q}/\mathbb{Z})^{m_Y}.$$

(v) $\mathrm{Br}(X) \cong (\mathbb{Q}/\mathbb{Z})^{m_Y}$.

Statements (iv) and (v) are due to Izquierdo. Statement (iv) is important, it is one of the building blocks for the Poitou–Tate duality theorems which Izquierdo establishes for finite commutative groups and for tori over K, with respect to just the completions at the points of codimension 1 of $\mathrm{Spec}(R)$. This leads to a proof [Izq19, §5.1] that for a principal homogeneous space E of a torus T over K, a suitable Brauer–Manin obstruction defined in [CTPS16] is the only obstruction to the existence of a K-point on E.

9.4 Smooth proper schemes over a henselian discrete valuation ring

The content of the present section was developed in [CTS13a].

Let R be a henselian discrete valuation ring with field of fractions K and residue field k. We assume that $\mathrm{char}(K) = 0$ and k is perfect. Let $\overline{K}$ be an algebraic closure of K, and let $K_{\mathrm{nr}} \subset \overline{K}$ be the maximal unramified extension of K. Let R_{nr} be the ring of integers of K_{nr}. Let

$$\mathfrak{g} = \mathrm{Gal}(\overline{K}/K), \quad G = \mathrm{Gal}(K_{\mathrm{nr}}/K), \quad I = \mathrm{Gal}(\overline{K}/K_{\mathrm{nr}}).$$

The valuation of K gives rise to a split exact sequence of G-modules

$$1 \longrightarrow R_{\mathrm{nr}}^* \longrightarrow K_{\mathrm{nr}}^* \longrightarrow \mathbb{Z} \longrightarrow 0.$$

We have $\mathrm{Br}(K_{\mathrm{nr}}) = 0$ (Theorem 1.2.13), which implies $\mathrm{H}^2(G, K_{\mathrm{nr}}^*) = \mathrm{Br}(K)$.

Let $\pi : \mathcal{X} \to \mathrm{Spec}(R)$ be a faithfully flat proper morphism of integral schemes with geometrically integral generic fibre $X = \mathcal{X} \times_R K$. Write

$$X_{\mathrm{nr}} = X \times_K K_{\mathrm{nr}}, \quad \mathcal{X}_{\mathrm{nr}} = \mathcal{X} \times_R R_{\mathrm{nr}}, \quad \overline{X} = X \times_K \overline{K}.$$

Lemma 9.4.1 *If the proper R-scheme $\mathcal{X}$ is smooth over R with geometrically integral fibres, then the following natural map is surjective:*

$$\mathrm{Br}(K) \oplus \mathrm{Ker}[\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{X}_{\mathrm{nr}})] \longrightarrow \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathrm{nr}})].$$

Proof. The map is well defined since $\mathrm{Br}(K_{\mathrm{nr}}) = 0$, so that the composition $\mathrm{Br}(K) \to \mathrm{Br}(X) \to \mathrm{Br}(X_{\mathrm{nr}})$ is zero.

The restriction map $\mathrm{Pic}(\mathcal{X}_{\mathrm{nr}}) \to \mathrm{Pic}(X_{\mathrm{nr}})$ is surjective since $\mathcal{X}_{\mathrm{nr}}$ is regular. The kernel of this map is generated by the classes of components of the closed fibre of $\mathcal{X}_{\mathrm{nr}} \to \mathrm{Spec}(R)$. The closed fibre is a principal divisor in $\mathcal{X}_{\mathrm{nr}}$. Since we assume that it is integral, the restriction map gives an isomorphism of G-modules

$$\mathrm{Pic}(\mathcal{X}_{\mathrm{nr}}) \xrightarrow{\sim} \mathrm{Pic}(X_{\mathrm{nr}}). \tag{9.2}$$

There is a Hochschild–Serre spectral sequence attached to the morphism $\mathcal{X}_{\mathrm{nr}} \to \mathcal{X}$:

$$E_2^{pq} = \mathrm{H}^p(G, \mathrm{H}^q_{\text{ét}}(\mathcal{X}_{\mathrm{nr}}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(\mathcal{X}, \mathbb{G}_m),$$

and a similar sequence attached to the morphism $X_{\rm nr}\to X$, see [Mil80, Thm. III.2.20, Remark III.2.21 (b)]. By functoriality the maps in these sequences are compatible with the inclusions of the generic fibres $X\hookrightarrow\mathcal{X}$ and $X_{\rm nr}\hookrightarrow\mathcal{X}_{\rm nr}$. We have $\mathrm{H}^0_{\text{ét}}(\mathcal{X}_{\rm nr},\mathbb{G}_m)=R^*_{\rm nr}$ because the morphism $\pi:\mathcal{X}\to\mathrm{Spec}(R)$ is proper with geometrically integral fibres. The low degree terms of the two spectral sequences give rise to the following commutative diagram of exact sequences, where the equality is induced by (9.2):

$$\begin{array}{ccccccc}
\mathrm{H}^2(G,R^*_{\rm nr}) & \to & \mathrm{Ker}[\mathrm{Br}(\mathcal{X})\to\mathrm{Br}(\mathcal{X}_{\rm nr})] & \to & \mathrm{H}^1(G,\mathrm{Pic}(\mathcal{X}_{\rm nr})) & \to & \mathrm{H}^3(G,R^*_{\rm nr})\\
\downarrow & & \downarrow & & || & & \downarrow\\
\mathrm{H}^2(G,K^*_{\rm nr}) & \to & \mathrm{Ker}[\mathrm{Br}(X)\to\mathrm{Br}(X_{\rm nr})] & \to & \mathrm{H}^1(G,\mathrm{Pic}(X_{\rm nr})) & \to & \mathrm{H}^3(G,K^*_{\rm nr})
\end{array}$$

The inclusion of G-modules $R^*_{\rm nr}\hookrightarrow K^*_{\rm nr}$ has a G-module retraction, hence the map $\mathrm{H}^3(G,R^*_{\rm nr})\to\mathrm{H}^3(G,K^*_{\rm nr})$ is injective. Since $\mathrm{H}^2(G,K^*_{\rm nr})=\mathrm{Br}(K)$, the statement follows from the above diagram. □

Proposition 9.4.2 *Assume that the proper R-scheme $\mathcal{X}$ is smooth over R with geometrically integral fibres. Assume also that $\mathrm{H}^1(X,\mathcal{O}_X)=0$ and the Néron–Severi group $\mathrm{NS}(\overline{X})$ is torsion-free. Then*

$$\mathrm{Br}_1(X)=\mathrm{Ker}[\mathrm{Br}(X)\to\mathrm{Br}(X_{\rm nr})].$$

Proof. For any prime $\ell\neq\mathrm{char}(k)$ the smooth base change theorem in étale cohomology for the smooth and proper morphism $\pi:\mathcal{X}\to\mathrm{Spec}(R)$ implies that the natural action of the inertia subgroup I on $\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))$ is trivial. Indeed, by [Mil80, Ch. VI, Cor. 4.2] the étale sheaf $R^2\pi_*\mu_{\ell^m}$ is locally constant for every $m\geq 1$. Also, the fibre of $R^2\pi_*\mu_{\ell^m}$ at the generic geometric point $\mathrm{Spec}(\overline{K})\to\mathrm{Spec}(R)$ is $\mathrm{H}^2_{\text{ét}}(\overline{X},\mu_{\ell^m})$. Now it follows from Remark 1.2 (b) in [Mil80, Ch. V] that the action of $\mathfrak{g}$ on $\mathrm{H}^2_{\text{ét}}(\overline{X},\mu_{\ell^m})$ factors through

$$\pi_1(\mathrm{Spec}(R),\mathrm{Spec}(\overline{K}))=\mathrm{Gal}(K_{\rm nr}/K)=G=\mathfrak{g}/I,$$

see [Mil80, Ch. I, Ex. 5.2(b)]. Thus I acts trivially on $\mathrm{H}^2_{\text{ét}}(\overline{X},\mu_{\ell^m})$ for every m, hence I acts trivially on $\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))$.

Since K has characteristic 0, for any prime ℓ the Kummer sequence gives a Galois equivariant embedding

$$\mathrm{NS}(\overline{X})\otimes\mathbb{Z}_\ell\hookrightarrow\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1)).$$

For any $\ell\neq\mathrm{char}(k)$ we conclude that I acts trivially on $\mathrm{NS}(\overline{X})\otimes\mathbb{Z}_\ell$, hence also on $\mathrm{Pic}(\overline{X})\cong\mathrm{NS}(\overline{X})\subset\mathrm{NS}(\overline{X})\otimes\mathbb{Z}_\ell$. Thus $\mathrm{H}^1(K_{\rm nr},\mathrm{Pic}(\overline{X}))=\mathrm{H}^1(I,\mathrm{NS}(\overline{X}))=0$. From the exact sequence

$$\mathrm{Br}(K_{\rm nr})\longrightarrow\mathrm{Ker}[\mathrm{Br}(X_{\rm nr})\to\mathrm{Br}(\overline{X})]\longrightarrow\mathrm{H}^1(K_{\rm nr},\mathrm{Pic}(\overline{X}))$$

we conclude that $\mathrm{Br}(X_{\rm nr})\to\mathrm{Br}(\overline{X})$ is injective. This implies the result. □

We are also interested in the situation when $\mathrm{H}^2(X,\mathcal{O}_X)$ is not necessarily zero, so we must take into account the transcendental Brauer group as well.

Proposition 9.4.3 *Let ℓ be a prime, $\ell \neq \mathrm{char}(k)$. Let $\pi : \mathcal{X} \to \mathrm{Spec}(R)$ be a smooth proper morphism with geometrically integral fibres, such that the closed geometric fibre has no connected unramified cyclic covering of degree ℓ. Then the group $\mathrm{Br}(X)\{\ell\}$ is generated by the images of $\mathrm{Br}(\mathcal{X})\{\ell\}$ and $\mathrm{Br}(K)\{\ell\}$.*

Proof. Let $Y = \mathcal{X} \times_R k$ be the closed fibre of π. We note that Y is a regular subscheme of codimension 1 of the regular scheme $\mathcal{X}$. Thus we can apply the exact sequence (3.16):

$$0 \longrightarrow \mathrm{Br}(\mathcal{X})[\ell^m] \longrightarrow \mathrm{Br}(X)[\ell^m] \longrightarrow \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/\ell^m). \tag{9.3}$$

Let $\overline{Y} = Y \times_k \bar{k}$, where $\bar{k}$ is an algebraic closure of k. As $\overline{Y}$ is connected, the spectral sequence

$$E_2^{pq} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(\overline{Y}, \mathbb{Z}/\ell^n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(\mathcal{X}_0, \mathbb{Z}/\ell^n)$$

gives rise to the exact sequence

$$0 \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/\ell^n) \longrightarrow \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/\ell^n) \longrightarrow \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/\ell^n).$$

By assumption $\overline{Y}$ has no connected unramified cyclic covering of degree ℓ, hence $\mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/\ell^n) = 0$.

Let $A \in \mathrm{Br}(X)\{\ell\}$. Take m such that $A \in \mathrm{Br}(X)[\ell^m]$. The image of A in $\mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/\ell^m)$ belongs to the injective image of $\mathrm{H}^1(k, \mathbb{Z}/\ell^m)$. We have the exact sequence (3.10)

$$0 \longrightarrow \mathrm{Br}(R)[\ell^m] \longrightarrow \mathrm{Br}(K)[\ell^m] \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/\ell^m) \longrightarrow 0,$$

with compatible maps to sequence (9.3). Hence there exists $\alpha \in \mathrm{Br}(K)[\ell^m]$ such that $A - \alpha \in \mathrm{Br}(X)[\ell^m]$ goes to zero in $\mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/\ell^m)$. By the exactness of (9.3) we have $A - \alpha \in \mathrm{Br}(\mathcal{X})[\ell^m]$. □

Remark 9.4.4 1. Let $\mathrm{char}(k) = p$. Already for π smooth and proper, it is an interesting problem to decide whether a similar statement for $\mathrm{Br}(X)\{p\}$ is true. For elements split by an unramified extension of K, including those of order divisible by p, this follows from Lemma 9.4.1 (see also [Bri07, Prop. 6]).

2. The hypotheses of Proposition 9.4.3 apply in particular when the fibres of $\pi : \mathcal{X} \to \mathrm{Spec}(R)$ are smooth complete intersections of dimension at least 2 in the projective space (an application of the weak Lefschetz theorem in étale cohomology, see [Kat04]). In particular they apply to smooth surfaces of arbitrary degree in $\mathbb{P}^3$.

9.5 Varieties over a local field

We start with the following statement, which is a generalisation of known results such as the implicit function theorem for varieties over a complete local field to the henselian case. Versions of this statement also hold for the fields of fractions of much more general henselian valuation rings, see [C12, Prop. 5.4]. See also [Mor12] and [GGMB14, §3.1].

Theorem 9.5.1 *Let A be a henselian discrete valuation ring with field of fractions K. Let X be a variety over K.*

(i) *There is a unique structure of a topological space on $X(K)$ which is functorial and compatible with fibre products, and such that open (respectively, closed) immersions in X give rise to open (respectively, closed) embeddings in $X(K)$, and étale morphisms give rise to local homeomorphisms. The topological space $X(K)$ is Hausdorff.*

If $f : X\to Y$ is a smooth morphism of varieties over K, then the induced map $X(K)\to Y(K)$ is topologically open.

(ii) *Assume further that K is locally compact, hence complete. Then $X(K)$ is locally compact. Moreover, if X is smooth, then $X(K)$ admits a unique functorial K-analytic manifold structure which agrees with the scheme structure and carries étale morphisms to K-analytic local isomorphisms.*

If $f : X\to Y$ is a proper morphism of varieties over K, then the induced map $X(K)\to Y(K)$ is topologically proper.

Recall that a discretely valued field is locally compact if and only if it is complete and has finite residue field, see [ANT67, Ch. II, §7].

9.5.1 Evaluation at rational and closed points

Proposition 9.5.2 *Let A be a henselian discrete valuation ring with field of fractions K. Let X be a variety over K and let $\mathcal{A} \in \mathrm{Br}(X)$. The evaluation map $X(K)\to\mathrm{Br}(K)$ sending $M \in X(K)$ to $\mathcal{A}(M) \in \mathrm{Br}(K)$ is locally constant and its image is annihilated by some positive integer.*

Proof. Take any $P \in X(K)$. Then $\alpha = \mathcal{A} - \mathcal{A}(P) \in \mathrm{Br}(X)$ is such that $\alpha(P) = 0$. By Corollary 3.4.4 there exists an étale morphism $f : U\to X$ such that $f^*\alpha = 0$ and P lifts to a point $M \in U(K)$. Then α vanishes on $f(U(K)) \subset X(K)$. Since $P \in f(U(K))$, this is an open neighbourhood of $P \in X(K)$ by the implicit function theorem (Theorem 9.5.1). The last statement is a special case of Lemma 3.4.5. □

It is clear that the same result also holds for a variety X over the field of real numbers $\mathbb{R}$.

By a p-adic field we understand a finite extension of $\mathbb{Q}_p$.

Theorem 9.5.3 *Let k be a p-adic field with ring of integers R. Let $\mathcal{X}$ be a regular, proper, integral, flat R-scheme with generic fibre X/k. If $\alpha \in \mathrm{Br}(X)$ vanishes at each closed point of a non-empty open set $U \subset X$, then α lies in $\mathrm{Br}(\mathcal{X}) \subset \mathrm{Br}(X)$.*

Proof. Here is a sketch of the proof for the prime to p-part of the statement [CTS96, Thm. 2.1]. Let ℓ be a prime, $\ell \neq p$. Using Chebotarev's theorem for varieties over a finite field, a suitable version of Hensel's lemma, and Theorem 3.7.4, one sees that the assumption implies that $\alpha \in \mathrm{Br}(X)\{\ell\}$ has trivial residues at the codimension 1 points of $\mathcal{X}$. Hence, by Gabber's purity theorem, α comes from an element of $\mathrm{Br}(\mathcal{X})$. For the general case, combine Saito and Sato's result [SS14, Thm. 1.1.3], which is conditional on purity for the Brauer group for regular schemes, with this purity theorem, proved recently by Česnavičius using previous work of Gabber (Theorem 3.7.5). □

The case when X is a curve goes back to Lichtenbaum [Lic69].

Corollary 9.5.4 (Lichtenbaum) *Let X be a smooth, projective, geometrically integral curve over a p-adic field k. If $\alpha \in \mathrm{Br}(X)$ vanishes at each closed point of X, then $\alpha = 0$.*

Proof. Let R be the ring of integers of k. There exists a regular proper flat model $\mathcal{X}\to\mathrm{Spec}(R)$ (as proved independently by Lipman and Shafarevich). By the previous theorem, α lies in $\mathrm{Br}(\mathcal{X}) \subset \mathrm{Br}(X)$. By Theorem 9.3.1, we have $\mathrm{Br}(\mathcal{X})\tilde{\longrightarrow}\mathrm{Br}(\mathcal{X}_0)$, where $\mathcal{X}_0$ is the closed fibre of $\mathcal{X}\to\mathrm{Spec}(R)$. But $\mathrm{Br}(\mathcal{X}_0) = 0$ by Theorem 4.5.1 (v), hence $\alpha = 0$. □

Remark 9.5.5 1. Evaluation on closed points of a smooth projective curve over a p-adic field induces a pairing

$$\mathrm{Br}(X) \times \mathrm{Pic}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

That the left kernel of this pairing is trivial (and, more precisely, the pairing induces a duality) was proved by Lichtenbaum as a consequence of the Tate duality theorems for abelian varieties over a p-adic field.

2. Let X be a smooth and geometrically integral curve over a p-adic field k. Let U be a non-empty open subset of X. If $\alpha \in \mathrm{Br}(U)$ vanishes at each closed point of U, then α lies in $\mathrm{Br}(X)$ and, moreover, $\alpha = 0$.

Let us explain this. Let P be a closed point in $X \smallsetminus U$ and let $K = k(P)$ be the residue field of P. Write $X_K = X \times_k K$. The morphism $P : \mathrm{Spec}(K)\to X$ gives rise to the morphism $\mathrm{Spec}(K \otimes_k K)\to X_K$ that can be precomposed with the dual morphism of the multiplication map $K \otimes_k K\to K$ to define a K-point $\tilde{P} : \mathrm{Spec}(K)\to X_K$ above P.

Suppose that α has a non-trivial residue $\chi \in \mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z})$ at P. Let $N > 1$ be the order of χ in $\mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z})$. Write α_K for the image of α in $\mathrm{Br}(X_K)$. The multiplicity of $\tilde{P}$ in the fibre $\mathrm{Spec}(K \otimes_k K)$ of $X_K\to X$ above P is 1, so by the functoriality of residues (Theorem 3.7.4) the residue of $\alpha_K \in \mathrm{Br}(U_K)$ at $\tilde{P}$ is $\chi \in \mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z})$.

Let π be a local equation at $\tilde{P} \in X_K$. Then α_K differs from the cup-product (χ, π) by an element $\beta \in \mathrm{Br}(V)$, where $V \subset X_K$ is a Zariski neighbourhood of $\tilde{P}$. One then finds a p-adic neighbourhood $W \subset V(K)$ of $\tilde{P}$ such that β is constant on W and π is invertible on $W \smallsetminus \tilde{P}$. The assumption on α then implies that (χ, π) takes a constant value on $W \smallsetminus \tilde{P}$. But for points $M \neq \tilde{P}$ in $W \subset U(K)$, the value $\pi(M) \in K$ takes all possible valuations. Thus $(\chi, \pi(M)) = v(\pi(M))\chi \in \mathrm{H}^1(K, \mathbb{Z}/N)$ is not constant, which is a contradiction. (For a similar and more detailed argument in a global context, we refer the reader to Theorem 12.6.1.) We conclude that α has zero residues on X, hence belongs to $\mathrm{Br}(X)$. Since α vanishes at all closed points of U, by the continuity of the evaluation map it vanishes at all closed points of X. One then applies Corollary 9.5.4.

3. There exist smooth, projective, geometrically integral curves X over a p-adic field with non-zero elements in $\mathrm{H}^1_{\text{ét}}(X,\mathbb{Z}/\ell)$ which vanish at each closed point of X, see [CTPS12, §6].

4. Let X be a variety over the field of real numbers $\mathbb{R}$. The natural pairing

$$X(\mathbb{R})\times \mathrm{Br}(X)\longrightarrow \mathrm{Br}(\mathbb{R})=\mathbb{Z}/2$$

is locally constant on $X(\mathbb{R})$ hence induces a map $\mathrm{Br}(X)\to(\mathbb{Z}/2)^S$, where S is the set of connected components of $X(\mathbb{R})$ for the real topology.

The real analogue of Tate's duality theorem for abelian varieties over a p-adic field and of Corollary 9.5.4 goes back to Witt (1934). For a smooth, projective, geometrically connected curve over $\mathbb{R}$, evaluating elements of $\mathrm{Br}(X)$ on the real points induces an isomorphism $\mathrm{Br}(X)\xrightarrow{\sim}(\mathbb{Z}/2)^S$. In particular $\mathrm{Br}(X)=0$ if $X(\mathbb{R})=\emptyset$. If X is a quasi-projective but possibly singular real curve, the map $\mathrm{Br}(X)\to(\mathbb{Z}/2)^S$ is injective [CTOP02, Prop. 1.13].

9.5.2 Index

Let R be the ring of integers of a p-adic field K with finite residue field k. Let $\mathcal{X}$ be a regular, connected, projective, flat R-scheme. Let X/K be the generic fibre of $\mathcal{X}$. We assume that X is geometrically integral. The closed fibre $\mathcal{X}_0/k$ is a divisor $\sum_i e_iD_i$, where e_i is a positive integer and D_i is an integral variety over k. Let f_i be the degree over k of the integral closure of k in the function field $k(D_i)$. In this context one defines the following positive integers.

(1) I_{Br} is the order of $\mathrm{Ker}[\mathrm{Br}(K)\to\mathrm{Br}(X)/\mathrm{Br}(\mathcal{X})]$.

(2) I is the g.c.d. of the degrees of the closed points of X.

(3) I_0 is the g.c.d. of the e_if_i.

The positive integer I is called the *index* of X. Note, by the way, that the kernel in (1) is cyclic; by the purity theorem it does not depend on the choice of $\mathcal{X}$.

Theorem 9.5.6 *We have* $I_{\mathrm{Br}}=I=I_0$.

Saito and Sato [SS14, Thm. 5.4.1] proved this theorem assuming purity for the Brauer group of regular schemes, a result which is now known in full generality (Theorem 3.7.5). Earlier results had been obtained by Lichtenbaum [Lic69] (in the case of a curve), then in [CTS96, Thm. 3.1] (for the prime-to-p part, in arbitrary dimension) and in [GLL13, Cor. 9.1] which shows $I=I_0$. The paper [GLL13] studies the case of a henselian discrete valuation ring R with an arbitrary residue field k.

9.5.3 Finiteness results for the Brauer group

In the good reduction case, Section 9.4 gives some control on the Brauer group of a smooth proper variety over a p-adic field. Here are two general results under weaker assumptions.

Proposition 9.5.7 *Let X be a variety over a p-adic field K. Then for any positive integer n the group* $\mathrm{Br}(X)[n]$ *is finite.*

Proof. The Kummer exact sequence shows that $\mathrm{Br}(X)[n]$ is a quotient of $\mathrm{H}^2_{\text{ét}}(X,\mu_n)$. Consider the spectral sequence

$$E_2^{pq}=\mathrm{H}^p(K,\mathrm{H}^q_{\text{ét}}(\overline{X},\mu_n))\Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X,\mu_n).$$

The groups $\mathrm{H}^q_{\text{ét}}(\overline{X},\mu_n)$ are finite for any $q\geq 0$ (see [Mil80, Ch. VI, Cor. 4.5]). The Galois cohomology groups $\mathrm{H}^p(K,M)$, where K is a p-adic field and M is finite, are finite for all $p\geq 0$ [SerCG, Ch. 2, §5, Prop. 14]. □

Proposition 9.5.8 *Let X be a smooth, proper and geometrically integral variety over a p-adic field K. Let $X_{\rm nr}=X\times_K K_{\rm nr}$, where $K_{\rm nr}$ is the maximal unramified extension of K. Then the group*

$$\mathrm{Ker}[\mathrm{Br}(X)\to\mathrm{Br}(X_{\rm nr})]/\mathrm{Br}_0(X)$$

is finite.

Proof. [CTS13a, Prop. 2.1] We assume that $K_{\rm nr}\subset\overline{K}$ and use the previous notation $\mathfrak{g}=\mathrm{Gal}(\overline{K}/K)$, $G=\mathrm{Gal}(K_{\rm nr}/K)$, $I=\mathrm{Gal}(\overline{K}/K_{\rm nr})$. Consider the Hochschild–Serre spectral sequence [Mil80, Thm. III.2.20, Remark III.2.21 (b)] attached to the morphism $X_{\rm nr}\to X$:

$$E_2^{pq}=\mathrm{H}^p(G,\mathrm{H}^q_{\text{ét}}(X_{\rm nr},\mathbb{G}_m))\Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X,\mathbb{G}_m). \tag{9.4}$$

Since $\mathrm{H}^2(G,K_{\rm nr}^*)=\mathrm{Br}(K)$, the exact sequence of low degree terms of (9.4) shows that the group under consideration is a subgroup of $\mathrm{H}^1(G,\mathrm{Pic}(X_{\rm nr}))$. There is an exact sequence of continuous discrete $\mathfrak{g}$-modules

$$0\longrightarrow \mathrm{Pic}^0(\overline{X})\longrightarrow \mathrm{Pic}(\overline{X})\longrightarrow \mathrm{NS}(\overline{X})\longrightarrow 0.$$

By the representability of the Picard functor, and since $\mathrm{char}(K)=0$, there exists an abelian variety A over K such that $A(\overline{K})$ is isomorphic to $\mathrm{Pic}^0(\overline{X})$ as a $\mathfrak{g}$-module (Theorem 4.1.1). Thus we rewrite the previous sequence as

$$0\longrightarrow A(\overline{K})\longrightarrow \mathrm{Pic}(\overline{X})\longrightarrow \mathrm{NS}(\overline{X})\longrightarrow 0. \tag{9.5}$$

The Hochschild–Serre spectral sequence attached to $\overline{X}\to X_{\rm nr}$ is

$$E_2^{pq}=\mathrm{H}^p(I,\mathrm{H}^q_{\text{ét}}(\overline{X},\mathbb{G}_m))\Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X_{\rm nr},\mathbb{G}_m).$$

By Hilbert's theorem 90 we have $\mathrm{H}^1(I, \overline{K}^*) = 0$. Since $\mathrm{Br}(K_{\mathrm{nr}}) = 0$ we obtain that the natural map $\mathrm{Pic}(X_{\mathrm{nr}}) \to \mathrm{Pic}(\overline{X})^I$ is an isomorphism. Now, by taking I-invariants in (9.5) we obtain the exact sequence of G-modules

$$0 \longrightarrow A(K_{\mathrm{nr}}) \longrightarrow \mathrm{Pic}(X_{\mathrm{nr}}) \longrightarrow \mathrm{NS}(\overline{X})^I.$$

The group $\mathrm{NS}(\overline{X})$ is finitely generated by the theorem of Néron and Severi, hence so is $\mathrm{NS}(\overline{X})^I$. Thus there is a G-module N, finitely generated as an abelian group, that fits into the exact sequence of continuous discrete G-modules

$$0 \longrightarrow A(K_{\mathrm{nr}}) \longrightarrow \mathrm{Pic}(X_{\mathrm{nr}}) \longrightarrow N \longrightarrow 0.$$

The resulting exact sequence of cohomology groups gives us an exact sequence

$$\mathrm{H}^1(G, A(K_{\mathrm{nr}})) \longrightarrow \mathrm{H}^1(G, \mathrm{Pic}(X_{\mathrm{nr}})) \longrightarrow \mathrm{H}^1(G, N). \tag{9.6}$$

We note that G is canonically isomorphic to the profinite completion $\hat{\mathbb{Z}}$, with the Frobenius as a topological generator. If M is a continuous discrete G-module which is finitely generated as an abelian group, then $\mathrm{H}^1(G, M)$ is finite. To see this, let G' be a finite index subgroup of G that acts trivially on M. The group $G' \simeq \hat{\mathbb{Z}}$ has a dense subgroup $\mathbb{Z}$ generated by a power of the Frobenius. Now $\mathrm{H}^1(G', M)$ is the group of continuous homomorphisms

$$\mathrm{Hom}_{\mathrm{cont}}(G', M) = \mathrm{Hom}_{\mathrm{cont}}(G', M_{\mathrm{tors}}) = M_{\mathrm{tors}},$$

which is visibly finite. An application of the restriction-inflation sequence finishes the proof of the finiteness of $\mathrm{H}^1(G, M)$.

To complete the proof of the proposition it remains to prove the finiteness of $\mathrm{H}^1(G, A(K_{\mathrm{nr}}))$. By [Mil86, Prop. I.3.8] this group is isomorphic to $\mathrm{H}^1(G, \pi_0(A_0))$, where $\pi_0(A_0)$ is the group of connected components of the closed fibre A_0 of the Néron model of A over $\mathrm{Spec}(R)$. Since $\pi_0(A_0)$ is finite, we see that $\mathrm{H}^1(G, \pi_0(A_0))$ is finite. □

Chapter 10

The Brauer group and families of varieties

In this section we are interested in the following question. Let $f : X\to Y$ be a dominant morphism of regular integral varieties. Can one compute the Brauer group $\mathrm{Br}(X)$ and its elements from the Brauer group of the base $\mathrm{Br}(Y)$ and the Brauer group of the generic fibre $\mathrm{Br}(X_\eta)$, in terms of the geometry of varieties X, Y and the morphism f? For example, when is the induced map $f^* : \mathrm{Br}(Y)\to\mathrm{Br}(X)$ surjective or injective? Recall that $\mathrm{Br}(X)$ is naturally a subgroup of $\mathrm{Br}(X_\eta)$. If $\mathrm{Br}(X_\eta)$ is known, then computing $\mathrm{Br}(X)$ involves determining the elements of $\mathrm{Br}(X_\eta)$ that are unramified on X. In general, this is a hard problem even if the generic fibre has very simple geometry, for instance, X_η is finite or X_η is a projective quadric.

The focus of Section 10.1 is the so called *vertical* subgroup $\mathrm{Br}_{\mathrm{vert}}(X/Y)$ of $\mathrm{Br}(X)$. It is defined as the set of elements of $\mathrm{Br}(X)$ whose restriction to $\mathrm{Br}(X_\eta)$ belongs to the image of $\mathrm{Br}(k(Y))$, where $k(Y) = k(\eta)$ is the function field of Y. There are several reasons to be interested in $\mathrm{Br}_{\mathrm{vert}}(X/Y)$.

- In some cases there are clean-cut algebraic formulae for $\mathrm{Br}_{\mathrm{vert}}(X/Y)$, whereas it may be difficult to give such formulae for the full Brauer group $\mathrm{Br}(X)$. For example, when $Y = \mathbb{P}^1_k$ and X_η is geometrically integral, generators of $\mathrm{Br}_{\mathrm{vert}}(X/Y)$ are explicitly computed in terms of the structure of the degenerate fibres of $f : X\to Y$.
- For certain types of morphisms, e.g. for families of quadrics of relative dimension at least one or for families of Severi–Brauer varieties, the full Brauer group is vertical, that is, the natural map $\mathrm{Br}_{\mathrm{vert}}(X/Y)\xrightarrow{\sim}\mathrm{Br}(X)$ is an isomorphism.
- Over a number field k, the vertical Brauer group $\mathrm{Br}_{\mathrm{vert}}(X/Y)$ appears in the definition of an obstruction to the existence of a rational point $P \in Y(k)$ such that the fibre X_P is smooth and has points in all completions of k.

In Section 10.3 we give a description of the 2-torsion subgroup $\mathrm{Br}(X)[2]$, where $f : X\to Y$ is a double cover of a rational surface Y over an algebraically closed field k. In Section 10.4 we study a universal family of cyclic twists and the associated vertical Brauer group; this construction has useful arithmetic applications. The main result of Section 10.5 is a formula for $\mathrm{Br}(X)$ in the case when X_η is a conic and Y is a rational surface over $\mathbb{C}$. This is used in Section 10.6 to recover the Artin–Mumford examples of unirational non-rational threefolds, along with several other examples.

10.1 The vertical Brauer group

Definition 10.1.1 *Let Y be an integral scheme with generic point $i : \eta\to Y$. Let $f : X\to Y$ be a dominant morphism, and let $X_\eta = X\times_Y \eta$ be the generic fibre of f. Write $j : X_\eta\to X$ for the natural inclusion, so that there is a cartesian square*

$$\begin{array}{ccc} X_\eta & \xrightarrow{j} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \eta & \xrightarrow{i} & Y \end{array}$$

The **vertical Brauer group** *of X/Y is*

$$\mathrm{Br}_{\mathrm{vert}}(X/Y) = \{A\in \mathrm{Br}(X) | j^*(A)\in \mathrm{Im}[\mathrm{Br}(\eta)\to\mathrm{Br}(X_\eta)]\}.$$

A formal consequence of the definition of $\mathrm{Br}_{\mathrm{vert}}(X/Y)$ is the exact sequence

$$0\longrightarrow \mathrm{Br}_{\mathrm{vert}}(X/Y)\longrightarrow \mathrm{Br}(X)\longrightarrow \mathrm{Br}(X_\eta)/\mathrm{Im}[\mathrm{Br}(\eta)\to\mathrm{Br}(X_\eta)].$$

If X is regular and integral, then by Theorem 3.5.4 we have inclusions

$$\mathrm{Br}(X)\subset \mathrm{Br}(X_\eta)\subset \mathrm{Br}(\eta'),$$

where η' is the generic point of X.

We shall mostly consider the case when X and Y are smooth, proper and geometrically integral varieties over a field k, so that $\eta = \mathrm{Spec}(k(Y))$ and $\eta' = \mathrm{Spec}(k(X))$. Then $\mathrm{Br}_{\mathrm{vert}}(X/Y)\subset \mathrm{Br}(X)\subset \mathrm{Br}(k(X))$ is the intersection of $\mathrm{Br}(X) = \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$ (see Proposition 5.2.2) with the image of the restriction map $\mathrm{Br}(k(Y))\to\mathrm{Br}(k(X))$. In other words, the elements of $\mathrm{Br}_{\mathrm{vert}}(X/Y)$ are the restrictions to $k(X)$ of the (possibly, ramified) classes in $\mathrm{Br}(k(Y))$ that become unramified in $k(X)$.

The following standard lemma will be useful.

Lemma 10.1.2 *Let $k\subset K$ be an extension of fields such that k is separably closed in K. Then the restriction map $\mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^1(K,\mathbb{Q}/\mathbb{Z})$ is injective.*

Proof. The assumption implies that the natural map $\mathrm{Gal}(K_{\mathrm{s}}/K)\to\mathrm{Gal}(k_{\mathrm{s}}/k)$ is surjective. Thus a non-trivial character $\chi : \mathrm{Gal}(k_{\mathrm{s}}/k)\to\mathbb{Q}/\mathbb{Z}$ gives rise to a non-trivial character $\mathrm{Gal}(K_{\mathrm{s}}/K)\to\mathbb{Q}/\mathbb{Z}$. □

Lemma 10.1.3 *Let n be a positive integer. For $i = 1, \ldots, n$ let k_i be a finite separable extension of k and let m_i be a positive integer. Let m be the g.c.d. of $m_1, \ldots, m_n$, and let r be the g.c.d. of $[k_1 : k]m_1, \ldots, [k_n : k]m_n$. Define*

$$\mathcal{L} = \bigcap_{i=1}^{n} \mathrm{Ker}[m_i \mathrm{res}_{k_i/k} : \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^1(k_i, \mathbb{Q}/\mathbb{Z})].$$

Then $r\mathcal{L} = 0$. Moreover, $\mathcal{L}$ is an extension of a finite abelian group by an abelian group of exponent m.

Proof. The first statement is clear since $\mathrm{cores}_{k_i/k}\mathrm{res}_{k_i/k} = [k_i : k]$. We note that $\mathcal{L}$ is the kernel of the composition of multiplication by m on $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ and the direct sum of maps

$$(m_i/m)\mathrm{res}_{k_i/k} : \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^1(k_i, \mathbb{Q}/\mathbb{Z})$$

for $i = 1, \ldots, n$. Thus it is enough to prove that $\mathcal{L}$ is finite if $m = 1$. So we now assume $m = 1$. Let K be a finite Galois extension of k that contains k_i for $i = 1, \ldots, n$. Then $k_i \otimes_k K \cong K^{[k_i:k]}$. It is clear that the kernel of the direct sum of multiplication by m_i maps on $\mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z})$ is trivial. Extending k to K we can conclude the proof since the kernel of $\mathrm{res}_{K/k} : \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z})$ is the finite group $\mathrm{H}^1(\mathrm{Gal}(K/k), \mathbb{Q}/\mathbb{Z})$. □

Let k be a field of characteristic 0. Let X and Y be smooth, integral varieties over k and let $f : X \to Y$ be a dominant morphism with generic fibre X_η, where $\eta = \mathrm{Spec}(k(Y))$ is the generic point of Y. Then there is a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Br}(X) & \longrightarrow & \mathrm{Br}(X_\eta) & \longrightarrow & \bigoplus_{P \in Y^{(1)}} \bigoplus_{V \subset X_P} \mathrm{H}^1(k(V), \mathbb{Q}/\mathbb{Z}) \\ & & \uparrow{\scriptstyle f^*} & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathrm{Br}(Y) & \longrightarrow & \mathrm{Br}(k(Y)) & \longrightarrow & \bigoplus_{P \in Y^{(1)}} \mathrm{H}^1(k(P), \mathbb{Q}/\mathbb{Z}) \end{array} \quad (10.1)$$

The bottom sequence is the exact sequence (3.13); here P ranges over all codimension 1 points of Y. The top exact sequence is obtained from (3.12) by taking the inductive limit over all open subsets $f^{-1}(U) \subset X$, where U is a non-empty open subset of Y. Here $V \subset X_P$ ranges over the irreducible components of the fibre X_P. Then the map $\mathrm{H}^1(k(P), \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(k(V), \mathbb{Q}/\mathbb{Z})$ is $m_V \mathrm{res}_{k(V)/k(P)}$, where m_V is the multiplicity of V_P in X_P. The diagram commutes by the functoriality of residues (Theorem 3.7.4).

For an irreducible component V of X_P we define κ_V as the algebraic closure of $k(P)$ in $k(V)$. For $f : X \to Y$ as above, the fibre X_P at a codimension 1 point $P \in Y$ is a variety over $k(P)$ which is split if and only if it contains an irreducible component V of multiplicity $m_V = 1$ such that $\kappa_V = k(P)$.

When X_η is geometrically integral, there is a non-empty Zariski open subset $U \subset Y$ such that the fibres of f at the points of U are geometrically integral.

Thus all but finitely many fibres of f over the points of codimension 1 in Y are geometrically integral, hence split.

Proposition 10.1.4 *Let $f : X \to Y$ be a dominant morphism of smooth, integral varieties over a field k of characteristic 0 with geometrically integral generic fibre. Let S be the finite set of points $P \in Y$ of codimension 1 such that the fibre X_P is not split (for example, X_P can be empty). Then every element of $\mathrm{Br}_{\mathrm{vert}}(X/Y)$ can be written as $f^*(\alpha)$, where $\alpha \in \mathrm{Br}(k(Y))$ is such that if $P \notin S$, then $\partial_P(\alpha) = 0$, and if $P \in S$, then*

$$\partial_P(\alpha) \in \bigcap_{V \subset X_P} \mathrm{Ker}[m_V \mathrm{res}_{k(V)/k(P)} : \mathrm{H}^1(k(P), \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^1(\kappa_V, \mathbb{Q}/\mathbb{Z})]. \quad (10.2)$$

Proof. This follows from the above diagram in view of Lemma 10.1.2. □

If X_P is empty, then the condition in (10.2) is vacuous.

This proposition shows, in particular, that split fibres can be disregarded, that is, counted as 'good' fibres for the determination of the vertical Brauer group attached to a morphism of varieties.

Corollary 10.1.5 *Let $f : X \to Y$ be a dominant morphism of smooth, integral varieties over a field k of characteristic 0 with geometrically integral generic fibre.*

(i) *Assume that for each point $P \in Y$ of codimension 1, the g.c.d. of the multiplicities m_V, where V is an irreducible component of X_P, is equal to 1. (This condition is satisfied if the fibres of f over all points of Y of codimension 1 are geometrically split.) Then $\mathrm{Br}_{\mathrm{vert}}(X/Y)/f^*\mathrm{Br}(Y)$ is finite.*

(ii) *Assume that for each point $P \in Y$ of codimension 1, the g.c.d. of the integers $m_V[\kappa_V : k(P)]$, where V is an irreducible component of X_P, is equal to 1. (This condition is satisfied if the fibres of f over all points of Y of codimension 1 are split.) Then $\mathrm{Br}_{\mathrm{vert}}(X/Y) = f^*\mathrm{Br}(Y)$.*

Proof. Diagram (10.1) implies that $\mathrm{Br}_{\mathrm{vert}}(X/Y)/f^*\mathrm{Br}(Y)$ is a quotient of

$$\bigoplus_{P \in Y^{(1)}} \mathrm{Ker}\left[\mathrm{H}^1(k(P), \mathbb{Q}/\mathbb{Z}) \longrightarrow \bigoplus_{V \subset X_P} \mathrm{H}^1(k(V), \mathbb{Q}/\mathbb{Z})\right],$$

where the map to $\mathrm{H}^1(k(V), \mathbb{Q}/\mathbb{Z})$ is $m_V\, \mathrm{res}_{k(V)/k(P)}$. Now both statements follow from Lemma 10.1.3. □

Remark 10.1.6 The proof of Corollary 10.1.5 (i) actually shows that the subgroup of $\mathrm{Br}(k(Y))$ consisting of the classes α such that $f^*(\alpha) \in \mathrm{Br}(k(X))$ lies in the image of $\mathrm{Br}(X)$ is finite modulo the image of $\mathrm{Br}(Y)$.

Exercise 10.1.7 *Let t be a coordinate function on $\mathbb{A}^1_{\mathbb{Q}} \subset \mathbb{P}^1_{\mathbb{Q}}$. Let X be a smooth, projective, geometrically integral surface over $\mathbb{Q}$ with a morphism $X \to \mathbb{P}^1_{\mathbb{Q}}$ whose generic fibre X_η is the smooth plane cubic curve over $\mathbb{Q}(t)$ defined by*

$$u^3 + tv^3 + t^2w^3 = 0.$$

Show that the group $\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}_0(X)$ is infinite.

Hint. Use a valuative argument to show that $\mathrm{div}_X(t) = 3D$ for some divisor D on X. This implies that all symbols $(\chi, t) \in \mathrm{Br}(\mathbb{Q}(t))$ with $\chi \in \mathrm{H}^1(\mathbb{Q}, \mathbb{Z}/3)$ and t is viewed in $\mathbb{Q}(t)^*/\mathbb{Q}(t)^{*3} = \mathrm{H}^1(\mathbb{Q}(t), \mu_3)$ are unramified over X. Thus the image of the composition, where the first arrow sends χ to (χ, t),

$$\mathrm{H}^1(\mathbb{Q}, \mathbb{Z}/3) \hookrightarrow \mathrm{Br}(\mathbb{Q}(t)) \longrightarrow \mathrm{Br}(X_\eta) \hookrightarrow \mathrm{Br}(\mathbb{Q}(X))$$

belongs to the subgroup $\mathrm{Br}(X) \subset \mathrm{Br}(\mathbb{Q}(X))$. The group $\mathrm{H}^1(\mathbb{Q}, \mathbb{Z}/3)$ is infinite. The map $\mathrm{H}^1(\mathbb{Q}, \mathbb{Z}/3) \hookrightarrow \mathrm{Br}(\mathbb{Q}(t))$ defined by $\chi \mapsto (\chi, t)$ is injective, as one sees by taking the residue at $t = 0$. By the exact sequence (4.9) the kernel of $\mathrm{Br}(\mathbb{Q}(t)) \to \mathrm{Br}(X_\eta)$ is the image of $\mathrm{Pic}(X_\eta \times_{\mathbb{Q}(t)} \overline{\mathbb{Q}(t)})^G$, where G is the absolute Galois group of $\mathbb{Q}(t)$. By the theorems of Mordell–Weil and Néron, the group $\mathrm{Pic}(X_\eta \times_{\mathbb{Q}(t)} \overline{\mathbb{Q}(t)})^G$ is finitely generated, hence its image in the torsion group $\mathrm{Br}(\mathbb{Q}(t))$ is finite.

In this example the divisor of the function t is divisible by 3. In fact, we can consider the following general situation.

Proposition 10.1.8 *Let X be a smooth, projective, geometrically integral variety over a field k of characteristic 0. Let $F \in k(X)^*$ be a non-constant rational function. Write the divisor of F as $\sum_{i=1}^n m_i D_i$, where each $D_i \subset X$ is an integral divisor. Let m be the g.c.d. of $m_1, \ldots, m_n$. Let k_i be the algebraic closure of k in the function field $k(D_i)$. Define*

$$\mathcal{L}(F) = \bigcap_{i=1}^n \mathrm{Ker}[m_i \mathrm{res}_{k_i/k} : \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^1(k_i, \mathbb{Q}/\mathbb{Z})].$$

Let $\mathrm{Br}_F(X)$ be the intersection of $\mathrm{Br}(X)$ with the image of the homomorphism $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Br}(k(X))$ associating to $\chi \in \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ the class of the cyclic algebra (F, χ) of degree d in $\mathrm{Br}(k(X))$, where d is the order of χ. Then we have the following statements.

(i) *The group $\mathrm{Br}_F(X)$ consists of the classes (F, χ), where $\chi \in \mathcal{L}(F)$.*

(ii) *If $m = 1$, then $\mathrm{Br}_F(X)$ is finite modulo $\mathrm{Br}(k)$. If $m > 1$ and k is finitely generated over $\mathbb{Q}$, then $\mathrm{Br}_F(X)$ is infinite modulo $\mathrm{Br}(k)$.*

(iii) *Let $\widetilde{X} \to X$ be a proper birational morphism such that F defines a surjective morphism $f : \widetilde{X} \to \mathbb{P}^1_k$. Then $\mathrm{Br}_F(X) \subset \mathrm{Br}_{\mathrm{vert}}(\widetilde{X}/\mathbb{P}^1_k)$.*

Proof. (i) is immediate by a computation of residues of (F, χ).

(ii) Lemma 10.1.3 gives that $\mathcal{L}(F)$ is finite if $m = 1$. For $m > 1$ use the hint to Exercise 10.1.7.

(iii) is obvious, because $(F, \chi) = (t, \chi)$, where t be the coordinate on $\mathbb{P}^1_k$ such that $F = t \circ f$. □

Using diagram (10.1) one can compute the Brauer group of a product of two varieties, under a simplifying assumption on the geometry of one of them. (For more general statements see Sections 4.6 and 15.4.)

Proposition 10.1.9 *Let k be a field of characteristic 0. Let X and Y be smooth, projective, geometrically integral varieties over k. Assume $X(k) \neq \emptyset$. If $\mathrm{Pic}(\overline{X})$ is torsion-free and $\mathrm{Br}(\overline{X}) = 0$, then $\mathrm{Br}(X \times_k Y)$ is generated by the inverse images of $\mathrm{Br}(X)$ and $\mathrm{Br}(Y)$ with respect to the maps induced by projections. If, moreover, $\mathrm{H}^1(k, \mathrm{Pic}(\overline{X})) = 0$, then the map $\mathrm{Br}(Y) \to \mathrm{Br}(X \times_k Y)$ is an isomorphism.*

Proof. The assumptions on X imply that $\mathrm{Br}(X) = \mathrm{Br}_1(X)$ and give a split exact sequence (see Section 4.3)

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(\overline{X})) \longrightarrow 0.$$

If one extends the ground field from k to the function field $K = k(Y)$ of Y, the assumptions on the geometric Picard group and on the geometric Brauer group of X_K over the algebraic closure $\overline{K}$ of K are preserved. For the Picard group, see Section 4.1. For the Brauer group, see Proposition 4.2.2. Thus one still has the analogous exact sequence for the K-variety X_K. Moreover, the map $\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(X_{\overline{K}})$ is an isomorphism and the absolute Galois group of $k(X)$ acts on these finitely generated free abelian groups via its quotient Γ_k, which gives an isomorphism $\mathrm{H}^1(k, \mathrm{Pic}(\overline{X})) \tilde{\longrightarrow} \mathrm{H}^1(K, \mathrm{Pic}(X_{\overline{K}}))$. The compatible split exact sequences

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(\overline{X})) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \mathrm{Br}(X_K) \longrightarrow \mathrm{H}^1(K, \mathrm{Pic}(X_{\overline{K}})) \longrightarrow 0$$

thus give that the natural map $\mathrm{Br}(X)/\mathrm{Br}(k) \to \mathrm{Br}(X_K)/\mathrm{Br}(K)$ is an isomorphism. All fibres of the projection $X \times_k Y \to Y$ are geometrically integral. Diagram 10.1 applied to the projection $X \times_k Y \to Y$ then immediately gives that $\mathrm{Br}(X \times_k Y)$ is generated by the sum of the images of $\mathrm{Br}(Y)$ and $\mathrm{Br}(X)$ under the two projections. Compare with [GA18]. □

We conclude this section by proving a statement announced in Chapter 8.

Theorem 10.1.10 *Let k be a field of characteristic 0. Let $f : X \to Y$ be a dominant morphism of smooth, projective, geometrically integral varieties over k. Assume that the generic fibre X_η is birationally equivalent to a $k(Y)$-torsor for a simply connected semisimple group over $k(Y)$. Then the map $f^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is an isomorphism.*

Proof. We have a commutative diagram of natural pullback maps

$$\begin{array}{ccc} \mathrm{Br}(X) & \hookrightarrow & \mathrm{Br}(X_\eta) \\ \uparrow & & \uparrow \cong \\ \mathrm{Br}(Y) & \hookrightarrow & \mathrm{Br}(k(Y)) \end{array}$$

The injectivity of horizontal arrows is due to the fact that X and Y are smooth and integral. The right hand vertical arrow is an isomorphism by Proposition 8.2.1. Thus $\mathrm{Br}(X) = \mathrm{Br}_{\mathrm{vert}}(X/Y)$. Now the result follows from Proposition 9.1.13 in view of Corollary 10.1.5 (ii). □

10.2 Conic bundles over the projective line

In this section we assume that the ground field k has characteristic zero. With extra care, one could extend most results over an arbitrary ground field.

Definition 10.2.1 *A* **conic bundle** *over $\mathbb{P}^1_k$ is a smooth, projective, geometrically integral surface X over a field k equipped with a morphism $X\to\mathbb{P}^1_k$ whose generic fibre X_η is a smooth conic over $K = k(\mathbb{P}^1_k)$.*

Let $A \in \mathrm{Br}(K)$ be the class of the quaternion algebra associated to the conic X_η. If $A = 0$, then $X_\eta \simeq \mathbb{P}^1_K$, hence $\mathrm{Br}(X_\eta) = \mathrm{Br}(K)$.

If $A \neq 0$, or, equivalently, X_η has no K-point, i.e., $X_\eta \not\simeq \mathbb{P}^1_K$, then, by Proposition 6.2.1, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Br}(K) \longrightarrow \mathrm{Br}(X_\eta) \longrightarrow 0, \tag{10.3}$$

where $1 \in \mathbb{Z}/2$ is mapped to $A \in \mathrm{Br}(K)$.

After suitable birational transformations respecting the projection to $\mathbb{P}^1_k$, one may assume that for each closed point $P = \mathrm{Spec}(k(P)) \in \mathbb{P}^1_k$ the fibre X_P is a reduced conic and that $X\to\mathbb{P}^1_k$ is relatively minimal. The last property means that no fibre X_P contains a curve that can be contracted onto a smooth point.

Let S be the finite set of closed points $P \in \mathbb{P}^1_k$ such that the fibre X_P is not smooth over $k(P)$, or, equivalently, is not geometrically integral. To such a point P one associates a quadratic field extension $F_P/k(P)$ over which X_P decomposes as a pair of transversal lines, with a unique intersection point defined over $k(P)$. Let us write $F_P = k(P)(\sqrt{a_P})$, where $a_P \in k(P)^*$.

By Proposition 10.1.4, since A goes to 0 in $\mathrm{Br}(X_\eta)$, for each point $P \in S$ the residue $\partial_P(A) \in \mathrm{H}^1(k(P), \mathbb{Q}/\mathbb{Z})$ lies in the subgroup

$$\mathrm{H}^1(F_P/k(P), \mathbb{Z}/2) = \mathrm{Ker}[\mathrm{H}^1(k(P), \mathbb{Z}/2) \longrightarrow \mathrm{H}^1(F_P, \mathbb{Z}/2)] \cong \mathbb{Z}/2.$$

A local calculation with a diagonal equation of X_η shows that $\partial_P(A)$ is the generator of this group, i.e. the class of a_P in $k(P)^*/k(P)^{*2} = \mathrm{H}^1(k(P), \mathbb{Z}/2)$.

Lemma 10.2.2 *Let $X\to\mathbb{P}^1_k$ be a relatively minimal conic bundle as above. Then $\mathrm{Br}_{\mathrm{vert}}(X) = \mathrm{Br}(X)$. The following properties are equivalent.*

(a) *The class A is in the image of $\mathrm{Br}(k)\to\mathrm{Br}(K)$.*

(b) *There exists a smooth conic C over k such that X is birationally equivalent over $\mathbb{P}^1_k$ to the constant conic bundle $C \times \mathbb{P}^1_k\to\mathbb{P}^1_k$.*

(c) *For each closed point $P \in \mathbb{P}^1_k$, the fibre X_P is smooth.*

If these properties do not hold, then the map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is injective.

Proof. Since $\mathrm{Br}(K)\to\mathrm{Br}(X_\eta)$ is surjective and $\mathrm{Br}(X)\to\mathrm{Br}(X_\eta)$ is injective (as X is smooth over k), we have $\mathrm{Br}_{\mathrm{vert}}(X) = \mathrm{Br}(X)$. The class of A in $\mathrm{Br}(K)$ is given by a quaternion (a, b), with $a, b \in K^* = k(\mathbb{P}^1)^*$. Since k is infinite, there exists a k-point P in $\mathbb{P}^1_k$ where a and b are invertible. Under assumption (a), the class $(a, b) \in \mathrm{Br}(k(t))$ coincides with the image of $(a(P), b(P)) \in \mathrm{Br}(k)$ in

$\mathrm{Br}(k(\mathbb{P}^1))$. Thus (a) implies (b). Under assumption (b), the generic fibre of $X\to\mathbb{P}^1_k$ is isomorphic to $C_{k(\mathbb{P}^1)}$. This gives (a).

The equivalence of (a) and (c) follows from the Faddeev exact sequence (1.26) via the interpretation of the class a_P in $k(P)^*/k(P)^{*2}$ as the residue of A at P.

The kernel of $\mathrm{Br}(k)\to\mathrm{Br}(X)\hookrightarrow\mathrm{Br}(k(X))$ is equal to the kernel of the composition

$$\mathrm{Br}(k)\hookrightarrow\mathrm{Br}(K)\to\mathrm{Br}(X_\eta)\hookrightarrow\mathrm{Br}(k(X)).$$

By (10.3), this map is injective unless $X_\eta = C\times_k K$, where C is a conic over k. □

Proposition 10.2.3 *Let $f : X\to\mathbb{P}^1_k$ be a relatively minimal conic bundle as above. If the class of X_η is not in the image of $\mathrm{Br}(k)\to\mathrm{Br}(K)$, then there is an exact sequence*

$$0\longrightarrow\mathrm{Br}(k)\longrightarrow\mathrm{Br}(X)\longrightarrow\bigoplus_{P\in S}(\mathbb{Z}/2)_P/\langle\partial(A)\rangle\longrightarrow k^*/k^{*2},$$

where $\partial(A)\in\bigoplus_{P\in S}(\mathbb{Z}/2)_P=\bigoplus_{P\in S}\mathrm{H}^1(F_P/k(P),\mathbb{Z}/2)$ is the vector with coordinates $\partial_P(A)$. The last map sends $1\in(\mathbb{Z}/2)_P$ to the class of the norm $\mathrm{N}_{k(P)/k}(a_P)$ in k^*/k^{*2}.

Proof. When the base is $\mathbb{P}^1_k$, the commutative diagram (10.1) whose bottom row is extended to the right as in the Faddeev exact sequence (1.26) and the middle column is extended as in (10.3), takes the following form:

$$\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & & \uparrow & & & & & & \\
0 & \to & \mathrm{Br}(X) & \to & \mathrm{Br}(X_\eta) & \to & \bigoplus_P \mathrm{H}^1(k(X_P),\mathbb{Q}/\mathbb{Z}) & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \to & \mathrm{Br}(k) & \to & \mathrm{Br}(K) & \to & \bigoplus_P \mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z}) & \to & \mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z}) & \to & 0 \\
 & & & & \uparrow & & & & & & \\
 & & & & \mathbb{Z}/2 & & & & & & \\
 & & & & \uparrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array}$$

The corestriction map $\mathrm{cores}_{k(P)/k} : \mathrm{H}^1(k(P),\mathbb{Z}/2)\longrightarrow\mathrm{H}^1(k,\mathbb{Z}/2)$ is the map $k(P)^*/k(P)^{*2}\to k^*/k^{*2}$ induced by the norm $\mathrm{N}_{k(P)/k}$. Indeed, the Kummer exact sequence in view of Hilbert's theorem 90 shows that this map comes from $\mathrm{cores}_{k(P)/k} : \mathrm{H}^0(k(P),k_s^*)\longrightarrow\mathrm{H}^1(k,k_s^*)$ which is $\mathrm{N}_{k(P)/k} : k(P)^*\to k^*$.

The proposition then follows from a diagram chase and Lemma 10.2.2. □

Corollary 10.2.4 *Let $f : X\to\mathbb{P}^1_k$ be a relatively minimal conic bundle as above. Assume that the class of X_η is not in the image of $\mathrm{Br}(k)\to\mathrm{Br}(K)$. Fix a k-point $M\in\mathbb{P}^1_k$ with smooth fibre. Let $S\subset\mathbb{P}^1_k$ be the finite set of closed points with* *singular fibre. Let* $\mathbb{A}^1_k \subset \mathbb{P}^1_k$ *be the complement to* M*. Let* t *be the coordinate on* $\mathbb{A}^1_k$*. Then we have a direct sum decomposition*

$$\mathrm{Br}(X) = \mathrm{Br}(k) \oplus f^*B,$$

where $B \subset \mathrm{Br}(K) = \mathrm{Br}(k(t))$ *is a finite subgroup whose elements have the following explicit description.*

A closed point $P \in S$ *is the zero set of a monic irreducible polynomial* $P(t) \in k[t]$*. Let* $\tau_P \in k(P)$ *be the image of* t *in* $k(P) = k[t]/(P(t))$*. Consider the subgroup* $B \subset \mathbb{F}_2^{|S|}$ *of vectors* $\varepsilon = (\varepsilon_P)$ *such that*

$$\prod_{P \in S} \mathrm{N}_{k(P)/k}(a_P)^{\varepsilon_P} = 1 \in k^*/k^{*2}.$$

The injective map $B \to \mathrm{Br}(K)$ *sends* ε *to*

$$A_\varepsilon = \sum_{P \in S} \varepsilon_P \,\mathrm{cores}_{k(P)/k}(t - \tau_P, a_P),$$

where $(t - \tau_P, a_P)$ *is the class of the quaternion algebra* $Q(t - \tau_P, a_P)$ *in* $\mathrm{Br}(k(P))$*.*

Proof. A calculation based on Proposition 1.4.6 shows that $\partial_P(A_\varepsilon) = a_P^{\varepsilon_P}$. This shows that the map $B \to \mathrm{Br}(K)$ is indeed injective. Then the statement follows from Proposition 10.2.3. □

Exercise 10.2.5 *Show that each* A_ε *is unramified at the point at infinity* M *and, moreover,* $A_\varepsilon(M) = 0$*.*

Exercise 10.2.6 *Let* $P(x) \in k[x]$ *be a separable polynomial and let* $a \in k$*,* $a \notin k^{*2}$*. Let* $f : X \to \mathbb{P}^1_k$ *be a smooth projective model of the generalised Châtelet surface given by the affine equation*

$$y^2 - az^2 = P(x).$$

Prove the following statements.

(a) *If* $P(x)$ *is irreducible, or is the product of two irreducible polynomials of odd degree, then* $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$*.*

(b) *If* $P(x)$ *the product of two non-constant irreducible polynomials of even degree, each of which is irreducible over* $k(\sqrt{a})$*, then* $\mathrm{Br}(X)/\mathrm{Br}(k) = \mathbb{Z}/2$*.*

(c) *Assume that the degree of* $P(x)$ *is even. Let* n *be the number of monic irreducible factors of* $P(x)$ *of even degree which remain irreducible over* $k(\sqrt{a})$*. Let* m *be the number of monic factors of* $P(x)$ *of odd degree. Then* $\mathrm{Br}(X)/\mathrm{Br}(k) = (\mathbb{Z}/2)^s$*, where* $s = n - 1$ *if* $m = 0$*,* $s = n + m - 1$ *if* m *is odd, and* $s = n + m - 2$ *if* $m > 0$ *is even.*

Proposition 10.2.7 *Let $f : X \to Y$ be a proper surjective morphism of smooth geometrically integral varieties over a field k of characteristic 0 such that the generic fibre X_η is a smooth quadric of dimension at least 1. Suppose that all fibres over points of codimension 1 in Y are split, or $\dim(X_\eta) \geq 3$. Then the map $f^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is surjective.*

Proof. From Proposition 6.2.3 we see that $\mathrm{Br}(X) = \mathrm{Br}_{\mathrm{vert}}(X/Y)$. By Corollary 10.1.5 (ii) we have $\mathrm{Br}_{\mathrm{vert}}(X/Y) = f^*\mathrm{Br}(Y)$, whenever all fibres over points of codimension 1 of Y are split. It remains to show that the splitness condition is satisfied when $\dim(X_\eta) \geq 3$. Recall that, for $P \in Y$ of codimension 1, the property that X_P is split does not depend on the choice of a smooth and proper model $X \to Y$ over the local ring $\mathcal{O}_{P,Y}$, see Corollary 9.1.11. If $\dim(X_\eta) \geq 3$, then working with a diagonal quadratic form one constructs a model whose closed fibre is split. □

In Section 10.5 we shall consider quadric bundles $f : X \to Y$ for which the map $f^* : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is not surjective.

Remark 10.2.8 (1) Another way to compute $\mathrm{Br}(X)/\mathrm{Br}(k)$ is to identify the Galois module $\mathrm{Pic}(X^{\mathrm{s}})$, then to compute $\mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}}))$. By Remark 4.3.3 the last group is $\mathrm{Br}_1(X)/\mathrm{Br}(k)$, which coincides with $\mathrm{Br}(X)/\mathrm{Br}(k)$.

This method produces the finer invariant given by the Galois module $\mathrm{Pic}(X^{\mathrm{s}})$ but is slightly less effective for producing explicit generators of the group $\mathrm{Br}(X)$. Further references: [CTSS87], [Sko96], [Sko01, §7.1].

(2) Vertical Brauer groups have been computed in various set-ups, including some cases where the generic fibre is given but there is no explicit smooth projective model for the total space. Examples include families of quadrics of relative dimension 2 over $\mathbb{P}^1_k$ and families of Severi–Brauer varieties, see [Sko90] and [CTS94]. In these two cases one has $\mathrm{Br}_{\mathrm{vert}}(X) = \mathrm{Br}(X)$.

(3) More generally, one would like to compute $\mathrm{Br}(X)$ for a smooth, projective and geometrically integral variety X equipped with a morphism $X \to \mathbb{P}^1_k$ whose generic fibre is geometrically integral and contains an open subset isomorphic to a homogeneous space of a connected linear algebraic group G over $K = k(\mathbb{P}^1)$. In this case the fibration admits a section over a finite extension of k.

Already in the case when $G = T \times_k K$, where T is a k-torus, it is difficult to compute $\mathrm{Br}(X)$. The quotient of $\mathrm{Br}(X)$ by the subgroup $\mathrm{Br}_{\mathrm{vert}}(X)$ is a subgroup of a known group, namely the unramified Brauer group of the K-torus T_K modulo $\mathrm{Br}(K)$, but in general one does not know which subgroup. A concrete case is when the generic fibre of $X \to \mathbb{P}^1_k$ is birationally equivalent to the affine K-variety with equation

$$\mathrm{N}_{L/k}(\Xi) = P(t)$$

for a finite separable extension L/k and a non-zero polynomial $P(t) \in k[t]$. (The projection to $\mathbb{P}^1_k$ is given by the coordinate t.) For some computations in this direction see [CTHS03] and [Wei12]; see also [VV12].

10.3 Double covers

The following theorem is a special case of [Sko17, Thm. 1.1]. We refer to [Sko17] for the proof of this theorem and more general results.

Theorem 10.3.1 *Let k be an algebraically closed field of characteristic different from 2. Let S be a smooth, projective, integral surface such that $\mathrm{Pic}(S)[2] = 0$ and $\mathrm{Br}(S)[2] = 0$, for instance a rational surface. Let $\pi : X \to S$ be a double cover ramified exactly along a smooth irreducible curve C. Let $j : C \hookrightarrow X$ be the natural closed embedding. There is a natural map $\Phi : \mathrm{Pic}(C)[2] \to \mathrm{Br}(X)[2]$, which gives rise to an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(C)[2]/j^*(\mathrm{Pic}(X)[\pi_*]) \longrightarrow \mathrm{Br}(X)[2] \longrightarrow \mathrm{Pic}(S)/\pi_*(\mathrm{Pic}(X)) \longrightarrow 0.$$

Here one writes $\mathrm{Pic}(X)[\pi_*]$ for the kernel of $\pi_* : \mathrm{Pic}(X) \to \mathrm{Pic}(S)$.

In the special case when $S = \mathbb{P}^2_k$ we have $\mathrm{Pic}(S) = \mathrm{Pic}(\mathbb{P}^2_k) = \mathbb{Z}$, hence $\mathrm{Pic}(S)/\pi_*(\mathrm{Pic}(X))$ is 0 or $\mathbb{Z}/2$.

Here we content ourselves with giving the definition of the map Φ. It comes from the comparison of the Gysin sequences for étale cohomology groups of S and X with coefficients $\mu_2 = \mathbb{Z}/2$:

$$\begin{array}{ccccccc}
\mathrm{H}^2(X,\mu_2) & \longrightarrow & \mathrm{H}^2(X\setminus C,\mu_2) & \longrightarrow & \mathrm{H}^1(C,\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^3(X,\mu_2) \\
\uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle [0]} & & \uparrow{\scriptstyle \pi^*} \\
\mathrm{H}^2(S,\mu_2) & \longrightarrow & \mathrm{H}^2(S\setminus C,\mu_2) & \longrightarrow & \mathrm{H}^1(C,\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^3(S,\mu_2)
\end{array}$$

The morphism $\pi : X \to S$ is ramified along C with ramification index 2, hence the induced map $\mathrm{H}^1(C,\mathbb{Z}/2) \to \mathrm{H}^1(C,\mathbb{Z}/2)$ is zero.

Since S and X are smooth, the restriction maps

$$\mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(S\setminus C), \quad \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X\setminus C)$$

are surjective, and the restriction maps

$$\mathrm{Br}(S) \longrightarrow \mathrm{Br}(S\setminus C), \quad \mathrm{Br}(X) \longrightarrow \mathrm{Br}(X\setminus C)$$

are injective. Using the Kummer sequences with coefficients μ_2, one obtains a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Br}(X)[2] & \longrightarrow & \mathrm{Br}(X\setminus C)[2] & \longrightarrow & \mathrm{H}^1(C,\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^3(X,\mu_2) \\
 & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle [0]} & & \uparrow{\scriptstyle \pi^*} \\
0 & \longrightarrow & \mathrm{Br}(S)[2] & \longrightarrow & \mathrm{Br}(S\setminus C)[2] & \longrightarrow & \mathrm{H}^1(C,\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^3(S,\mu_2)
\end{array}$$

We thus get a map

$$\Phi : \mathrm{Ker}[\mathrm{H}^1(C,\mathbb{Z}/2) \to \mathrm{H}^3(S,\mu_2)] \longrightarrow \mathrm{Br}(X)[2]/\pi^*(\mathrm{Br}(S)[2]).$$

Assuming $\mathrm{Pic}(S)[2] = 0$, we have $\mathrm{H}^1(S, \mu_2) = 0$ and thus by Poincaré duality $\mathrm{H}^3(S, \mu_2) = 0$. If, moreover, $\mathrm{Br}(S)[2] = 0$, then we get a map

$$\Phi : \mathrm{Pic}(C)[2] = \mathrm{H}^1(C, \mathbb{Z}/2) \longrightarrow \mathrm{Br}(X)[2].$$

Remark 10.3.2 (1) The group $\mathrm{H}^1(C, \mathbb{Z}/2)$ is represented by rational functions $f \in k(C)^*$ such that $\mathrm{div}(f) = 2D$ for a divisor D on C. Such a rational function thus gives rise to an element of $\mathrm{Br}(X)[2] \subset \mathrm{Br}(k(X))[2]$. By Merkurjev's theorem every such class is a sum of quaternion algebras. When k is algebraically closed, $k(X)$ is a C_2-field by Tsen's theorem. By Albert's criterion [GS17, Thm. 1.5.5] the class of a sum of quaternion algebras in $\mathrm{Br}(k(X))$ is equal to the class of a single quaternion algebra. It seems quite a challenge to construct such a quaternion algebra explicitly starting from f.

(2) Other papers have been concerned with double and more generally cyclic covers [F92, vG05, CV15, IOOV17]. In [IOOV17] for a double cover of $S = \mathbb{P}^2$ as above, one constructs an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X)/(\mathbb{Z}\pi^*\mathcal{O}(1) + 2\mathrm{Pic}(X)) \longrightarrow (\mathrm{Pic}(C)/\mathbb{Z}K_C)[2] \longrightarrow \mathrm{Br}(X)[2] \longrightarrow 0,$$

where $K_C \in \mathrm{Pic}(C)$ is the canonical class. The map $(\mathrm{Pic}(C)/\mathbb{Z}K_C)[2] \to \mathrm{Br}(X)[2]$ has a description in terms of a geometric construction of Azumaya algebras on X. See also [CV15].

Remark 10.3.3 In a different direction, one can ask the following question. Suppose $X \to S$ is a double cover of smooth, projective, complex surfaces. Can one compute the kernel of the restriction map $\mathrm{Br}(S) \to \mathrm{Br}(X)$? A restriction-corestriction argument shows that this kernel is contained in $\mathrm{Br}(S)[2]$. An interesting case is that of an Enriques surface S and its unramified double $K3$-covering $X \to S$ over an algebraically closed field of characteristic zero. Here $\mathrm{Pic}(S) = \mathbb{Z}^{10} \oplus \mathbb{Z}/2$, $\mathrm{Br}(S) = \mathbb{Z}/2$, $\mathrm{Pic}(X)$ is torsion-free, and $\mathrm{Br}(X) \simeq (\mathbb{Q}/\mathbb{Z})^s$ for some integer $s > 0$. Beauville [Bea09] showed that the kernel of the map $\mathbb{Z}/2 = \mathrm{Br}(S) \to \mathrm{Br}(X) = (\mathbb{Q}/\mathbb{Z})^s$ depends on the Enriques surface S. He proved that in the (coarse) moduli space of Enriques surfaces, the surfaces S for which the kernel is non-zero, hence equal to $\mathbb{Z}/2$, form a countable, infinite union of non-empty algebraic hypersurfaces. In [HS05] one finds an example definable over $\mathbb{Q}$ for which the map $\mathrm{Br}(S) \to \mathrm{Br}(X)$ is injective.

One step in Beauville's proof is the following general result [Bea09, Prop. 4.1]. Let $\pi : X \to S$ be an étale, cyclic covering of smooth projective varieties over an algebraically closed field k. Let σ be a generator of the Galois group G of $\pi : X \to S$, and let $\mathrm{N} = \pi_* : \mathrm{Pic}(X) \to \mathrm{Pic}(S)$ be the natural norm homomorphism. Then the kernel of $\pi^* : \mathrm{Br}(S) \to \mathrm{Br}(X)$ is isomorphic to $\mathrm{Ker}(\mathrm{N})/(1 - \sigma^*)\mathrm{Pic}(X)$.

10.4 The universal family of cyclic twists

Let X be a smooth, proper and geometrically integral variety over a field k of characteristic 0 equipped with an action of μ_n. Assume that there is a dense open subset $U \subset X$ such that the morphism $\pi : U \to V$, where $V = U/\mu_n$, is a μ_n-torsor. This implies that V is smooth and geometrically integral.

For $a \in k^*$ let X_a be the cyclic twist of X by a, that is, the quotient of $X \times_k T_a$ by the diagonal action of μ_n, where T_a is the μ_n-torsor over k given by $x^n = a$. The twists are naturally parameterised by the points of $\mathbb{G}_{m,k}$ and there is a universal family of cyclic twists $\mathcal{X} \to \mathbb{G}_{m,k}$. More precisely, one defines $\mathcal{X}$ as the quotient of $X \times_k \mathbb{G}_{m,k}$ by the diagonal action of μ_n, where $\mu_n \subset \mathbb{G}_{m,k}$ acts on $\mathbb{G}_{m,k}$ by multiplication. Then $\mathcal{U} = (U \times_k \mathbb{G}_{m,k})/\mu_n$ is Zariski open in $\mathcal{X}$. The projection $U \times_k \mathbb{G}_{m,k} \to U$ gives rise to a map $\mathcal{U} \to V$ which is a $\mathbb{G}_{m,k}$-torsor. We have the following commutative diagram, where the vertical arrows are quotients by μ_n and the arrows pointing left are $\mathbb{G}_{m,k}$-torsors:

$$\begin{array}{ccccc} U & \longleftarrow & U \times_k \mathbb{G}_{m,k} & \longrightarrow & \mathbb{G}_{m,k} \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & \mathcal{U} & \longrightarrow & \mathbb{G}_{m,k} \end{array}$$

By Hilbert's theorem 90 any $\mathbb{G}_m$-torsor is trivial over the generic point. Hence $\mathcal{U}$, and thus $\mathcal{X}$, is stably birationally equivalent to V.

Using Hironaka's theorem, we can compactify $\mathcal{X}$ to a regular proper variety W equipped with a morphism $f : W \to \mathbb{P}^1_k$ so that $\mathcal{X} = f^{-1}(\mathbb{G}_{m,k})$. In particular, the generic fibre of $W \to \mathbb{P}^1_k$ is geometrically integral and the closed fibres away from 0 and ∞ are smooth (these fibres are twists of X, e.g., the fibre over $a \in k^*$ is X_a).

Let Y be a smooth proper variety over k containing V as a dense open subset. Since $\mathcal{X}$ and W are stably birationally equivalent to Y, we have a isomorphism $\mathrm{Br}(W) \cong \mathrm{Br}(Y)$.

In this section we compute the vertical Brauer group $\mathrm{Br}_{\mathrm{vert}}(W/\mathbb{P}^1_k)$ as a subgroup of $\mathrm{Br}(Y)$. The motivation for this comes from arithmetic. Suppose that k is a number field and Y is everywhere locally solvable. If $\mathrm{Br}_{\mathrm{vert}}(W/\mathbb{P}^1_k)$ gives no Brauer–Manin obstruction to the Hasse principle on Y, for example, if $\mathrm{Br}_{\mathrm{vert}}(W/\mathbb{P}^1_k) = \mathrm{Br}(k)$, then, under an appropriate assumption on the ramification, there is an $a \in k^*$ such that X_a is everywhere locally solvable, see Theorem 13.2.23. Moreover, an unobstructed family of points $P_v \in V(k_v)$ can be lifted to a family of points $Q \in U_a(k_v)$ on a twisted form of the torsor $\pi : U \to V$.

Let $[U/V]$ be the class of the torsor $\pi : U \to V$ in $\mathrm{H}^1_{\text{ét}}(V, \mu_n)$. Let $F \in k(Y)^*$ be a non-zero rational function such that the generic fibre of π is given by the equation $x^n = F$. Since U is geometrically integral, F is not a constant function. In $k(\mathcal{X}) = k(W)$ we have the relation $tu^n = F$, where $u \in k(W)^*$. Write $\mathrm{div}(F) = \sum_D m_D D$, where each D is an integral divisor in Y and m_D is a positive integer. Let k_D be the integral closure of k in $k(D)$. Recall from Proposition 10.1.8 the notation

$$\mathcal{L}(F) = \bigcap_D \mathrm{Ker}[m_D \,\mathrm{res}_{k_D/k} : \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^1(k_D, \mathbb{Q}/\mathbb{Z})]. \qquad (10.4)$$

For $\chi \in \mathrm{H}^1(k, \mathbb{Z}/n)$ we denote by $[U/V] \cup \chi \in \mathrm{H}^2_{\text{ét}}(V, \mu_n)$ the element obtained via the cup-product

$$\mathrm{H}^1_{\text{ét}}(V, \mu_n) \times \mathrm{H}^1(k, \mathbb{Z}/n) \longrightarrow \mathrm{H}^1_{\text{ét}}(V, \mu_n) \times \mathrm{H}^1(V, \mathbb{Z}/n) \longrightarrow \mathrm{H}^2_{\text{ét}}(V, \mu_n).$$

Let $A_\chi \in \mathrm{Br}(V)$ be the image of this element under the map $\mathrm{H}^2_{\text{ét}}(V, \mu_n) \to \mathrm{Br}(V)$ coming from the Kummer sequence. The restriction of A_χ to $\mathrm{Br}(k(Y))$ is the class of the cyclic algebra (χ, F). For each irreducible divisor D supported in $\mathrm{div}(F)$ we have $\partial_D(A_\chi) = m_D \mathrm{res}_{k_D/k}(\chi)$. (We have $\partial_D(A_\chi) = 0$ if D is not contained in $Y \smallsetminus V$.) Thus $A_\chi \in \mathrm{Br}(Y)$ if and only if $\chi \in \mathcal{L}(F)[n]$.

Proposition 10.4.1 (i) *The group* $\mathrm{Br}_{\mathrm{vert}}(W/\mathbb{P}^1_k) \subset \mathrm{Br}(k(W))$ *is generated by* $\mathrm{Br}(k)$ *and the classes* $A_\chi = (\chi, F) = f^*(\chi, t)$, *where* $\chi \in \mathcal{L}(F)[n]$.

(ii) *Let m be the g.c.d. of the integers m_D, for all integral divisors D in the support of* $\mathrm{div}(F)$. *If $(m, n) = 1$, then* $\mathrm{Br}_{\mathrm{vert}}(W/\mathbb{P}^1_k)$ *is finite modulo* $\mathrm{Br}(k)$.

(iii) *If $(m_D, n) = 1$ for some integral divisor D in the support of* $\mathrm{div}(F)$, *then each fibre of $f : W \to \mathbb{P}^1_k$ is geometrically split.*

Proof. (i) Let $\varphi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be the finite morphism given by $t = z^n$. By the definitions of $\mathcal{X}$ and W the base change of $W/\mathbb{P}^1_k$ along φ is a variety birationally equivalent to $X \times_k \mathbb{P}^1_k$ over $\mathbb{P}^1_k$. We have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Br}(k(z)) & \longrightarrow & \mathrm{Br}(k(X \times_k \mathbb{P}^1_k)) & \hookleftarrow & \mathrm{Br}(X \times_k \mathbb{P}^1_k) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Br}(k(t)) & \longrightarrow & \mathrm{Br}(k(W)) & \hookleftarrow & \mathrm{Br}(W)
\end{array}$$

where the Brauer groups in the right hand column are identified with the unramified (over k) subgroups of their ambient groups.

By definition, any element of $\mathrm{Br}_{\mathrm{vert}}(W/\mathbb{P}^1_k)$ comes from some $A \in \mathrm{Br}(k(t))$ whose image in $\mathrm{Br}(k(W))$ lies in $\mathrm{Br}(W)$. The fibres of $\mathcal{X} \to \mathbb{G}_{m,k}$ are geometrically integral, thus A can be ramified only at 0 and ∞. Let $\chi \in \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ be the residue of A at ∞. By the diagram, $\varphi^* A \in \mathrm{Br}(k(z))$ gives an element of $\mathrm{Br}(k(X \times_k \mathbb{P}^1_k))$ that lies in $\mathrm{Br}(X \times_k \mathbb{P}^1_k)$. However, all fibres of the projection $X \times_k \mathbb{P}^1_k \to \mathbb{P}^1_k$ are geometrically integral, which implies that already $\varphi^* A$ is unramified over k, so that $\varphi^* A \in \mathrm{Br}(\mathbb{P}^1_k) = \mathrm{Br}(k)$. The covering $\varphi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is ramified at ∞ with ramification index n, hence $n\chi = 0$. Thus $\chi \in \mathrm{H}^1(k, \mathbb{Z}/n)$. The Faddeev exact sequence implies that up to addition of an element of $\mathrm{Br}(k)$, the class A is represented by the cyclic algebra (χ, t). In $k(W)$ we have the relation $tu^n = F$, so the image of (χ, t) in $\mathrm{Br}(k(W)) = \mathrm{Br}(k(Y \times_k \mathbb{P}^1_k))$ is the image of $(\chi, F) \in \mathrm{Br}(k(Y))$, which is exactly $A_\chi \in \mathrm{Br}(V) \subset \mathrm{Br}(k(Y))$. Thus $(\chi, t) \in \mathrm{Br}(W)$ if and only if $A_\chi \in \mathrm{Br}(Y)$. We have seen that $A_\chi \in \mathrm{Br}(Y)$ if and only if $\chi \in \mathcal{L}(F)$. This proves (i).

(ii) Lemma 10.1.3 implies that $\mathcal{L}(F)[n]$ is finite in this case.

(iii) Since the fibres over the points other than $t = 0$ and $t = \infty$ are geometrically integral, it is enough to consider the fibre above 0 (the fibre above ∞ is treated similarly). This fibre has an integral component of multiplicity 1 if and only if the morphism $W\to\mathbb{P}^1_k$ has a section over $\bar{k}[[t]]$. By the valuative criterion of properness, it suffices to show that the generic fibre of $W\to\mathbb{P}^1_k$ has a $\bar{k}((t))$-point. The generic fibre is the cyclic cover of $Y\times_k\bar{k}((t))$ given by $x^n=t^{-1}F$. By assumption, there is an irreducible divisor $D\subset Y$ with $\mathrm{val}_D(F)=m$ such that $(m,n)=1$. Take $a,b\in\mathbb{Z}$ such that $am-bn=1$ and $a>0$. Consider the 'constant' $\bar{k}[[t]]$-scheme $\mathcal{Y}=Y\times_k\bar{k}[[t]]$ and let $\mathcal{D}=D\times_k\bar{k}[[t]]\subset\mathcal{Y}$. We can find a section s of $\mathcal{Y}\to\mathrm{Spec}(\bar{k}[[t]])$ such that the value of s at the generic point $\mathrm{Spec}(\bar{k}((t)))$ is outside the support of $\mathrm{div}(F)$ and the value of s at the closed point $\mathrm{Spec}(\bar{k})$ is contained in D but not in any other irreducible component of $\mathrm{div}(F)$; moreover, we can arrange that the intersection index of $\mathcal{D}$ and s in $\mathcal{Y}$ equals a. Let v be the valuation of the discrete valuation ring $\bar{k}[[t]]$. By the construction of s we have $v(F(s))=am$, hence $v(t^{-1}F(s))=am-1=bn$. Thus s lifts to a $\bar{k}((t))$-point on the cyclic cover of $Y\times_k\bar{k}((t))$ given by $x^n=t^{-1}F$. This means that the generic fibre of $W\to\mathbb{P}^1_k$ has a $\bar{k}((t))$-point. $\square$

We compute the group in Proposition 10.4.1 in two concrete situations. Let $p(x)$ and $q(y)$ be separable non-constant polynomials with coefficients in k, and let $n\geq 2$ be a positive integer. Let C_1 and C_2 be smooth, projective curves with affine equations $u^n=p(x)$ and $v^n=q(y)$, respectively.

Example A Let $n=2$. Consider the affine surface with equation $z^2=p(x)q(y)$. It is birationally equivalent to the quotient of $C_1\times_k C_2$ by the diagonal action of μ_2 on u and v. Indeed, $z=uv$ is invariant and satisfies $z^2=p(x)q(y)$. For example, if $p(x)$ and $q(x)$ are of degree 3 or 4, we obtain a K3 surface. If $\deg p(x)=\deg q(x)=3$, we obtain the Kummer surface associated to the product of elliptic curves C_1 and C_2. If $\deg p(x)=\deg q(x)=4$, we obtain the Kummer surface associated to a 2-covering of the product of Jacobians of C_1 and C_2. Such a situation occurs in [SkS05].

Example B Here we assume that $n=\deg p(x)=\deg q(x)$. Let $P(x,y)$ and $Q(z,w)$ be homogeneous forms of degree n such that $p(x)=P(x,1)$ and $q(x)=Q(x,1)$. The smooth surface $S\subset\mathbb{P}^3_k$ of degree n given by $P(x,y)=Q(z,w)$ is birationally equivalent to the quotient of $C_1\times_k C_2$ by the diagonal action of μ_n on u and v. Indeed, $z=u/v$ is invariant under this action of μ_n and satisfies $p(x)=q(y)z^n$. If $n=3$, then S is a smooth cubic surface; such a situation occurs in Swinnerton-Dyer's paper [SwD01]. If $n=4$, then S is a quartic K3 surface.

Let us consider both examples at the same time. In Example A, to fix ideas, we assume that the degrees of $p(x)$ and $q(y)$ are even. The ramification locus of the projection $C_1\to\mathbb{P}^1_k$ given by x is exactly the zero set of $p(x)$, and similarly for C_2. Define

$$L_1=k[x]/(p(x)),\quad L_2=k[y]/(q(y)),\quad L=L_1\otimes_k L_2.$$

Let $Z=\mathrm{Spec}(L)\subset C_1\times_k C_2$ be the closed subset given by $p(x)=q(y)=0$; this is the fixed locus of the action of μ_n. Let U be the complement to Z in $C_1\times_k C_2$. It is clear that U is the largest open subset of $C_1\times_k C_2$ on which the diagonal action of μ_n is free. The singular locus of the quotient $(C_1 \times_k C_2)/\mu_n$ is $Z/\mu_n \cong Z$. We define Y as the minimal resolution of this quotient. Each singular $\bar{k}$-point of $(C_1 \times_k C_2)/\mu_n$ is an isolated quotient singularity with a well known resolution. Over the completion of its local ring, it is isomorphic to the vertex of the affine cone over the rational normal curve of degree n. The exceptional divisor of the resolution is a smooth irreducible rational curve E with $(E^2) = -n$.

Let X be the blow-up of Z in $C_1 \times_k C_2$. Then we have a finite morphism $\pi : X \to Y$ of smooth projective varieties whose restriction to U is a torsor $\pi : U \to V$ with structure group μ_n. We have the following commutative diagram where the vertical arrows are quotient morphisms by the action of μ_n and the horizontal arrows are birational morphisms:

$$\begin{array}{ccccc} C_1 \times_k C_2 & \longleftarrow & X & \longrightarrow & S' \\ \downarrow & & \downarrow & & \downarrow \\ (C_1 \times_k C_2)/\mu_n & \longleftarrow & Y & \longrightarrow & S \end{array}$$

The surfaces S and S' feature only in Example B: here $S' \subset \mathbb{P}^4_k$ is given by $t^n = P(x,y) = Q(z,w)$ and the action of μ_n on S' is by multiplication on the coordinate t. The natural projection $S' \to S$ is a torsor for μ_n away from its ramification divisor D which is given by $P(x,y) = Q(z,w) = 0$. (Geometrically this is the union of n^2 lines joining two sets of n points each. So $D_{\text{sing}}(\bar{k})$ consists of $2n$ points.) Note that S'_{sing} is the union of closed subsets $x = y = 0$ and $z = w = 0$; the image of S'_{sing} in S is D_{sing}. The morphism $X \to S'$ is obtained by blowing-up S'_{sing}, and the morphism $Y \to S$ is obtained by blowing-up D_{sing}.

With notation as before we can take $F = P(x,y)/x^n$, then

$$\mathcal{L}(F)[n] = \mathrm{H}^1(L/k, \mathbb{Z}/n) = \mathrm{Ker}[\mathrm{res}_{L/k} : \mathrm{H}^1(k, \mathbb{Z}/n) \to \mathrm{H}^1(L, \mathbb{Z}/n)].$$

Proposition 10.4.2 *Assume that we are in the situation of Example A, with* $n = 2$ *and* $\deg(p(x))$, $\deg(q(x))$ *even, or Example B, with* $n = \deg(p(x)) = \deg(q(x))$.

(i) *If* $\mathcal{L}$ *is generated by the subgroups* $\mathrm{H}^1(L_1/k, \mathbb{Z}/n)$ *and* $\mathrm{H}^1(L_2/k, \mathbb{Z}/n)$, *then* $\mathrm{Br}_{\mathrm{vert}}(W/\mathbb{P}^1_k) = \mathrm{Br}_0(Y)$.

(ii) *For* $n = 2$ *the condition of* (i) *is satisfied when each of* $p(x)$ *and* $q(y)$ *is irreducible with a pluriquadratic splitting field.*

(iii) *If* n *is a prime number, the condition of* (i) *is satisfied when*

$$p(x) = a_1 x^n + a_2, \quad q(y) = a_3 y^n + a_4, \quad \text{where} \quad a_1, a_2, a_3, a_4 \in k^*.$$

Proof. (i) Recall that C_1 and C_2 are curves with affine equations $u^n = p(x)$ and $v^n = q(y)$, respectively. We have two natural morphisms $Y \to C_i \to \mathbb{P}^1_k$ given by the projections to the coordinates x and y, respectively. The rational function F on Y can be represented by either $p(x)$ or $q(y)$ modulo n-th powers. Thus, if $\chi \in \mathrm{H}^1(L_1/k, \mathbb{Z}/n)$, then $A_\chi = (p(x), \chi) \in \mathrm{Br}(Y)$ belongs to the image of $\mathrm{Br}(k(x))$

in $\mathrm{Br}(k(Y))$. As an element of $\mathrm{Br}(k(x))$, the class $(p(x), \chi)$ is unramified away from the closed points of $\mathbb{A}^1_k$ given by the monic irreducible factors $r(x)$ of $p(x)$. The residue at the closed point $r \in \mathbb{A}^1_k$ given by $r(x) = 0$ is the restriction $\mathrm{res}_{k_r/k}(\chi) \in \mathrm{H}^1(k_r, \mathbb{Z}/n)$, where $k_r = k[x]/(r(x))$. Since $L_1 = \oplus_r k_r$, where the sum is over all monic irreducible $r(x)$ dividing $p(x)$, we have $\mathrm{res}_{k_r/k}(\chi) = 0$. Hence $(p(x), \chi)$ is unramified everywhere on $\mathbb{A}^1_k$. This implies that $(p(x), \chi) \in \mathrm{Br}(k)$. Similar considerations apply to the case $\chi \in \mathrm{H}^1(L_2/k, \mathbb{Z}/n)$. This proves (i).

(ii) In this case L is the direct sum of copies of L_1L_2, the compositum of L_1 and L_2. All these fields are pluriquadratic extensions of k, and the statement follows at once.

(iii) In this case n is coprime to $[k(\zeta) : k] = n - 1$, where ζ is a primitive n-th root of unity. A restriction-corestriction argument then shows that it is enough to establish (i) for $k = k(\zeta)$, but this is straightforward. □

If $p(x)$ and $q(y)$ are very general, then the map $k^*/k^{*2} \to L^*/L^{*2}$ is injective. Such is the case if $p(x)$ and $q(y)$ are both irreducible of degree 4, the Galois closure of each of the extensions $k[x]/(p(x))$ and $k[y]/(q(y))$ is an extension of k whose Galois group is the symmetric group S_4, and these Galois extensions are linearly disjoint. See [HS16, Prop. 3.1, Lemma 2.1].

For a proof of (iii) in terms of valuations which avoids discussing the geometry of underlying varieties, see [CT03, Prop. 3.5].

10.5 Conic bundles over a complex surface

The following result is due to Artin and Mumford [AM72, §3, Thm. 1]. We give a different proof based on the Bloch–Ogus theory.

Theorem 10.5.1 (Artin–Mumford) *Let S be a smooth, projective, rational surface over $\mathbb{C}$. There is an exact sequence*

$$0 \to \mathrm{Br}(\mathbb{C}(S)) \to \bigoplus_{x \in S^{(1)}} \mathrm{H}^1(\mathbb{C}(x), \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{y \in S^{(2)}} \mathbb{Q}/\mathbb{Z}(-1) \to \mathbb{Q}/\mathbb{Z}(-1) \to 0.$$

The map $\mathrm{Br}(\mathbb{C}(S)) \to \mathrm{H}^1(\mathbb{C}(x), \mathbb{Q}/\mathbb{Z})$, is the residue map ∂_x attached to $x \in S^{(1)}$. The map $\partial_y : \mathrm{H}^1(\mathbb{C}(x), \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}(-1)$ attached to $y \in S^{(2)}$ is zero when y is not in the closure of x, otherwise it is the sum of residue maps computed on the normalisation of the closure of x. The last map is the sum.

Proof. Let X be a smooth integral variety over $\mathbb{C}$. We write η for the generic point $\mathrm{Spec}(\mathbb{C}(X))$. Let $n > 0$, $q \geq 0$ and j be integers. Let $\mathcal{H}^q(\mu_n^{\otimes j})$ be the Zariski sheaf on X associated to the presheaf

$$U \mapsto \mathrm{H}^q_{\text{ét}}(U, \mu_n^{\otimes j}).$$

Then there is the local-to-global spectral sequence

$$E_2^{pq} = \mathrm{H}^p_{\mathrm{zar}}(X, \mathcal{H}^q(\mu_n^{\otimes j})) \Rightarrow \mathrm{H}^n_{\text{ét}}(X, \mu_n^{\otimes j}). \tag{10.5}$$

By the Gersten conjecture for étale cohomology proved by Bloch and Ogus in 1974 (see [CT95a, CTKH97]) there is an exact sequence of Zariski sheaves

$$0\to\mathcal{H}^q(\mu_n^{\otimes j})\to i_{\eta *}\mathrm{H}^q_{\text{ét}}(\mathbb{C}(X),\mu_n^{\otimes j})\to \bigoplus_{x\in X^{(1)}} i_{x*}\mathrm{H}^{q-1}_{\text{ét}}(\mathbb{C}(x),\mu_n^{\otimes (j-1)})\to\dots \quad (10.6)$$

which is a flasque resolution of the sheaf $\mathcal{H}^q(\mu_n^{\otimes j})$. Here $i_{\eta *}$ is the map induced by the natural map $i:\eta\to X$, and similarly for i_{x*}. The maps in the exact sequence are residue maps, as explained by Kato in [Kat86]. In particular, we obtain

$$\mathrm{H}^p_{\text{zar}}(X,\mathcal{H}^q(\mu_n^{\otimes j}))=0,\quad p>q.$$

Together with the spectral sequence (10.5) this gives an injective map

$$\mathrm{H}^1_{\text{zar}}(X,\mathcal{H}^2(\mu_n^{\otimes j}))\hookrightarrow \mathrm{H}^3_{\text{ét}}(X,\mu_n^{\otimes j}). \quad (10.7)$$

Now set $q=2$ and $j=2$. Taking global sections of the flasque resolution (10.6) we obtain a complex

$$0\longrightarrow \mathrm{H}^2_{\text{ét}}(\mathbb{C}(X),\mu_n^{\otimes 2})\longrightarrow \bigoplus_{x\in X^{(1)}}\mathrm{H}^1_{\text{ét}}(\mathbb{C}(x),\mu_n)\longrightarrow \bigoplus_{x\in X^{(2)}}\mathbb{Z}/n\longrightarrow 0. \quad (10.8)$$

By the purity theorem for the Brauer group we have an exact sequence

$$0\longrightarrow \mathrm{Br}(X)[n]\longrightarrow \mathrm{Br}(\mathbb{C}(X))[n]\longrightarrow \bigoplus_{x\in X^{(1)}}\mathrm{H}^1_{\text{ét}}(\mathbb{C}(x),\mathbb{Z}/n).$$

It shows that the cohomology group of (10.8) at $\mathrm{H}^2_{\text{ét}}(\mathbb{C}(X),\mu_n^{\otimes 2})$ is canonically isomorphic to $\mathrm{Br}(X)[n]\otimes\mu_n$. The cohomology group at the middle term is $\mathrm{H}^1_{\text{zar}}(X,\mathcal{H}^2(\mu_n^{\otimes j}))$. Finally, the cohomology group at the right term is the cokernel of the map

$$\bigoplus_{x\in X^{(1)}}\mathbb{C}(x)^*/\mathbb{C}(x)^{*n}\longrightarrow \bigoplus_{x\in X^{(2)}}\mathbb{Z}/n$$

induced by the divisor map on the normalisation of the closure of x in X. This group is $\mathrm{CH}^2(X)/n$, the mod n quotient of the codimension 2 Chow group $\mathrm{CH}^2(X)$.

Let us specialise to the case when $X=S$ is a smooth and projective rational surface. Since S is simply connected, we have $\mathrm{H}^1_{\text{ét}}(S,\mathbb{Z}/n)=0$. By Poincaré duality this implies $\mathrm{H}^3_{\text{ét}}(S,\mathbb{Z}/n)=0$. Now the inclusion (10.7) gives $\mathrm{H}^1_{\text{zar}}(S,\mathcal{H}^2(\mu_n^{\otimes j}))=0$. For any smooth, projective, integral variety over $\mathbb{C}$ the Chow group of zero-cycles of *degree zero* is divisible, as one sees by reducing to the case of curves. Hence the degree map $\mathrm{CH}^2(S)\to\mathbb{Z}$ induces an isomorphism $\mathrm{CH}^2(S)/n\xrightarrow{\sim}\mathbb{Z}/n$.

Since S is a smooth and projective rational variety, we have $\mathrm{Br}(S)=0$ by Corollary 5.2.6. The complex (10.8) then gives the exact sequence

$$0\to\mathrm{Br}(\mathbb{C}(S))[n]\otimes\mu_n\to\oplus_{x\in X^{(1)}}\mathbb{C}(x)^*/\mathbb{C}(x)^{*n}\to\oplus_{x\in X^{(2)}}\mathbb{Z}/n\to\mathbb{Z}/n\to 0.$$

Twisting by $\mu_n^{\otimes(-1)}$ and passing to the direct limit over all integers n gives the exact sequence of the theorem. □

Theorem 10.5.2 *Let S be an integral surface over $\mathbb{C}$. Any element of $\mathrm{Br}(\mathbb{C}(S))[2]$ is the class of a quaternion algebra.*

Proof. Any element of order 2 in the Brauer group of a field of characteristic not equal to 2 is the class of a tensor product of quaternion algebras. This is a special case of Merkurjev's theorem, itself a special case of the Merkurjev–Suslin theorem. In the special case when the field is the field of rational functions on a surface over $\mathbb{C}$, this was proved earlier by S. Bloch.

The tensor product of two quaternion algebras over $\mathbb{C}(S)$ is similar to a quaternion algebra. This follows from Albert's criterion [GS17, Thm. 1.5.5] and the fact that a quadratic form in at least 5 variables over $\mathbb{C}(S)$ has a nontrivial zero (Tsen–Lang). □

Corollary 10.5.3 *Let S be a smooth and projective rational surface over $\mathbb{C}$. Suppose that $\{\gamma_x\} \in \oplus_{x\in S^{(1)}}\mathrm{H}^1(\mathbb{C}(x),\mathbb{Z}/2)$ has trivial image in $\oplus_{y\in S^{(2)}}\mathbb{Z}/2$. Then there exists a quaternion algebra α over $\mathbb{C}(S)$ whose class in $\mathrm{Br}(\mathbb{C}(S))$ has residue $\gamma_x \in \mathrm{H}^1(\mathbb{C}(x),\mathbb{Z}/2)$ at each $x \in S^{(1)}$. The class of α in $\mathrm{Br}(\mathbb{C}(S))$ is uniquely defined.*

Proof. This follows from Theorems 10.5.1 and 10.5.2. □

Note that the above proof is far from constructive: it is not clear how to find rational functions f and g in $\mathbb{C}(S)^*$ such that $\alpha = (f,g) \in \mathrm{Br}(\mathbb{C}(S))[2]$.

Proposition 10.5.4 *Let S be a smooth surface over $\mathbb{C}$. Let $\pi : X\to S$ be a proper morphism. If X is smooth and all the fibres of π are conics, then the morphism π is flat and the ramification locus $C \subset S$ is a curve with at most ordinary quadratic singularities.*

Proof. See [Bea77, Ch. I, Prop. 1.2]. □

We could not find the following general formula in the literature.

Theorem 10.5.5 *Let S be a smooth and projective rational surface over $\mathbb{C}$. Let X be a smooth threefold equipped with a dominant morphism $\pi : X\to S$ whose generic fibre is a smooth conic. Let $\alpha \in \mathrm{Br}(\mathbb{C}(S))[2]$ be the associated quaternion algebra class. Assume that $\alpha \neq 0$. Let $C_1,\ldots,C_n$ be the integral curves in S such that the residue of α at the generic point of C_i is non-zero:*

$$0 \neq \partial_{C_i}(\alpha) \in \mathrm{H}^1(\mathbb{C}(C_i),\mathbb{Z}/2) = \mathbb{C}(C_i)^*/\mathbb{C}(C_i)^{*2}.$$

Assume that each C_i is smooth and the ramification locus $C = \cup_{i=1}^n C_i$ of α is a curve with at most ordinary quadratic singularities. Consider the subgroup $H \subset (\mathbb{Z}/2)^n$ consisting of the elements $(r_1,\ldots,r_n)$ such that for $i \neq j$ we have $r_i = r_j$ when there is a point $p \in C_i \cap C_j$ with the property that $\partial_p(\partial_{C_i}(\alpha)) = \partial_p(\partial_{C_j}(\alpha)) \in \mathbb{Z}/2$ is non-zero. Then $\mathrm{Br}(X)$ is the quotient of H by the diagonal element $(1,\ldots,1)$ which is the image of α.

Proof. The generic fibre X_η of π is a smooth conic over the function field $\mathbb{C}(S)$. By a result going back to Witt, the natural map $\mathrm{Br}(\mathbb{C}(S)) \to \mathrm{Br}(X_\eta)$ is surjective with kernel $\mathbb{Z}/2$ spanned by $\alpha \neq 0$ (Proposition 6.2.1). Pick any $\beta \in \mathrm{Br}(X)$. The image of β in $\mathrm{Br}(X_\eta)$ is the image of some $\rho \in \mathrm{Br}(\mathbb{C}(S))$. For $x \in S^{(1)}$ write $\gamma_x = \partial_x(\alpha)$. Comparing the residues of ρ on S and on X we see that for any $x \in S^{(1)}$ the residue of ρ in $\mathrm{H}^1(\mathbb{C}(x), \mathbb{Q}/\mathbb{Z})$ lies in the subgroup of $\mathrm{H}^1(\mathbb{C}(x), \mathbb{Z}/2)$ generated by γ_x. From Theorem 10.5.1 we conclude that ρ is of exponent 2, hence $\mathrm{Br}(X)$ is of exponent 2. Moreover, the injective image of $\mathrm{Br}(X)$ in $\mathrm{Br}(\mathbb{C}(X))$ coincides with the image of a certain subgroup of $\mathrm{Br}(\mathbb{C}(S))[2]$ under the natural map $\mathrm{Br}(\mathbb{C}(S))[2] \to \mathrm{Br}(\mathbb{C}(X))[2]$ (whose kernel $\mathbb{Z}/2$ is generated by α).

Let us prove that this subgroup consists of the classes $\rho \in \mathrm{Br}(\mathbb{C}(S))[2]$ unramified outside the C_i's and with the property that

$$(\partial_{C_1}(\rho), \ldots, \partial_{C_n}(\rho)) = (r_1\gamma_1, \ldots, r_n\gamma_n) \in \bigoplus_{i=1}^n \mathrm{H}^1(\mathbb{C}(C_i), \mathbb{Z}/2)$$

is in the kernel of the map

$$\bigoplus_{i=1}^n \mathrm{H}^1(\mathbb{C}(C_i), \mathbb{Z}/2) \to \oplus_{y \in S^{(2)}} \mathbb{Z}/2.$$

Indeed, let v be a discrete, rank one valuation on the function field $\mathbb{C}(X)$ of X. Let F_v be its residue field, which contains $\mathbb{C}$. Since π is proper, the valuation v centered at a point M of S. If M does not lie on one of the C_i's, then clearly $\pi^*(\rho)$ is unramified at v. If M is the generic point of one of the C_i's, then the residue of ρ at v is a multiple of the residue of α at v, hence is zero since $\alpha = 0$ in $\mathrm{Br}(\mathbb{C}(X))$.

Assume M is a closed point which lies on exactly one C_i. The residue γ_i can be represented by the class of a rational function which is invertible at M. One may lift this function to a rational function h on S invertible at M. If u is a local equation for $C_i \subset S$ at M, the difference $\alpha - (h, u)$ is in the Brauer group of the local ring of S at M, because its residues on the curves passing through M vanish. Thus the image of (h, u) in $\mathrm{Br}(\mathbb{C}(X))$ is unramified at v. Similarly, the difference $\rho - r_i(h, u)$ is in the Brauer group of the local ring of S at M. Hence the image of ρ in $\mathrm{Br}(\mathbb{C}(X))$ is unramified at v.

Let us now consider the case when M lies at the intersection of two curves C_1 and C_2.

Suppose first that $\partial_M(\partial_{C_1}(\alpha)) = \partial_M(\partial_{C_2}(\alpha)) = 0 \in \mathbb{Z}/2$. Let u, resp. v, be a local equation for $C_1 \subset S$, resp. $C_2 \subset S$ at M. One may find rational functions h_1 and h_2 invertible at M with the property that $\rho - r_1(h_1, u) - r_2(h_2, v)$ is in the Brauer group of the local ring of S at M. The residue of $\rho_{\mathbb{C}(X)}$ at v is then the class of a product of powers of $h_1(M)$ and $h_2(M)$ in F_v^*/F_v^{*2}, and that is 1, since $h_1(M)$ and $h_2(M)$ are in $\mathbb{C}^*$.

Suppose now that $\partial_M(\partial_{C_1}(\alpha)) = \partial_M(\partial_{C_2}(\alpha)) = 1 \in \mathbb{Z}/2$. By assumption, we then have $r_1 = r_2$. Thus locally around M, the residue of ρ is a multiple of the residue of α, hence there exists an integer s (equal to 0 or 1) such that $\rho - s\alpha$ is in the Brauer group of the local ring of S at M. Since α vanishes in $\mathrm{Br}(\mathbb{C}(X))$, we conclude that $\rho_{\mathbb{C}(X)}$ is unramified at v. □.

Remark 10.5.6 By a definition common in the literature on complex algebraic geometry, a "standard conic bundle" over a surface S is a proper flat morphism $f : X \to S$ of smooth projective varieties such that each fibre is a conic, the ramification locus is a simple normal crossings divisor with smooth components, and the fibration is relatively minimal. Assume that $X \to S$ is a standard conic bundle – this is a more stringent assumption than the hypothesis of Theorem 10.5.5. Then by [Bea77, Lemme 1.5.2] in each connected component of the ramification divisor the integers r_i are equal. Theorem 10.5.5 then gives the formula $\mathrm{Br}(X) \simeq (\mathbb{Z}/2)^{c-1}$, where c is the number of connected components of C. This result is mentioned by V.A. Iskovskikh [Isk97, Teorema, p. 206]; it can also be extracted from [Zag77].

10.6 Variations on the Artin–Mumford example

Now let us take $S = \mathbb{P}^2_{\mathbb{C}}$. Let E_1 and E_2 be two transversal smooth cubic curves E_1 and E_2. Let $\gamma_i \in \mathrm{H}^1_{\text{ét}}(E_i, \mathbb{Z}/2)$, $\gamma_i \neq 0$, for $i = 1, 2$. By Corollary 10.5.3 there exists a unique quaternion algebra class $\alpha \in \mathrm{Br}(\mathbb{C}(\mathbb{P}^2))[2]$ unramified outside of $E_1 \cup E_2$, with residues γ_1 on E_1 and γ_i on E_2. Let X be a smooth threefold with a morphism $X \to \mathbb{P}^2_{\mathbb{C}}$ whose generic fibre is a conic corresponding to $\alpha \in \mathrm{Br}(\mathbb{C}(\mathbb{P}^2))$. By Proposition 6.2.1 (Witt's theorem) the kernel of the map $\mathrm{Br}(\mathbb{C}(\mathbb{P}^2)) \to \mathrm{Br}(\mathbb{C}(X))$ is $\mathbb{Z}/2$ generated by α. Theorem 10.5.5 gives $\mathrm{Br}(X) = \mathbb{Z}/2$.

Artin and Mumford [AM72] provided a concrete example of such a situation and proved that $\mathrm{Br}(X) \neq 0$ by computing $\mathrm{H}^3(X(\mathbb{C}), \mathbb{Z})_{\mathrm{tors}}$ on an explicit smooth projective model. In [AM72, §2] they construct a singular variety V which is a double cover of $\mathbb{P}^3_{\mathbb{C}}$ ramified along a special quartic surface with 10 nodes. They compute an explicit resolution of singularities $\tilde{V} \to V$ and determine $\mathrm{H}^3(\tilde{V}, \mathbb{Z})_{\mathrm{tors}}$. In [AM72, §3, §4], they study general conic bundles over rational complex surfaces. At the end of §4, they come back to the variety V of §2 and show that it is birational to a conic bundle, and look at it from this point of view. Here are some details (cf. [CTO89]).

Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth conic with a homogeneous quadratic equation $q(x, y, t) = 0$. Fix three distinct points P_1, Q_1, R_1 on C and consider the divisor $2P_1 + 2Q_1 + 2R_1$ on C. The restriction map $\mathrm{H}^0(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}(3)) \to \mathrm{H}^0(C, \mathcal{O}_C(3))$ is surjective, since $\mathrm{H}^1(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}(j)) = 0$ for any $j \in \mathbb{Z}$. Thus there exists a cubic curve E_1 which meets C in the divisor $2P_1 + 2Q_1 + 2R_1$, that is, a cubic curve through P_1, Q_1, R_1, which is tangent to C at these points and whose equation is not a multiple of q. Repeat this construction for a disjoint triple of points P_2, Q_2, R_2 on C to obtain a cubic curve E_2. A Bertini argument shows that one can choose E_1 and E_2 which are smooth and intersect each other transversally outside of C. Let $h_1 = 0$ and $h_2 = 0$ be the equations of these cubic curves.

Let $l = 0$ be a general tangent line to C. Then it is not hard to check (see [CTO89]) that the quaternion algebra $(q/l^2, h_1h_2/l^6)$ defines a conic bundle over $\mathbb{P}^2_{\mathbb{C}}$ unramified outside $E_1 \cup E_2$ such that the residue at E_i is a non-zero element $\gamma_i \in \mathrm{H}^1(E_i, \mathbb{Z}/2)$, for $i = 1, 2$. Similarly, the unique non-trivial residue of the quaternion algebra $(q/l^2, h_1/l^3)$ is $\gamma_1 \in \mathrm{H}^1(E_1, \mathbb{Z}/2)$; thus this algebra defines a non-trivial element in $\mathrm{Br}(\mathbb{C}(X))$ which is actually in $\mathrm{Br}(X)$.

One advantage of this concrete representation is that it leads to a proof of the unirationality of this particular X. Indeed, the conic bundle acquires a rational section after the base change from $\mathbb{P}^2_{\mathbb{C}}$ to the double cover $z^2 = q(x, y, t)$. This equation defines a smooth quadric in $\mathbb{P}^3_{\mathbb{C}}$ which is a rational variety.

In Section 11.1.2 we shall use this very special example for a deformation argument.

Similar examples are given in [CTO89]. The ramification locus in [CTO89, Example 2.4] is a union of eight lines.

Exercise 10.6.1 *Let $X \to \mathbb{P}^2_{\mathbb{C}}$ be a smooth conic bundle as in Theorem 10.5.5. If the ramification locus $C = \cup_{i=1}^n C_i$ is a union of $n \leq 5$ lines without triple intersections, then* $\mathrm{Br}(X) = 0$.

In fact, one can drop the assumption about triple intersections. For this, blow up $\mathbb{P}^2_{\mathbb{C}}$ in the points where more than two lines meet. We obtain a surface S, where the reduced total transform of the 5 lines (including the exceptional curves produced in the process) is a divisor C with normal crossings. We also obtain a smooth conic bundle $X' \to S$ unramified outside C. *Check that for any initial configuration of 5 lines, we have* $\mathrm{Br}(X) = 0$.

Exercise 10.6.2 *Construct smooth conic bundles $X \to \mathbb{P}^2_{\mathbb{C}}$ with $\mathrm{Br}(X) \neq 0$ ramified exactly in the union of six lines.*

It is enough to take six lines in general position and partition them into two triples, say L_1, L_2, L_3 and M_1, M_2, M_3. Choose $\gamma_{L_1} \in \mathbb{C}(L_1)^*/\mathbb{C}(L_1)^{*2}$ to be the class of a rational function whose divisor on L_1 is $(L_1 \cap L_2) - (L_1 \cap L_3)$, and similarly for the other lines. One immediately checks that the assumptions of Corollary 10.5.3 are fulfilled for the family γ_x with $\gamma_x = \gamma_{L_1}$ at $x = L_1$, similarly at the other 5 lines, and $1 \in \mathbb{C}(x)^*/\mathbb{C}(x)^{*2}$ at other codimension 1 points. There thus exists a quaternion algebra (a, b) over $\mathbb{C}(S)$ which has exactly these residues. One may thus produce a conic bundle $X \to S = \mathbb{P}^2_{\mathbb{C}}$ with ramification locus the union of these 6 lines in $\mathbb{P}^2_{\mathbb{C}}$.

Choosing six lines tangent to a given smooth conic, one produces a degenerate version of the Artin–Mumford example.

Exercise 10.6.3 *Let (u, v) be the coordinates in $\mathbb{A}^2_{\mathbb{C}}$. Let $X \to \mathbb{A}^2_{\mathbb{C}}$ be the conic bundle given in $\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ by the equation*

$$S^2 v(v^2 - 1) - T^2 u(u^2 - 1) + uv(u^2 - v^2)W^2 = 0.$$

Let $Y \to \mathbb{A}^2_{\mathbb{C}}$ be the conic bundle given in $\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ by the equation

$$S^2 - uT^2 - vR^2 = 0.$$

By computing residues on $\mathbb{A}^2_{\mathbb{C}}$ *show that* X *and* Y *are birationally equivalent over* $\mathbb{A}^2_{\mathbb{C}}$. Hint. Use the fact that if two quaternions algebras have the same class in the Brauer group, then the associated conics are isomorphic. Conclude that X is rational over $\mathbb{C}$. For background and a detailed proof, see [CT15].

Exercise 10.6.4 *A construction of a unirational but not stably rational variety fibred in Severi–Brauer varieties over* $\mathbb{P}^2_{\mathbb{C}}$. In [CTO89, Exemple 2.4] one constructs a non-trivial unramified Brauer class in the function field of a conic bundle over $\mathbb{P}^2_{\mathbb{C}}$ without actually producing a nice explicit model. This example can be generalised.

Let p be a prime. Let L_1, respectively L_2, be the line in $\mathbb{P}^2_{\mathbb{C}}$ given by the affine equation $u = 0$, respectively by $v = 0$. Choose p distinct points on each of these affine lines. Join each of these p points on L_1 to all the p points on L_2. Let g_1 be an equation of the union of these p^2 lines. Do this construction again using disjoint sets of points. Let g_2 be an equation of the union of the second family of p^2 lines. Let ζ be a primitive p-th root of unity. Let $X \to \mathbb{P}^2_{\mathbb{C}}$ be a proper morphism such that X is smooth and the generic fibre X_η is the Severi–Brauer variety over $\mathbb{C}(\mathbb{P}^2) = \mathbb{C}(u, v)$ attached to the cyclic algebra $(u/v, g_1g_2)_\zeta$.

By Amitsur's theorem ([GS17, Thm. 5.4.1], see also Section 6.1), the kernel of the restriction map $\mathrm{Br}(\mathbb{C}(\mathbb{P}^2)) \to \mathrm{Br}(\mathbb{C}(X))$ is the $\mathbb{Z}/p$-module generated by the class $(u/v, g_1g_2)_\zeta$. Comparing the residues of $\alpha = (u/v, g_1g_2)_\zeta$ and $\beta = (u/v, g_1)_\zeta$ at codimension 1 points of $\mathbb{P}^2_{\mathbb{C}}$, one sees that β is not a multiple of α, hence its image $\beta_{\mathbb{C}(X)} \in \mathrm{Br}(\mathbb{C}(X))$ does not vanish. One then shows that the residue of $\beta_{\mathbb{C}(X)}$ is trivial at any point x of codimension 1 of X by studying the behaviour of β at the point $y \in \mathbb{P}^2$ which is the image of x. (Note that y can have dimension 0, 1 or 2.) Thus $\mathrm{Br}(X) \neq 0$. This implies that X is not stably rational.

Let $K = \mathbb{C}(u, v) = \mathbb{C}(\mathbb{P}^2)$. Let $L = K(\sqrt[p]{g_1g_2})$. By Proposition 6.1.7 the generic fibre X_η is birationally equivalent to the affine K-variety with equation $u/v = \mathrm{N}_{L/K}(\Xi)$. Let $E = K(\sqrt[p]{u/v})$. We have $E = \mathbb{C}(u, z)$, where $z^p = u/v$, so E is a purely transcendental extension of $\mathbb{C}$. The variety $X_E = X_\eta \times_K E$ is then birationally equivalent to the affine variety over E with equation $1 = \mathrm{N}_{EL/E}(\Xi)$. As is well-known (Hilbert's theorem 90 for a cyclic extension, see the proof of Proposition 6.1.6), the latter variety is an E-torus isomorphic to the cokernel of the diagonal embedding $\mathbb{G}_{m,E} \to R_{EL/E}(\mathbb{G}_m)$. But this is an open set of a projective space over E, hence the function field of X_E is purely transcendental over E, hence over $\mathbb{C}$. Thus the function field $\mathbb{C}(X)$ is contained in a purely transcendental extension of $\mathbb{C}$, hence X is unirational.

For some recent computations of unramified Brauer groups of conic bundles over threefolds, see [ABBP].

Chapter 11

Rationality in a family

The specialisation method allows one to prove that a smooth and projective complex variety is not stably rational if it can be deformed into a singular variety whose desingularisation has a non-zero Brauer group. The original idea is due to C. Voisin who stated it in terms of the decomposition of the diagonal. In this chapter we present this method in the set-up proposed by Colliot-Thélène and Pirutka [CTP16] and later simplified by S. Schreieder. In this form the method can be applied under very mild additional assumptions. As an example of application, we construct a conic bundle over $\mathbb{P}^2_{\mathbb{C}}$ ramified in a smooth sextic curve which is not stably rational.

In Section 11.2 we consider smooth projective fourfolds X with a dominant morphism $X \to \mathbb{P}^2_{\mathbb{C}}$ such that the generic fibre is a quadric. Using a calculation of $\mathrm{Br}(X)$ in this case, we present the striking recent example of Hassett, Pirutka and Tschinkel of an algebraic family of smooth projective fourfolds some of whose elements are rational, whereas others not even stably rational.

Most of the material in this chapter follows the exposition in [CT18].

11.1 Specialisation method

11.1.1 Main theorem

The following theorem is S. Schreieder's improvement [Sch18, Prop. 26] of the specialisation method. The assumptions in [Sch18, Prop. 26] are weaker than in this section. The same proof also works in the more general setting of higher unramified cohomology with torsion coefficients in place of the Brauer group.

Schreieder's proof is cast in the geometric language of the decomposition of the diagonal. We give here a more "field-theoretic" proof. It is known that both points of view are equivalent, cf. [ACTP17, CTP16].

Theorem 11.1.1 *Let R be a discrete valuation ring with field of fractions K and algebraically closed residue field κ of characteristic 0. Let $\mathcal{X}$ be an integral* *projective scheme over R, whose generic fibre $X = \mathcal{X}_K$ is smooth and geometrically integral and whose closed fibre Z/κ is geometrically integral. Assume that*

(i) *there exist a non-empty open set $U \subset Z$ and a projective, birational desingularisation $f : \tilde{Z} \to Z$ such that $V := f^{-1}(U) \to U$ is an isomorphism and such that $\tilde{Z} \setminus V$ is a union $\cup_i Y_i$ of smooth irreducible divisors of $\tilde{Z}$;*

(ii) *$X_{\overline{K}}$ is stably rational, where $\overline{K}$ is an algebraic closure of K.*

Then the restriction map $\mathrm{Br}(\tilde{Z}) \to \oplus_i \mathrm{Br}(\kappa(Y_i))$ is injective. In particular, if each $\mathrm{Br}(Y_i) = 0$, then $\mathrm{Br}(\tilde{Z}) = 0$.

Proof. We can go over to the completion of R and thus assume that $R = \kappa[[t]]$ and $K = \kappa((t))$. Since $X_{\overline{K}}$ is stably rational, there exists a finite extension $K_1 = \kappa((t^{1/n}))$ of K such that X_{K_1} is K_1-stably rational. We replace $\mathcal{X}/R$ by $\mathcal{X} \times_R \kappa[[t^{1/n}]]$. This does not affect the closed fibre.

Now $\mathcal{X}/R$ is an integral projective scheme whose generic fibre X/K is stably rational over K and whose closed fibre Z/κ satisfies (i). Since X is stably rational over K, for any field extension $K \subset F$ the degree map $\mathrm{CH}_0(X_F) \to \mathbb{Z}$ is an isomorphism.

Let $L = \kappa(Z)$. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} \bigoplus_i \mathrm{CH}_0(Y_{i,L}) & \to & \mathrm{CH}_0(\tilde{Z}_L) & \to & \mathrm{CH}_0(V_L) & \to & 0 \\ & & \downarrow & & \downarrow \cong & & \\ & & \mathrm{CH}_0(Z_L) & \to & \mathrm{CH}_0(U_L) & \to & 0. \end{array}$$

Let us explain how this diagram is constructed. For each i, the closed embedding $\rho_i : Y_i \to \tilde{Z}$ induces a map $\rho_{i*} : \mathrm{CH}_0(Y_{i,L}) \to \mathrm{CH}_0(\tilde{Z}_L)$. The top exact sequence is the classical localisation sequence for the Chow group. The map $f_* : \mathrm{CH}_0(\tilde{Z}_L) \to \mathrm{CH}_0(Z_L)$ is induced by the proper map $f : \tilde{Z} \to Z$. The map $\mathrm{CH}_0(V_L) \xrightarrow{\sim} \mathrm{CH}_0(U_L)$ is the isomorphism induced by the isomorphism[1] $f : V \xrightarrow{\sim} U$. Finally, $\mathrm{CH}_0(Z_L) \to \mathrm{CH}_0(U_L)$ is the restriction map.

Let ξ be the generic point of $\tilde{Z}$ and let η be the generic point of Z. Choose $m \in V(\kappa)$ and let $n = f(m) \in U(\kappa)$. Thus η and n_L are smooth L-points of Z_L.

Let $S = L[[t]]$ and let F be the field of fractions of S. The extension $R \subset S$ of complete discrete valuation rings is compatible with the extension $\kappa \subset L$ of their residue fields. By Hensel's lemma, the points η and n_L lift to F-points of the generic fibre X_F of $\mathcal{X}_S/S$. Since the degree map $\mathrm{CH}_0(X_F) \to \mathbb{Z}$ is an isomorphism, these two points are rationally equivalent in X_F. By Fulton's specialisation theorem for the Chow group of a proper scheme over a discrete valuation ring [Ful98, Prop. 20.3], we obtain $\eta = n_L \in \mathrm{CH}_0(Z_L)$. Then from the above diagram we conclude that

$$\xi = m_L + \sum_i \rho_{i*}(z_i) \in \mathrm{CH}_0(\tilde{Z}_L),$$

[1] Instead of assuming that $f^{-1}(U) \to U$ is an isomorphism, it would be enough, as in [Sch19], to assume that this morphism is a universal CH_0-isomorphism.

where $z_i \in \mathrm{CH}_0(Y_{i,L})$. There is a natural bilinear pairing (5.2)

$$\mathrm{CH}_0(\tilde{Z}_L) \times \mathrm{Br}(\tilde{Z}) \longrightarrow \mathrm{Br}(L).$$

Suppose that $\alpha \in \mathrm{Br}(\tilde{Z})$ goes to zero in $\mathrm{Br}(\kappa(Y_i))$, for each i. Since Y_i is smooth and integral, already the image of α in $\mathrm{Br}(Y_i)$ is zero. The value $\alpha(m_L) \in \mathrm{Br}(L)$ is just the image of $\alpha(m) \in \mathrm{Br}(\kappa) = 0$. Now the above equality implies $\alpha(\xi) = 0 \in \mathrm{Br}(L)$. But since $\tilde{Z}$ is smooth and integral, the pairing of $\mathrm{Br}(\tilde{Z})$ with the generic point $\xi \in \tilde{Z}_L(L)$ induces the embedding $\mathrm{Br}(\tilde{Z}) \hookrightarrow \mathrm{Br}(\kappa(Z)) = \mathrm{Br}(L)$. Thus $\alpha = 0 \in \mathrm{Br}(\tilde{Z})$. □

Remark 11.1.2 (a) One may replace condition (ii) in the above theorem by the weaker hypothesis that $X_{\overline{K}}$ is universally CH_0-trivial. The same proof works.

(b) In the proof of the theorem, under the assumption of (ii), one can replace the use of specialisation of the Chow group by specialisation of R-equivalence on rational points. See [CTP16] and [CT18].

11.1.2 Irrational conic bundles with smooth ramification

The Artin–Mumford example was used by Voisin [Voi15] to prove that very general double coverings of $\mathbb{P}^3_{\mathbb{C}}$ ramified in a smooth quartic hypersurface are not stably rational. It was used by Colliot-Thélène and Pirutka [CTP16] to prove that very general quartic hypersurfaces in $\mathbb{P}^4_{\mathbb{C}}$ are not stably rational. The specialisation method was applied in [BB18] and [HKT16] to prove that for $d \geq 6$ very general conic bundles over $\mathbb{P}^2_{\mathbb{C}}$ ramified in a smooth curve of degree d are not stably rational. Let us show how the Artin–Mumford example can be used to establish the following special case of this result.

Proposition 11.1.3 *There exists a standard conic bundle $X \to \mathbb{P}^2_{\mathbb{C}}$ ramified in a smooth curve of degree 6 such that X is not stably rational.*

Proof. In Deligne's Bourbaki talk [Del71] we find the following presentation of the Artin–Mumford example. As in Section 10.6 we are given two transversal smooth cubic curves with homogeneous equations $h_1 = 0$ and $h_2 = 0$ and a smooth conic $q = 0$ which is tangent to the each cubic $h_i = 0$ in three points P_i, Q_i, R_i, where $i = 1, 2$. Moreover, the points $P_1, Q_1, R_1, P_2, Q_2, R_2$ are distinct and disjoint from the intersection points of the two cubics. Let $g = 0$ be a cubic curve that meets the conic in the divisor $P_1 + Q_1 + R_1 + P_2 + Q_2 + R_2$. Multiplying g by a non-zero number we arrange that the curve $h_1 h_2 - g^2 = 0$ contains the conic as an irreducible component, so that

$$h_1 h_2 = g^2 + qc$$

for some homogeneous polynomial c of degree 4. Consider the vector bundle $\mathcal{V} = \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}$ on $\mathbb{P}^2_{\mathbb{C}}$ and the quadratic form $\Phi : \mathcal{V} \to \mathcal{O}$ given by

$$\Phi(x, y, z) = cx^2 + 2gxy - qy^2 - z^2.$$

The vanishing of Φ defines a flat conic bundle $X \subset \mathbb{P}(\mathcal{V}^*)$ over $\mathbb{P}^2_{\mathbb{C}}$ whose total space has nine singular points, which are ordinary quadratic singularities. Resolving the singularities gives a birational map $X' \to X$. There are many ways to prove that $\mathrm{Br}(X') \neq 0$, see Section 10.6.

One then considers the family of all quadratic forms $\Phi : \mathcal{V} \to \mathcal{O}$ given by

$$\Phi(x, y, z) = Cx^2 + 2Gxy - Qy^2 - z^2,$$

where C, G, Q are homogeneous forms of respective degrees $4, 3, 2$. We claim that for a very general triple of such forms, the vanishing of the discriminant $G^2 + QC = 0$ defines a smooth curve in $\mathbb{P}^2_{\mathbb{C}}$. (Then the total space X is smooth.) More precisely, suppose that $C = 0$, $G = 0$, $Q = 0$ are smooth curves such that the closed set $C = G = Q = 0$ is empty. We claim that for almost all $\lambda \in \mathbb{C}$, the curve $G^2 + \lambda QC = 0$ is smooth. By one of the Bertini theorems, since G^2 and QC have no common factor, it is enough to show that for $\lambda \neq 0$, the curve $G^2 + \lambda QC = 0$ has no singular point with $G^2 = 0$ and $QC = 0$. Any such point would satisfy $2GG'_x + \lambda Q'_x C + \lambda QC'_x = 0$ and the similar equations with respect to the variables y and z. If the point lies on $G = C = 0$ it then satisfies $QC'_x = 0, QC'_y = 0, QC'_z = 0$, hence $Q = 0$ by the non-singularity of the curve $C = 0$. However, the set $G = C = Q = 0$ is empty, so we have a contradiction. A similar argument shows that the point cannot lie on $G = Q = 0$.

Voisin's deformation argument in its original form [Voi15] can now be applied: by specialising to the Artin–Mumford example in the version recalled above, we see that the very general conic bundle in the family defined by C, G, Q is not stably rational. Alternatively, one can use [CTP16, Thm. 1.17] or Theorem 11.1.1 together with [CTP16, §2] to establish this result. □

11.2 Quadric bundles over the complex plane

Hassett, Pirutka and Tschinkel [HPT18] used the specialisation method to give the first examples of families $X \to B$ of smooth, projective, integral complex varieties with some fibres rational and some other fibres not even stably rational. A simplified version of the specialisation method, as proposed by Schreieder [Sch18, Sch], gives a streamlined proof of the main result of [HPT18] which avoids explicit resolution of singularities. This simplified specialisation method was described in Section 11.1. In this section, following [CT18], we give examples from [HPT18] in their simplest form.

11.2.1 A special quadric bundle

The references for this section are [HPT18], [Pir18], [CT18].

Let x, y, z be homogeneous coordinates in $\mathbb{P}^2_k$, and U, V, W, T be homogeneous coordinates in $\mathbb{P}^3_k$. Let

$$F(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx).$$

Let $X \subset \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ be the hypersurface given by the bihomogeneous equation

$$yzU^2 + zxV^2 + xyW^2 + F(x, y, z)T^2 = 0.$$

Let $p : X \to \mathbb{P}^2_{\mathbb{C}}$ be the morphism given by the projection $\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$. The fibres of $p : X \to \mathbb{P}^2_{\mathbb{C}}$ are 2-dimensional quadrics; in particular, p is a flat morphism. The morphism p is smooth over the complement to the plane octic curve defined by the vanishing of the determinant

$$x^2y^2z^2F(x, y, z) = 0.$$

Note that this equation describes the union of the smooth conic $F = 0$ and three tangents to this conic taken with multiplicity 2. Note that X has singular points over the singular points of the curve $xyzF(x, y, z) = 0$.

Part (a) of the following proposition is a result of Hassett, Pirutka, and Tschinkel [HPT18, Prop. 11]. Part (b) is a special case of the general statement [Sch18, Prop. 7], the proof of which builds upon results of Pirutka ([Pir18, Thm. 3.17], [Sch18, Thm. 4]). As we shall now see, the proof of (a) can be modified to simultaneously give a proof of (b).

Proposition 11.2.1 *Let $\tilde{X} \to X$ be a projective birational desingularisation of X. Let*

$$\alpha = (x/z, y/z) \in \mathrm{Br}(\mathbb{C}(\mathbb{P}^2))$$

and let β be the image of α under the map $p^* : \mathrm{Br}(\mathbb{C}(\mathbb{P}^2)) \to \mathrm{Br}(\mathbb{C}(X))$.

(a) *We have $\beta \in \mathrm{Br}(\tilde{X})$ and $\beta \neq 0$.*

(b) *For each irreducible divisor $Y \subset \tilde{X}$ the restriction of β to $\mathrm{Br}(\mathbb{C}(Y))$ is 0.*

Proof. The equation of X is symmetric in (x, y, z). In view of this symmetry, it is enough to consider the open set $z = 1$ with affine coordinates x and y. In the rest of the proof we consider only this open set. Then $\alpha = (x, y)$ has non-trivial residues precisely at $x = 0$ and $y = 0$. In particular, $\alpha \neq 0$.

Let $K = \mathbb{C}(\mathbb{P}^2) = \mathbb{C}(x, y)$, let $L = \mathbb{C}(X)$, and let X_η/K be the generic fibre of $p : X \to \mathbb{P}^2_{\mathbb{C}}$. The discriminant of the quadratic form $\langle y, x, xy, F(x, y, 1)\rangle$ is not a square in K, thus the map $\mathrm{Br}(K) \to \mathrm{Br}(X_\eta)$ is an isomorphism by Proposition 6.2.3 (c), so that the composition $\mathrm{Br}(K) \tilde{\longrightarrow} \mathrm{Br}(X_\eta) \hookrightarrow \mathrm{Br}(L)$ is injective. Thus $\beta = p^*(\alpha) \in \mathrm{Br}(L)$ is non-zero.

Let v be a discrete valuation $L^* \to \mathbb{Z}$, let S be the valuation ring of v and let κ_v be the residue field. If $K \subset S$, then $v(x) = v(y) = 0$, hence (x, y) is unramified. If $K \not\subset S$, then $S \cap K = R$ is a discrete valuation ring with field of fractions K. The image of the closed point of $\mathrm{Spec}(R)$ in $\mathbb{P}^2_{\mathbb{C}}$ is then either a point m of codimension 1 or a (complex) closed point m of $\mathbb{P}^2_{\mathbb{C}}$.

Consider the first case. If the codimension 1 point m does not belong to $xy = 0$, then $\alpha = (x, y) \in \mathrm{Br}(K)$ is unramified at m, hence $\beta \in \mathrm{Br}(L)$ is unramified at v. Moreover, the evaluation of β in $\mathrm{Br}(\kappa_v)$ is just the image under $\mathrm{Br}(\mathbb{C}(m)) \to \mathrm{Br}(\kappa_v)$ of the evaluation of α in $\mathrm{Br}(\mathbb{C}(m))$. By Tsen's theorem $\mathrm{Br}(\mathbb{C}(m)) = 0$, hence the image of β in $\mathrm{Br}(\kappa_v)$ is zero.

Suppose that m is a generic point of a component of $xy = 0$, say m is the generic point of $x = 0$. In $L = \mathbb{C}(X)$ we have an identity

$$yU^2 + xV^2 + xyW^2 + F(x, y, 1) = 0$$

with $yU^2 + xV^2 \neq 0$. In the completion of K at the generic point of $x = 0$, $F(x, y, 1)$ is a square, because $F(x, y, 1)$ modulo x is equal to $(y-1)^2$, a non-zero square. Thus, in the completion L_v, the quadratic form $\langle y, x, xy, 1\rangle$ has a non-trivial zero, hence (x, y) goes to zero in $\mathrm{Br}(L_v)$. Hence β is unramified at v, thus $\beta \in \mathrm{Br}(S)$ and the image of β in $\mathrm{Br}(\kappa_v)$ is zero.

Now consider the second case, i.e. m is a closed point of $\mathbb{P}^2_{\mathbb{C}}$. There is a local homomorphism of local rings $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}},m} \to S$ which induces an embedding $\mathbb{C} \to \kappa_v$ of residue fields. If $x(m) \neq 0$, then x becomes a non-zero square in the residue field $\mathbb{C}$ hence in κ_v. This implies that the residue of $\beta = (x, y) \in \mathrm{Br}(L)$ at v is trivial. The analogous argument holds if $y(m) \neq 0$. It remains to discuss the case $x(m) = y(m) = 0$. We have $F(0, 0, 1) = 1 \in \mathbb{C}^*$. Thus $F(x, y, 1)$ reduces to 1 in κ_v, hence is a square in the completion L_v. As above, in the completion L_v, the quadratic form $\langle y, x, xy, 1\rangle$ has a non-trivial zero, hence (x, y) goes to zero in $\mathrm{Br}(L_v)$. Hence β is unramified at v, thus $\beta \in \mathrm{Br}(S)$ and the image of β in $\mathrm{Br}(\kappa_v)$ is zero. □

As in the reinterpretation [CTO89] of the Artin–Mumford examples, the intuitive idea behind the above result is that the quadric bundle $X \to \mathbb{P}^2_{\mathbb{C}}$ is ramified along $xyzF(x, y, z) = 0$ and the ramification of the symbol $(x/z, y/z)$, which is "contained" in the ramification of the quadric bundle $X \to \mathbb{P}^2_{\mathbb{C}}$, disappears over smooth projective models of X: ramification eats up ramification (Abhyankar's lemma). Here one also uses the fact that the smooth conic defined by $F(x, y, z) = 0$ is tangent to each of the lines $x = 0, y = 0, z = 0$, and does not vanish at the intersection point of any two of these three lines.

11.2.2 Rationality is not constant in a family

In this section we complete the simplified proof of the theorem of Hassett, Pirutka and Tschinkel [HPT18].

Theorem 11.2.2 *There exist a smooth projective family of complex fourfolds $X \to T$, where T is an open subset of the affine line $\mathbb{A}^1_{\mathbb{C}}$, and points $m, n \in T(\mathbb{C})$ such that the fibre X_n is rational whereas the fibre X_m is not stably rational.*

Proof. Consider the universal family of quadric bundles over $\mathbb{P}^2_{\mathbb{C}}$ given in $\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ by a bihomogeneous form of bidegree $(2, 2)$. This is given by a symmetric (4×4)-matrix whose entries $a_{i,j}(x, y, z)$ are homogeneous quadratic forms in x, y, z. If the determinant of the matrix is non-zero, it is a homogeneous polynomial of degree 8. The parameter space is $B = \mathbb{P}^{59}_{\mathbb{C}}$ (the corresponding vector space is given by the coefficients of 10 quadratic forms in three variables). We have the map $X \to B$ whose fibres X_m are quadric bundles $X_m \to \mathbb{P}^2_{\mathbb{C}}$, where $X_m \subset \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{C}$ is the zero set of a non-zero complex bihomogeneous form of bidegree $(2, 2)$.

Using Bertini's theorem, one shows that there exists a non-empty open set $B_0 \subset B$ such that the fibres of $X \to B$ over the points $m \in B_0$ are flat quadric bundles $X_m \to \mathbb{P}^2_{\mathbb{C}}$ which are smooth as complex varieties.

Using Bertini's theorem, one also shows that there exist points $m \in B_0$ with the property that the corresponding quadric bundle has $a_{1,1} = 0$. This implies that the morphism $X_m \to \mathbb{P}^2_{\mathbb{C}}$ has a rational section given by the point $(1, 0, 0, 0)$, hence the generic fibre of $X_m \to \mathbb{P}^2_{\mathbb{C}}$ is rational over $\mathbb{C}(\mathbb{P}^2)$, so that the complex variety X_m is rational over $\mathbb{C}$. [Warning. This Bertini argument uses the fact that we consider families of quadric surfaces over $\mathbb{P}^2_{\mathbb{C}}$. It does not work for families of conics over $\mathbb{P}^2_{\mathbb{C}}$.]

These Bertini arguments are briefly described in [Sch19, Lemma 20, Thm. 47] and are tacitly used in [Sch18, p. 3].

The special example in Section 11.2.1 defines a point $P_0 \in B(\mathbb{C})$. Let $Z = X_{P_0}$. Using Proposition 11.2.1, one finds a projective birational desingularisation $f : \tilde{Z} \to Z$ and a non-empty open set $U \subset Z$ such that

- the induced map $V := f^{-1}(U) \to U$ is an isomorphism;
- $\tilde{Z} \setminus V$ is a union $\cup_i Y_i$ of smooth irreducible divisors of $\tilde{Z}$;
- there is a non-trivial element in $\mathrm{Br}(\tilde{Z})$ which vanishes on each Y_i.

Theorem 11.1.1 then implies that the generic fibre of $X \to B$ is not geometrically stably rational. There are various ways to conclude from this that there are many points $m \in B_0(\mathbb{C})$ such that the fibre X_m is not stably rational.

Take one such point $m \in B_0(\mathbb{C})$ and a point $n \in B_0(\mathbb{C})$ such that X_n is rational. Over an open set of the line joining m and n we get a projective family of smooth varieties with one fibre rational and another fibre not stably rational. □

The proof by Hassett, Pirutka and Tschinkel [HPT18] uses an explicit desingularisation of the variety Z in Section 11.2.1, with a description of the exceptional divisors appearing in the process. Schreieder's improvement of the specialisation method enables one to bypass this explicit desingularisation. Papers [HPT18] and [Sch18] contain many other results about families of quadric surfaces over the projective plane. For further developments the reader is referred to [ABBP], which gives a different approach to [HPT18] as well as some generalisations, to [ABP] and [Sch19, Sch].

Let us summarise the current state of knowledge about the behaviour of rationality and stable rationality in the fibres of an algebraic family of proper and smooth varieties. Let T be a smooth connected variety over $\mathbb{C}$ and let $X \to T$ be a proper and smooth morphism with connected, projective fibres of relative dimension d. It has been known for a while that the set of points t such that X_t is rational, respectively, stably rational, is a countable union of locally closed subsets of T, see [dFF13, Prop. 2.3].

- For $d \leq 2$, stable rationality is equivalent to rationality, and either all fibres are rational or no fibre is rational.

- For arbitrary d stable rationality specialises. Thus the set of points t such that X_t is stably rational, is a countable union of *closed* subsets of T (Nicaise–Shinder [NSh]).
- For arbitrary d rationality specialises. Thus the set of points t such that X_t is rational, is a countable union of *closed* subsets of T (Kontsevich–Tschinkel [KTsc]).
- By the examples discussed in this section, for $d \geq 4$, neither rationality nor stable rationality extends by generisation (Hassett, Pirutka and Tschinkel [HPT18]).
- For $d = 3$ stable rationality does not extend by generisation (Hassett, Kresch and Tschinkel [HKT]).
- For $d = 3$ it is not known if rationality extends by generisation.

Recall that a property P of varieties over algebraically closed fields, which is stable under extensions of such fields, *extends by generisation* if for any smooth projective scheme X over $\mathrm{Spec}(\mathbb{C}[[t]])$, if P holds for the closed fibre, then P holds for the geometric generic fibre, that is, the fibre over an algebraic closure of $\mathbb{C}((t))$.

Chapter 12

The Brauer–Manin set and the formal lemma

This is the first of three chapters which deal with applications of the Brauer group to the arithmetic of varieties over a number field k. Section 12.1 is a collection of preliminary results from algebraic number theory and class field theory. In Section 12.2 we discuss the Hasse principle, weak and strong approximation. Section 12.3 contains the definition and basic properties of the Brauer–Manin obstruction, which is the fundamental reason why the knowledge of the Brauer group is necessary for the study of local-to-global principles for rational points. When the cokernel of the natural map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is finite, the Brauer–Manin obstruction on X involves only finitely many primes; the set of these primes is studied in Section 12.4. Explicit examples of calculation of the Brauer–Manin obstruction to the Hasse principle and weak approximation are presented in Section 12.5. In Section 12.6 we state and prove Harari's formal lemma, which is a fundamental tool in studying the variation of the Brauer–Manin obstruction in a family of varieties.

12.1 Number fields

Let k be a number field. We write Ω for the set of places of k. The completion of k at a place v is denoted by k_v. For a finite (=non-archimedean) place v we will also use the notation v to denote the associated normalised valuation.

12.1.1 Primes and approximation

Dirichlet's theorem on primes in an arithmetic progression can be extended to number fields in the following form.

Theorem 12.1.1 (Dirichlet, Hasse) *Let $S\subset\Omega$ be a finite set of* **finite** *places and let $\lambda_v\in k_v$ for each $v\in S$. For any $\varepsilon>0$ there exist $\lambda\in k^*$ and a finite place $v_0\notin S$ of absolute degree 1 such that*

(i) $|\lambda - \lambda_v|_v < \varepsilon$ *for each place* $v \in S$;
(ii) $\lambda > 0$ *in each real completion of* k;
(iii) λ *is a unit at any place* $v \notin S \cup \{v_0\}$ *whereas* $v_0(\lambda) = 1$.

Here v_0 may not be chosen at the outset.

The next statement (which is easy for $k = \mathbb{Q}$) enables one to approximate also at the archimedean places, if one accepts to lose control over an infinite set of places of k that can be chosen at the outset. Typically, this will be the set of places split in a given finite extension of k.

Theorem 12.1.2 (Dirichlet, Hasse, Waldschmidt, Sansuc) *Let* $S \subset \Omega$ *be a finite set of places and let* $\lambda_v \in k_v$ *for each* $v \in S$. *Let* V *be an infinite set of places of* k. *For any* $\varepsilon > 0$ *there exist* $\lambda \in k^*$ *and a finite place* $v_0 \notin S$ *of absolute degree* 1 *such that*
(i) $|\lambda - \lambda_v|_v < \varepsilon$ *for each* $v \in S$,
(ii) λ *is a unit at each finite place* $v \notin S \cup \{v_0\} \cup V$ *and* $v_0(\lambda) = 1$.

Proof. See [San82a]. □

Here again v_0 may not be chosen at the outset.

We recall a corollary of the celebrated Chebotarev density theorem.

Theorem 12.1.3 (Chebotarev) *Let* K/k *be a finite extension of number fields. There exists an infinite set of places* v *of* k *which are completely split in* K, *i.e. such that the* k_v*-algebra* $K \otimes_k k_v$ *is isomorphic to* $k_v^{[K:k]}$.

This special case of Chebotarev's theorem has an elementary proof (reference given in [HW15, Lemma 5.2]). Theorem 12.1.2 can be compared to the following proposition [HW15, Lemma 5.2].

Proposition 12.1.4 *Let* K/k *be an extension of number fields. Let* S *be a finite set of places of* k. *Let* $\xi_v \in \mathrm{N}_{K/k}(K \otimes_k k_v^*) \subset k_v^*$ *for each* $v \in S$. *Then there exists* $\xi \in k^*$ *arbitrarily close to* ξ_v *for* $v \in S$ *and such that* ξ *is a unit outside* S *except possibly at the places above which* K *has a place of degree* 1. *In addition, if* v_0 *is a place of* k *not in* S, *over which* K *possesses a place of degree* 1, *one can ensure that* ξ *is integral outside* $S \cup \{v_0\}$.

Chebotarev's theorem is used to prove the existence of such a place v_0, but the proof otherwise only uses the strong approximation theorem.

Here is another corollary of the Chebotarev density theorem.

Theorem 12.1.5 *Let* K/k *be a non-trivial finite extension of number fields. There exist infinitely many places* v *of* k *such that the* k_v*-algebra* $K \otimes_k k_v$ *has no direct summand isomorphic to* k_v. *In particular, given an irreducible polynomial* $P(t)$ *of degree at least* 2, *there exist infinitely many places* v *such that* $P(t)$ *has no root in* k_v.

It is well known that the second statement does not hold for reducible polynomials. A classical example is $P(t) = (t^2 - 13)(t^2 - 17)(t^2 - 221) \in \mathbb{Q}[t]$.

Here is another variation on the same theme [Har94, Prop. 2.2.1].

Theorem 12.1.6 *Let $L/K/k$ be finite extensions of number fields, with L/K cyclic. There exist infinitely many places w of K of degree 1 over k which are inert in the extension L/K.*

12.1.2 Class field theory and the Brauer group

There is a vast literature on class field theory. We refer here to Harari's recent book [Har17] both for proofs and for a list of references to classical literature. The Witt residue was introduced in Definition 1.4.9.

Definition 12.1.7 *For each place v of k define*

$$\mathrm{inv}_v : \mathrm{Br}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

as follows. If v is finite, let inv_v be the Witt residue $\mathrm{Br}(k_v)\to\mathrm{H}^1(\mathbb{F}_v,\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v),\mathbb{Q}/\mathbb{Z})$ followed by the evaluation at the Frobenius element. If v is real, define inv_v by $\mathrm{Br}(k_v) = \mathbb{Z}/2 \hookrightarrow \mathbb{Q}/\mathbb{Z}$. For a complex place v set $\mathrm{inv}_v = 0$.

The definition of $\mathrm{Br}(k_v)\to\mathbb{Q}/\mathbb{Z}$ given here is the one used in [SerCL, Ch. XIII, §2], [ANT67, Ch. VI], [NSW, Ch. VII, Cor. (7.1.4)], and [Har17, §8.2].

Theorem 12.1.8 (i) *For each finite place v of k, the map inv_v is an isomorphism. For each real place v, the map inv_v is the injection $\mathrm{Br}(k_v) = \mathbb{Z}/2 \hookrightarrow \mathbb{Q}/\mathbb{Z}$. For each complex place v we have $\mathrm{Br}(k_v) = 0$.*

(ii) *The diagonal map $\mathrm{Br}(k)\to\prod_{v\in\Omega}\mathrm{Br}(k_v)$ factors through the direct sum $\oplus_{v\in\Omega}\mathrm{Br}(k_v)$.*

(iii) *The maps inv_v fit into an exact sequence*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \bigoplus_{v\in\Omega}\mathrm{Br}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0, \tag{12.1}$$

where the map to $\mathbb{Q}/\mathbb{Z}$ is the sum of inv_v for all $v \in \Omega$.

The fact that (12.1) is a complex is a generalisation of the Gauss quadratic reciprocity law. Injectivity on the second arrow is a celebrated theorem of H. Hasse, R. Brauer and E. Noether, generalising results of Legendre and Hilbert.

Theorem 12.1.9 *Let K/k be an abelian extension of number fields, and let $G = \mathrm{Gal}(K/k)$. For each place $v \in \Omega$, let $G_v \subset G$ be the decomposion group of v. There is a well-defined isomorphism*

$$j_v : k_v^*/\mathrm{N}_{K/k}((K \otimes_k k_v)^*)\tilde{\longrightarrow}G_v$$

called the norm residue homomorphism, or the local Artin map [SerCL, Ch. XIII, §4], [Har17, Ch. 9]. *For a normalised valuation v of k unramified in K,* *this map sends an element $c \in k^*$ to $\mathrm{Frob}_v^{v(c)} \in G$ (ibid.). These maps fit into an exact sequence*

$$k^*/\mathrm{N}_{K/k}(K^*) \longrightarrow \bigoplus_{v\in\Omega} k_v^*/\mathrm{N}_{K/k}((K\otimes_k k_v)^*) \longrightarrow G \longrightarrow 1. \qquad (12.2)$$

If K/k is cyclic, we have an exact sequence

$$1 \longrightarrow k^*/\mathrm{N}_{K/k}(K^*) \longrightarrow \bigoplus_{v\in\Omega} k_v^*/\mathrm{N}_{K/k}((K\otimes_k k_v)^*) \longrightarrow G \longrightarrow 1. \qquad (12.3)$$

Corollary 12.1.10 *Let K/k be an abelian extension of number fields.*

(a) *If $c \in k^*$ is a local norm for K/k at all places of k except possibly one place v_0, then c is also a local norm at v_0.*

(b) (Hasse) *If K/k is cyclic, and $c \in k^*$ is a local norm for K/k at all places of k except possibly one place v_0, then it is a global norm.*

The above results are special cases of the following theorem.

Theorem 12.1.11 (Tate–Nakayama) *Let T be an algebraic k-torus. Write $\hat{T} = \mathrm{Hom}_{k-\mathrm{gp}}(T, \mathbb{G}_{m,k})$ for the Galois lattice defined by the character group of T. There is a natural exact sequence of abelian groups*

$$\mathrm{H}^1(k,T)\to\bigoplus_{v\in\Omega}\mathrm{H}^1(k_v,T)\to\mathrm{Hom}(\mathrm{H}^1(k,\hat{T}),\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^2(k,T)\to\bigoplus_{v\in\Omega}\mathrm{H}^2(k_v,T) \qquad (12.4)$$

and a perfect duality of finite abelian groups $Ш^1(k,T) \times Ш^2(k,\hat{T})\to\mathbb{Q}/\mathbb{Z}$.

The map $\mathrm{H}^1(k_v,T)\to\mathrm{Hom}(\mathrm{H}^1(k,\hat{T}),\mathbb{Q}/\mathbb{Z})$ is induced by a perfect pairing induced by the cup-product

$$\mathrm{H}^1(k_v,T)\times\mathrm{H}^1(k_v,\hat{T}) \longrightarrow \mathrm{H}^2(k_v,\mathbb{G}_m) = \mathrm{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}.$$

We refer the reader to the following references: [Tate66], [SerCG, Ch. II, §5.8, Thm. 6] (local duality), [NSW, Ch. VII, VIII], [Mil86, Ch. I, Thm. 4.20].

Remark 12.1.12 Using Theorem 12.1.11, one easily proves the following statement. If K/k, $K \neq k$, is a finite extension of number fields, then the quotient $k^*/\mathrm{N}_{K/k}(K^*)$ is infinite if and only if the kernel of the restriction map $\mathrm{Br}(k)\to\mathrm{Br}(K)$ is infinite. In fact, these groups are indeed infinite. The only known proof of this statement for an arbitrary extension K/k (due to Fein, Kantor, and Schacher) uses the classification of finite simple groups.

Note that *a priori* there are two possible definitions of the map $\mathrm{Br}(k_v)\to\mathbb{Q}/\mathbb{Z}$ (see Theorem 1.4.10). To discuss global problems, it is necessary to define the local maps $\mathrm{inv}_v : \mathrm{Br}(k_v)\to\mathbb{Q}/\mathbb{Z}$ and $j_v : k_v^*/\mathrm{N}_{K/k}((K\otimes_k k_v)^*)\to G$ in a uniform way. (It is not enough to define these maps up to sign, except obviously in the case where one deals with the 2-torsion subgroups).

Formulae for invariants of cup-products with values in $\mathrm{Br}(k_v) \subset \mathbb{Q}/\mathbb{Z}$ are called explicit reciprocity laws [SerCL, Ch. XIV], [Iwa68], [Har17, Ch. 9]. One should pay particular attention when applying the formulas. For instance, formulae for residues of cup-products in Section 1.4.1 in this book are given for the cohomological residue, and not for the Witt residue. By Theorem 1.4.10, the Witt residue is the negative of the cohomological residue.

See [CTKS87] for a concrete example where one handles 3-torsion elements.

12.1.3 Adèles and adelic points

In this section we use a very helpful article of B. Conrad [C12] to which we refer for many carefully worked out details.

If v is a non-archimedean place of k, we denote by $\mathcal{O}_v$ the ring of integers of the completion k_v. We shall write S for a finite set of places of k containing all the archimedean places. Let $\mathcal{O}$ be the ring of integers of k and let $\mathcal{O}_S$ be the ring of S-integers, i.e. the elements of k that belong to $\mathcal{O}_v$ for $v \notin S$.

The product $\prod_{v\in\Omega} k_v$ is a topological ring equipped with the product topology, where each k_v carries its natural archimedean or non-archimedean topology. The ring of adèles $\mathbf{A}_k$ is defined as a subring of $\prod_{v\in\Omega} k_v$ given by the condition that all but finitely many components are in $\mathcal{O}_v$. The topology of $\mathbf{A}_k$ induced by the topology of $\prod_{v\in\Omega} k_v$ is such that a base is given by the open sets $\prod_{v\in S} U_v \times \prod_{v\notin S} \mathcal{O}_v$, where U_v is open in k_v. We put

$$\mathbf{A}_S = \prod_{v\in S} k_v \times \prod_{v\notin S} \mathcal{O}_v.$$

Then $\mathbf{A}_k$ is the direct limit of the open subrings $\mathbf{A}_S$ over all finite $S \subset \Omega$ containing the archimedean places. We note that k is discrete in $\mathbf{A}_k$, and $\mathcal{O}_S$ is discrete in $\mathbf{A}_{k,S}$.

If $X \subset \mathbb{A}^n_k$ is an affine variety, then the set $X(\mathbf{A}_k)$ is identified with a closed subset of $\mathbf{A}_k^n$ and so acquires a locally compact Hausdorff subspace topology. This topology does not depend on the closed immersion $X \hookrightarrow \mathbb{A}^n_k$, see [C12, Prop. 2.1]. Since k is discrete in $\mathbf{A}_k$, the set $X(k)$ is discrete in $X(\mathbf{A}_k)$ if X is affine. Although a closed immersion $X \hookrightarrow X'$ gives rise to a closed embedding $X(\mathbf{A}_k) \hookrightarrow X'(\mathbf{A}_k)$ of topological spaces, this is not true for open immersions. The standard example is $\mathbb{G}_{m,k} \subset \mathbb{A}^1_k$. Indeed, the topology on the ring of idèles $\mathbf{A}_k^*$ coming from the closed immersion $\mathbb{G}_{m,k} \subset \mathbb{A}^2_k$ given by $xy = 1$, is not the topology induced from $\mathbf{A}_k$. (The elements $a, b \in \mathbf{A}_k^*$ are close when not only a and b are close, but a^{-1} and b^{-1} are close too.) This shows that to equip the set $X(\mathbf{A}_k)$ with the structure of a topological space when X is not affine one cannot proceed by gluing over the affine open subsets. Following Weil and Grothendieck, this goal is achieved by working with integral models.

Nevertheless, the approach via gluing works for a local topological ring R such that R^* is open in R and has continuous inversion, e.g. if $R = k_v$ or $R = \mathcal{O}_v$. This crucially uses the fact that if $\{U_i\}$ is an open covering of X, then $X(R)$ is the union of the sets $U_i(R)$. See [C12, Prop. 3.1, Prop. 5.4] and Theorem 9.5.1.

Let X be a variety over k (that is, a separated scheme of finite type over k). By [EGA, IV_3, §8.8] for some finite set T of places there exists a separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}_T$ with generic fibre X. Let $S \subset \Omega$ be a finite set containing T. It is clear that an $\mathbf{A}_{k,S}$-valued point of $\mathcal{X}$ gives rise to an $\mathbf{A}_k$-valued point of $\mathcal{X} \times_{\mathcal{O}_T} \mathbf{A}_k$. Since $\mathcal{O}_T \subset k \subset \mathbf{A}_k$, we have $\mathcal{X} \times_{\mathcal{O}_T} \mathbf{A}_k = X \times_k \mathbf{A}_k$, so an $\mathbf{A}_k$-valued point of $\mathcal{X} \times_{\mathcal{O}_T} \mathbf{A}_k$ is identified with an $\mathbf{A}_k$-valued point of X. This gives rise to a map of sets

$$\varinjlim \mathcal{X}(\mathbf{A}_{k,S}) \longrightarrow \mathcal{X}(\mathbf{A}_k) = X(\mathbf{A}_k). \tag{12.5}$$

Here the limit is over S, and it does not depend on T. An $\mathbf{A}_k$-valued point of X comes from an $\mathbf{A}_{k,S}$-valued point of $\mathcal{X}$ for some S, so this map is bijective.

gwrite $\mathcal{X}_{S,v}$ for $\mathcal{X}_S \times_{\mathcal{O}_S} \mathcal{O}_v$. The natural map of sets

$$\mathcal{X}(\mathbf{A}_{k,S}) \xrightarrow{\sim} \prod_{v \in S} X(k_v) \times \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v)$$

is a bijection [C12, Thm. 3.6]. This implies that $X(\mathbf{A}_k)$ is the restricted topological product of the sets $X(k_v)$, for $v \in \Omega$, with respect to their subsets $\mathcal{X}(\mathcal{O}_v)$ for $v \notin S$. Here $X(k_v)$ and $\mathcal{X}(\mathcal{O}_v)$ are topologised via gluing, as explained above. This makes $\mathcal{X}(\mathbf{A}_{k,S})$ a locally compact Hausdorff topological space. If $S \subset S'$, then $\mathcal{X}(\mathbf{A}_{k,S}) \to \mathcal{X}(\mathbf{A}_{k,S'})$ is an open embedding. Using (12.5) we make $X(\mathbf{A}_k)$ a topological space in such a way that a subset of $X(\mathbf{A}_k)$ is open if its intersection with each $\mathcal{X}(\mathbf{A}_{k,S})$ is open. Then $X(\mathbf{A}_k)$ is a locally compact Hausdorff topological space with a countable basis of open sets. We note that the sets $\mathcal{X}(\mathbf{A}_{k,S})$ form an open covering of $X(\mathbf{A}_k)$. A morphism $f : X \to Y$ of varieties over k gives rise to a continuous map $X(\mathbf{A}_k) \to Y(\mathbf{A}_k)$.

We refer to $X(\mathbf{A}_k)$ as the *adelic space* of X and call its elements the *adelic points* of X. If X is an affine variety over k, the topology of the adelic space $X(\mathbf{A}_k)$ is the natural topology defined earlier in the affine case.

If X is proper, we can take $\mathcal{X}$ to be proper over $\mathcal{O}_T$. For $v \notin S$, by the valuative criterion of properness, we have $X(k_v) = \mathcal{X}(\mathcal{O}_v)$, hence $X(\mathbf{A}_k)$ coincides with the product topological space $\prod_{v \in \Omega} X(k_v)$, and so is compact by Tychonoff's theorem. More generally, if $X \to Y$ is a proper morphism of varieties over k, then the continuous map of topological spaces $X(\mathbf{A}_k) \to Y(\mathbf{A}_k)$ is topologically proper: the inverse image of a compact set is compact. If $X \to Y$ is a smooth surjective morphism of separated schemes of finite type over k with geometrically integral fibres, then $X(\mathbf{A}_k) \to Y(\mathbf{A}_k)$ is open [C12, Thm. 4.5] (the proof uses the Lang–Weil–Nisnevich inequality [LW54], [Po18, Thm. 7.7.1]).

For a finite set of places $T \subset \Omega$ let $\mathbf{A}_k^T$ be the ring of T-adèles of k, i.e. the adèles without the components at the places in T. We have the topological space $X(\mathbf{A}_k^T)$ of T-adelic points.

12.2 Hasse principle and approximation

A variety X over k is called *everywhere locally soluble* if $\prod_{v \in \Omega} X(k_v) \neq \emptyset$.

Definition 12.2.1 (Hasse principle) *A variety X over k* **fails** *the Hasse principle if $\prod_{v\in\Omega} X(k_v) \neq \emptyset$ whereas $X(k)=\emptyset$. A class of algebraic varieties over k* **satisfies** *the Hasse principle if no variety in this class fails the Hasse principle.*

For each place v the set of local points $X(k_v)$ has a natural topology inherited from the topology of k_v. For any subset $S\subset\Omega$ we consider $\prod_{v\in S} X(k_v)$ as the topological space with respect to the product topology.

Definition 12.2.2 (Weak approximation) *Weak approximation holds for a variety X over k if the image of the diagonal map*

$$X(k) \longrightarrow \prod_{v\in\Omega} X(k_v)$$

is dense. Equivalently, for any finite set $S\subset\Omega$, the image of $X(k)$ under the diagonal embedding

$$X(k) \longrightarrow \prod_{v\in S} X(k_v)$$

is dense.

In particular, if an everywhere locally soluble variety over k satisfies weak approximation, then it has a k-point. Thus, *according to our definition*, if we have a class of varieties over k such that weak approximation holds for each variety of this class, then this class satisfies the Hasse principle. One should however be aware that for some classes of varieties it may be easy to prove weak approximation assuming the existence of a k-point, but hard to prove the Hasse principle. The simplest example is the class of quadrics.

Proposition 12.2.3 (Kneser) *Let X and Y be smooth and geometrically integral varieties over k such that X is everywhere locally soluble. If X and Y are birationally equivalent, then weak approximation holds for X if and only if it holds for Y.*

Proof. There exist non-empty open sets $U\subset X$ and $V\subset Y$ which are isomorphic. For a given place v of k, $U(k_v)$ is open in $X(k_v)$ for the v-adic topology. Since X is smooth, $U(k_v)$ is also dense in $X(k_v)$ by the implicit function theorem (Theorem 9.5.1). This is enough to conclude. □

In particular, it suffices to prove weak approximation for a non-empty open subset.

Definition 12.2.4 (Weak weak approximation) *A smooth, geometrically integral variety X over k with a k-point satisfies weak weak approximation if there exists a finite set $T\subset\Omega$ such that the image of the diagonal map*

$$X(k) \longrightarrow \prod_{v\in\Omega\smallsetminus T} X(k_v)$$

is dense. Equivalently, for any finite set $S \subset \Omega$ with $S \cap T = \emptyset$, the image of the diagonal map

$$X(k) \longrightarrow \prod_{v \in S} X(k_v)$$

is dense.

Let X be an integral variety over a number field k. A subset $H \subset X(k)$ is called a *Hilbert set* if there exists an integral variety Z over k and a dominant quasi-finite morphism $Z \to X$ such that H is the set of k-points P with connected fibre $Z_P = Z \times_X P$. The intersection of two Hilbert sets in $X(k)$ contains a Hilbert set.

Definition 12.2.5 (Hilbertian weak approximation) *A geometrically integral variety X over k satisfies hilbertian weak approximation if the image of any Hilbert set $H \subset X(k)$ under the diagonal map*

$$X(k) \longrightarrow \prod_{v \in \Omega} X(k_v)$$

is dense.

Assume $X_{\text{smooth}}(k) \neq \emptyset$. If X satisfies hilbertian weak approximation, then any Hilbert subset of $X(k)$ is Zariski dense in X, so is not empty. The following result shows that Hilbertian weak approximation holds for any non-empty Zariski open subset of the projective line.

Theorem 12.2.6 (Ekedahl) *Let $H \subset \mathbb{A}^1(k) = k$ be a Hilbert set. Let $S \subset \Omega$ be a finite set of places and let $\lambda_v \in k_v$ for each $v \in S$. Then for any $\varepsilon > 0$ there exists $\lambda \in H$ such that $|\lambda - \lambda_v|_v < \varepsilon$ for each $v \in S$.*

Proof. See [Eke90]. □

Recall that for a finite set of places $T \subset \Omega$ we denote by $\mathbf{A}_k^T$ the ring of T-adèles of k, i.e. the adèles without the components at the places of T.

Definition 12.2.7 (Strong approximation) *A variety X over k satisfies strong approximation with respect to a finite set $T \subset \Omega$ if the image of the diagonal map*

$$X(k) \longrightarrow X(\mathbf{A}_k^T)$$

is dense.

Definition 12.2.8 (Hilbertian strong approximation) *A variety X over k satisfies Hilbertian strong approximation with respect to a finite set $T \subset \Omega$ if for any Hilbert set $H \subset X(k)$ the image of H under the diagonal map $X(k) \to X(\mathbf{A}_k^T)$ is dense.*

If X is *proper*, then $X(\mathbf{A}_k^T)=\prod_{v\in\Omega\smallsetminus T}X(k_v)$. Thus if weak approximation holds for X, then strong approximation holds for X with respect to any finite set $S\subset\Omega$, in particular for $S=\emptyset$. The same is true in the Hilbertian case.

The following theorem from [Eke90] may be viewed as an extension of the Chinese remainder theorem.

Theorem 12.2.9 (Ekedahl) *Let $H\subset k$ be a Hilbert set. Let $S\subset\Omega$ be a finite set of places and let $\lambda_v\in k_v$ for each $v\in S$. Let v_0 be a place of k not in S. Then for any $\varepsilon>0$ there exists $\lambda\in H$ such that*

(i) *$|\lambda-\lambda_v|_v<\varepsilon$ for each $v\in S$, and*

(ii) *$v(\lambda)\geq 0$ at each finite place $v\notin S\cup\{v_0\}$.*

Thus Hilbertian strong approximation holds for the affine line $\mathbb{A}^1_k$ with respect to any *non-empty* finite set $T\subset\Omega$. Note that v_0 can be chosen to be any place outside of S.

Ekedahl's theorem [Eke90, Thm. 1.3] is actually more general.

Theorem 12.2.10 (Ekedahl) *Let R be the ring of integers of a number field k. Let $\pi: X\to\mathrm{Spec}(R)$ be a morphism of finite type and let $\rho: Y\to X$ be an étale covering such that the generic fibre of the composed morphism $\pi\rho$ is geometrically irreducible. If weak approximation holds for $X\times_R k$, then weak approximation holds for the set of points $x\in X(k)$ with connected fibres $\rho^{-1}(x)$.*

The same holds when weak approximation is replaced by strong approximation with respect to a finite set $T\subset\Omega$.

12.3 The Brauer–Manin set

Adelic evaluation map

Proposition 12.3.1 *Let k be a number field, let X be a variety over k and let $A\in\mathrm{Br}(X)$.*

(i) *There exist a finite set of places $T\subset\Omega$ containing all the archimedean places, a separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}_T$ with generic fibre X, and an element $\mathcal{A}\in\mathrm{Br}(\mathcal{X})$ with image $A\in\mathrm{Br}(X)$.*

(ii) *For $\mathcal{X}\to\mathrm{Spec}(\mathcal{O}_T)$ as in (i), for any finite place $v\notin T$ and for any point $M_v\in\mathcal{X}(\mathcal{O}_v)\subset X(k_v)$ we have $A(M_v)=\mathcal{A}(M_v)=0$.*

(iii) *If X is proper, there exists a finite set of places $T\subset\Omega$ such that for all $v\notin T$ and for any $M_v\in X(k_v)$ we have $A(M_v)=0$.*

(iv) *The map*

$$\mathrm{ev}_A: X(\mathbf{A}_k)\longrightarrow\mathbb{Q}/\mathbb{Z}$$

which sends an adelic point (M_v) to $\sum_{v\in\Omega}\mathrm{inv}_v A(M_v)\in\mathbb{Q}/\mathbb{Z}$ is a well-defined continuous map whose image is annihilated by a positive integer.

Proof. (i) We have $X=\varprojlim\mathcal{X}$, where the limit is over separated $\mathcal{O}_S$-schemes of finite type with generic fibre such that $S\subset\Omega$ is finite and contains all the archimedean places. By Section 2.2.4 we have $\mathrm{Br}(X) = \varinjlim \mathrm{Br}(\mathcal{X}_S)$, which implies (i).

(ii) This follows from $\mathrm{Br}(\mathcal{O}_v) = 0$ (Theorem 3.4.2 (ii)).

(iii) If X is proper, then in (i) we can take $\mathcal{X}\to\mathrm{Spec}(\mathcal{O}_T)$ to be proper. Then for any finite place $v \notin T$ we have $\mathcal{X}(\mathcal{O}_v) = X(k_v)$. Now (iii) follows from (ii).

(iv) Let $\mathcal{X}\to\mathrm{Spec}(\mathcal{O}_T)$ be as in (i). By Section 12.1.3 the sets

$$\prod_{v\in S} U_v \times \prod_{v\notin S} \mathcal{X}(\mathcal{O}_v),$$

where $S \subset \Omega$ is a finite set containing T and $U_v \subset X(k_v)$ is an open set for $v \in S$, form a basis of open sets of $X(\mathbf{A}_k)$. By (ii), the adelic evaluation map ev_A is well-defined on such open sets. It is continuous on each of these open sets. Indeed, the local evaluation map $\mathrm{ev}_A : \mathcal{X}(\mathcal{O}_v)\to\mathrm{Br}(k_v)$ is zero for $v \notin T$ and $\mathrm{ev}_A : X(k_v)\to\mathrm{Br}(k_v)$ is continuous for any place v (Corollary 9.5.2). That the image of the adelic evaluation map $\mathrm{ev}_A : X(\mathbf{A}_k) \longrightarrow \mathbb{Q}/\mathbb{Z}$ is annihilated by a positive integer is a consequence of Lemma 3.4.5. □.

Write $\mathbf{A}_k^{\mathbb{C}}$ for the ring of $\Omega_{\mathbb{C}}$-adèles $\mathbf{A}_k^{\Omega_{\mathbb{C}}}$, where $\Omega_{\mathbb{C}}$ is the set of complex places of k. In particular, $\mathbf{A}_k^{\mathbb{C}} = \mathbf{A}_k$ if k is totally real, e.g. if $k = \mathbb{Q}$. Since $\mathrm{Br}(\mathbb{C}) = 0$, the evaluation map $\mathrm{ev}_A : X(\mathbf{A}_k) \longrightarrow \mathbb{Q}/\mathbb{Z}$ factors through the evaluation map

$$\mathrm{ev}_A^{\mathbb{C}} : X(\mathbf{A}_k^{\mathbb{C}}) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

The Brauer–Manin pairing

By definition, *the Brauer–Manin pairing*

$$X(\mathbf{A}_k) \times \mathrm{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

sends $(M_v) \in X(\mathbf{A}_k)$ and $A \in \mathrm{Br}(X)$ to

$$\mathrm{ev}_A\big((M_v)\big) = \sum_{v\in\Omega} \mathrm{inv}_v A(M_v) \in \mathbb{Q}/\mathbb{Z}.$$

If X is proper, then $X(\mathbf{A}_k) = \prod_v X(k_v)$. In this case the pairing becomes

$$\prod_{v\in\Omega} X(k_v) \times \mathrm{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

For any subset $B \subset \mathrm{Br}(X)$, we denote by $X(\mathbf{A}_k)^B \subset X(\mathbf{A}_k)$ the set of adelic points orthogonal to B with respect to the Brauer–Manin pairing, that is, the intersection of $\mathrm{ev}_A^{-1}(0)$ for $A \in B$. By the continuity of the evaluation map, it is a closed subset of $X(\mathbf{A}_k)$. When B is finite, Proposition 12.3.1 (iv) shows that the map $X(\mathbf{A}_k)\to\mathrm{Maps}(B,\mathbb{Q}/\mathbb{Z})$ factors through $\mathrm{Maps}(B,\mathbb{Z}/n)$ for some n, hence $X(\mathbf{A}_k)^B$ is closed and open in $X(\mathbf{A}_k)$. The set $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ is called *the Brauer–Manin set* of X. We abbreviate this notation by $X(\mathbf{A}_k)^{\mathrm{Br}}$.

Similarly, the evaluation map without complex components gives rise to the Brauer–Manin set $X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}$.

If, moreover, X is proper, then $X(\mathbf{A}_k)$ is compact. When $X(\mathbf{A}_k)^{\mathrm{Br}}$ is empty, the compact set $X(\mathbf{A}_k)$ has a covering by open subsets $X(\mathbf{A}_k) \setminus X(\mathbf{A}_k)^b$, for all $b \in \mathrm{Br}(X)$. Hence there is a *finite* subset $B \subset \mathrm{Br}(X)$ such that

$$X(\mathbf{A}_k) = \bigcup_{b \in B} (X(\mathbf{A}_k) \setminus X(\mathbf{A}_k)^b),$$

and therefore $X(\mathbf{A}_k)^B = \emptyset$.

Let X be a variety over k. For any $A \in \mathrm{Br}(X)$ we have the basic commutative diagram:

$$\begin{array}{ccccc}
X(k) & \hookrightarrow & X(\mathbf{A}_k) & & \\
\downarrow{\scriptstyle \mathrm{ev}_A} & & \downarrow & \searrow^{\mathrm{ev}_A} & \\
\mathrm{Br}(k) & \longrightarrow & \bigoplus_{v \in \Omega} \mathrm{Br}(k_v) & \xrightarrow{\mathrm{inv}_v} & \mathbb{Q}/\mathbb{Z}
\end{array}$$

where the bottom line is the complex given by the class field theory exact sequence (12.1).

Theorem 12.3.2 (Manin) [Man71] *Let k be a number field and let X be a variety over k. The Brauer–Manin set $X(\mathbf{A}_k)^{\mathrm{Br}}$ contains the closure of the image of the diagonal map $X(k) \to X(\mathbf{A}_k)$.*

Proof. The inclusion $X(k) \subset X(\mathbf{A}_k)^{\mathrm{Br}}$ follows immediately from the above diagram. Since $X(\mathbf{A}_k)^{\mathrm{Br}}$ is closed in $X(\mathbf{A}_k)$, it contains the closure of $X(k)$. □

Manin's observation is that this simple theorem accounts for most counter-examples to the Hasse principle known at the time. In these examples, the rôle of sequence (12.1) is played by some explicit form of the reciprocity law, mostly the quadratic reciprocity law. We shall review some of these examples in Section 12.5.

It is common to use the following terminology:

If X is a variety over k such that $X(\mathbf{A}_k) \neq \emptyset$ but $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$, then one says that *there is a Brauer–Manin obstruction to the Hasse principle for X.*

If the inclusion $X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)$ is not an equality, then one says that *there is a Brauer–Manin obstruction to strong approximation for X.* If X is proper, then this is a Brauer–Manin obstruction to weak approximation.

The space $X(\mathbf{A}_k)$ is the union of subsets

$$\prod_{v \in S} X(k_v) \times \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v),$$

where $S \subset \Omega$ is a finite set containing all infinite places, and $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_S)$ is a separated scheme of finite type with generic fibre X. For each subset $B \subset \mathrm{Br}(X)$ there is an inclusion

$$\mathcal{X}(\mathcal{O}_S) \subset \left(\prod_{v \in S} X(k_v) \times \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v) \right)^B .$$

The Brauer–Manin pairing may sometimes be used to show the failure of strong approximation outside a finite set of places, or even to give counter-examples to the integral Hasse principle (proving the emptiness of the set $\mathcal{X}(\mathcal{O}_S)$ of $\mathcal{O}_S$-integral points).

As we have seen in Proposition 12.3.1 (i), for a given element $A \in \mathrm{Br}(\mathcal{X})$, the image of ev_A on the set $\prod_{v\in S} X(k_v) \times \prod_{v\notin S} \mathcal{X}(\mathcal{O}_v)$ is computed using only the places in S. In general (unless X is proper) one cannot reduce the computation of $X(\mathbf{A}_k)^A$ to calculations at finitely many places.

Remark 12.3.3 Let X be a variety over k. More generally, given a contravariant functor F from the category of k-schemes to the category of sets, any element $\xi \in F(X)$ gives rise to a commutative diagram

$$\begin{array}{ccc} X(k) & \to & X(\mathbf{A}_k) \\ \downarrow & & \downarrow \\ F(k) & \to & \prod_{v\in\Omega} F(k_v) \end{array}$$

where the vertical arrows are given by "evaluation" of the element ξ on rational points and on local points. This gives a restriction on the image of $X(k)$ in $X(\mathbf{A}_k)$. The Brauer–Manin obstruction corresponds to the case $F(X) = \mathrm{Br}(X)$ and $\xi \in \mathrm{Br}(X)$. Another useful example of such a functor is the étale cohomology set $\mathrm{H}^1_{\text{ét}}(\cdot, G)$, where G is an algebraic group over k.

Here are some observations on the Brauer–Manin obstruction.

(1) Recall that $\mathrm{Br}_0(X) \subset \mathrm{Br}(X)$ is the image of the map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ induced by the structure morphism $X\to\mathrm{Spec}(k)$. If $X(k) \neq \emptyset$, then the homomorphism $\mathrm{Br}(k)\to\mathrm{Br}(X)$ has a section and so is injective. Using the injective map of the exact sequence (12.1) one shows that if $X(\mathbf{A}_k) \neq \emptyset$, then the natural map $\mathrm{Br}(k)\to\mathrm{Br}_0(X)$ is an isomorphism.

(2) Let $B \subset \mathrm{Br}(X)$. The set $X(\mathbf{A}_k)^B$ only depends on the image of B in the quotient $\mathrm{Br}(X)/\mathrm{Br}_0(X)$.

(3) Let us write X_v for $X \times_k k_v$. Let $\text{Б}(X) \subset \mathrm{Br}(X)$ be the subgroup consisting of elements $A \in \mathrm{Br}(X)$ such that for each place $v \in \Omega$ there exists $\alpha_v \in \mathrm{Br}(k_v)$ whose image in $\mathrm{Br}(X_v)$ is the same as the image of A. Assume $X(\mathbf{A}_k) \neq \emptyset$. Then $\mathrm{Br}(k_v)\to\mathrm{Br}(X_v)$ is injective for each v, so that α_v is well defined and equal to the value of A at any k_v-point of X. By Proposition 12.3.1, $\alpha_v = 0$ for almost all v. For each adelic point $(M_v) \in X(\mathbf{A}_k)$ one then has

$$\sum_{v\in\Omega} \mathrm{inv}_v A(M_v) = \sum_{v\in\Omega} \mathrm{inv}_v(\alpha_v) \in \mathbb{Q}/\mathbb{Z}.$$

The value of this sum does not depend on (M_v). The Brauer–Manin obstruction attached to the "small" subgroup $\text{Б}(X) \subset \mathrm{Br}(X)$ plays a great rôle in the study of the Hasse principle for homogeneous spaces of connected linear algebraic groups [Bor96, BCS08, Witt08] – but it is too small to control weak approximation.

Recall that for any abelian variety A over a number field k, the Tate-Shararevich group of A is defined as

$$Ш(A) = \mathrm{Ker}[\mathrm{H}^1(k,A)\to \prod_{v\in\Omega} \mathrm{H}^1(k_v,A)].$$

The quotient group $Б(X)/\mathrm{Br}_0(X)$ is conjecturally finite. Indeed, we have the following proposition.

Proposition 12.3.4 [BCS08, Prop. 2.14] *Let X be a smooth, projective, geometrically integral variety over k such that $X(\mathbf{A}_k)\neq\emptyset$. If $Ш(\mathbf{Pic}^0_{X_K/K})$ is finite for any finite extension K/k, then $Б(X)/\mathrm{Br}_0(X)$ is finite.*

(4) Let X be a smooth, projective, geometrically integral variety over a number field k. Suppose there is a finite field extension K/k such that $X_K(\mathbf{A}_K)^{\mathrm{Br}}=\emptyset$. Then $X(K)=\emptyset$, hence $X(k)=\emptyset$. But can one conclude that $X(\mathbf{A}_k)^{\mathrm{Br}}=\emptyset$? This question is open in general. The answer is positive if $\mathrm{Pic}(\overline{X})$ is a finitely generated free abelian group and $\mathrm{Br}(\overline{X})=0$.

We finish this section with a remark about the functoriality of the Brauer–Manin set.

Proposition 12.3.5 *A morphism $f:X\to Y$ of varieties over a number field k induces a continuous map of their Brauer–Manin sets $X(\mathbf{A}_k)^{\mathrm{Br}}\to Y(\mathbf{A}_k)^{\mathrm{Br}}$.*

Proof. We have a continuous map of topological spaces $f: X(\mathbf{A}_k)\to Y(\mathbf{A}_k)$, see Section 12.1.3, and a map of Brauer groups $f^*:\mathrm{Br}(Y)\to\mathrm{Br}(X)$, see Section 3.2. For a point $(P_v)\in X(\mathbf{A}_k)$ and $A\in\mathrm{Br}(Y)$ we have $(f^*A)(P_v)=A(f(P_v))$, hence f sends $X(\mathbf{A}_k)^{f^*A}$ to $Y(\mathbf{A}_k)^A$. Thus f sends $X(\mathbf{A}_k)^{\mathrm{Br}}\subset X(\mathbf{A}_k)^{f^*\mathrm{Br}(Y)}$ to $Y(\mathbf{A}_k)^{\mathrm{Br}(Y)}=Y(\mathbf{A}_k)^{\mathrm{Br}}$. □

In particular, for varieties X and Y the Brauer–Manin set of $X\times_k Y$ is contained in $X(\mathbf{A}_k)^{\mathrm{Br}}\times Y(\mathbf{A}_k)^{\mathrm{Br}}$. In the crucial case this is an equality.

Theorem 12.3.6 *Let X and Y be smooth, projective, geometrically integral varieties over a number field k. Then we have*

$$(X\times Y)(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k)^{\mathrm{Br}}\times Y(\mathbf{A}_k)^{\mathrm{Br}}.$$

Proof. See [SZ14, Thm. C], which is based on the results of Sections 4.6 and 15.4, and uses torsors and the descent theory. □

12.4 The structure of the Brauer–Manin set

When $\mathrm{Br}(X)$ is finite modulo $\mathrm{Br}_0(X)$, the Brauer–Manin set of X is an open and closed subset of $X(\mathbf{A}_k)$. More precisely, we have the following

Lemma 12.4.1 *Let X be a proper variety over a number field k. Assume that $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite. Then there exists a finite set S of places of k such that*

$$X(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{v \notin S} X(k_v)$$

for an open and closed set $Z \subset \prod_{v\in S} X(k_v)$.

Proof. There is a finite set $B \subset \mathrm{Br}(X)$ that generates $\mathrm{Br}(X)$ modulo $\mathrm{Br}_0(X)$. By Proposition 12.3.1 (iii) there is a finite set of places S such that $A(M_v) = 0$ for each $A \in B$ and any $M_v \in X(k_v)$, where $v \notin S$. Thus for each $A \in B$ the evaluation map $\mathrm{ev}_A : X(\mathbf{A}_k) \to \mathbb{Q}/\mathbb{Z}$ is the composition of the projection $X(\mathbf{A}_k) \to \prod_{v\in S} X(k_v)$ and a continuous map $\prod_{v\in S} X(k_v) \to \mathbb{Q}/\mathbb{Z}$. The resulting map $\prod_{v\in S} X(k_v) \to (\mathbb{Q}/\mathbb{Z})^B$ is continuous with finite image (Proposition 12.3.1 (iv)) thus its kernel Z is an open and closed subset of $\prod_{v\in S} X(k_v)$. □

In this section we discuss how small the set S can be. We essentially follow the paper [CTS13a].

Question 12.4.2 *Let X be a smooth, projective and geometrically integral variety over a number field k. Assume that $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite. Can one choose S in Lemma 12.4.1 to be the union of the archimedean places of k and the places of bad reduction for X?*

The following result gives sufficient conditions under which the answer is positive.

Theorem 12.4.3 *Let k be a number field. Let S be a finite set of places of k containing the archimedean places, and let $\mathcal{O}_S$ be the ring of S-integers of k. Let $\pi : \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_S)$ be a smooth and proper $\mathcal{O}_S$-scheme with geometrically integral fibres. Let X/k be its generic fibre. Assume*

(i) $\mathrm{H}^1(X, \mathcal{O}_X) = 0$*;*

(ii) *the Néron–Severi group* $\mathrm{NS}(\overline{X})$ *has no torsion;*

(iii) *the transcendental Brauer group* $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ *is a finite abelian group of order invertible in* $\mathcal{O}_S$*.*

Then $X(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{v\notin S} X(k_v)$, where $Z \subset \prod_{v\in S} X(k_v)$ is an open and closed subset.

Proof. We claim that for any place $v \notin S$, the image of $\mathrm{Br}(X)$ in $\mathrm{Br}(X_v)$ is contained in the sum of the images of $\mathrm{Br}(k_v)$ and $\mathrm{Br}(\mathcal{X}_v)$. It is enough to prove this statement for the ℓ-primary component, for each prime ℓ.

Let p be the residual characteristic of v. The combination of assumptions (i) and (ii), Proposition 9.4.2 and Lemma 9.4.1 gives that the image of $\mathrm{Br}_1(X)$ in $\mathrm{Br}(X_v)$ is contained in the subgroup generated by the images of $\mathrm{Br}(k_v)$ and $\mathrm{Br}(\mathcal{X}_v)$. Assumption (iii) implies $\mathrm{Br}(X)\{p\} \subset \mathrm{Br}_1(X)$. Thus the image of $\mathrm{Br}(X)\{p\}$ in $\mathrm{Br}(X_v)$ is contained in the subgroup generated by the images of $\mathrm{Br}(k_v)\{p\}$ and $\mathrm{Br}(\mathcal{X}_v)\{p\}$.

Let us prove the analogous statement for any prime $\ell \neq p$. By Proposition 9.4.3 we only need to check that $\mathrm{H}^1_{\text{ét}}(\overline{\mathcal{X}}_0, \mathbb{Z}/\ell) = 0$, where $\overline{\mathcal{X}}_0$ is the closed geometric fibre of $\pi : \mathcal{X}_v \to \mathrm{Spec}(\mathcal{O}_v)$. By the smooth base change theorem for étale cohomology (see, e.g. [Mil80], VI, Cor. 4.2) the group $\mathrm{H}^1_{\text{ét}}(\overline{\mathcal{X}}_0, \mathbb{Z}/\ell)$ is isomorphic to $\mathrm{H}^1_{\text{ét}}(\overline{X}_v, \mathbb{Z}/\ell)$, which in turn is isomorphic to $\mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{Z}/\ell)$ by [Mil80], VI, Cor. 4.3. The Kummer exact sequence gives an isomorphism $\mathrm{H}^1_{\text{ét}}(\overline{X}, \mu_\ell) \tilde{\longrightarrow} \mathrm{Pic}(\overline{X})[\ell]$, and the vanishing of the latter group follows from conditions (i) and (ii).

We now complete the proof of the theorem here under the simplifying assumption $X(k) \neq \emptyset$. We refer to [CTS13a, Lemma 1.2] for the (easy) argument assuming only $X(\mathbf{A}_k) \neq \emptyset$.

Let thus fix a k-point P. The map $\mathrm{Br}(k) \to \mathrm{Br}_0(X)$ is then an isomorphism and the group $\mathrm{Br}(X)/\mathrm{Br}_0(X) = \mathrm{Br}(X)/\mathrm{Br}(k)$ is finite. This group $\mathrm{Br}(X)/\mathrm{Br}(k)$ is generated by the images of finitely many elements $A_i \in \mathrm{Br}(X)$ that can be assumed to satisfy $A_i(P) = 0$. For $v \notin S$, we have an equality

$$A_i \otimes_k k_v = \beta_i + \gamma_i \in \mathrm{Br}(X_v),$$

where $\beta_i \in \mathrm{Br}(\mathcal{X}_v)$ and $\gamma_i \in \mathrm{Br}(k_v)$. We have $\beta_i(P) = 0$ since $P \in X(k)$ extends to an $\mathcal{O}_v$-point of $\mathcal{X}_v$ by the properness of $\mathcal{X}/\mathcal{O}$ and $\mathrm{Br}(\mathcal{O}_v) = 0$ (Theorem 3.4.2 (ii) and Theorem 1.2.11). It follows that $\gamma_i = 0$. Hence $A_i \otimes_k k_v$ belongs to $\mathrm{Br}(\mathcal{X}_v)$, and so A_i vanishes at every point of $X(k_v) = \mathcal{X}_v(\mathcal{O}_v)$.

Let $B \subset \mathrm{Br}(X)$ be the finite group generated by the elements $A_i \in \mathrm{Br}(X)$. We now conclude that the Brauer–Manin pairing

$$X(\mathbf{A}_k) \times \mathrm{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is induced by the pairing

$$\prod_{v \in S} X(k_v) \times B \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

This concludes the proof. □

Remark 12.4.4 Conditions (i) and (ii) together are equivalent to the assumption that $\mathrm{Pic}(\overline{X})$ is a finitely generated torsion-free abelian group. We do not know if condition (iii) may be dropped from this theorem. The finiteness of the transcendental Brauer group is closely related to the Tate conjecture for divisors, see Theorem 15.2.1.

One can give purely geometric conditions under which the assumptions of Theorem 12.4.3 are satisfied.

Corollary 12.4.5 *Let $\pi : \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_S)$ be a smooth proper $\mathcal{O}_S$-scheme with geometrically integral fibres. Let X/k be its generic fibre. Assume*

(i) $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ *for* $i = 1, 2$;

(ii) *the Néron–Severi group* $\mathrm{NS}(\overline{X})$ *has no torsion;*

(iii) *either* $\dim X = 2$, *or* $\mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$ *is torsion-free for every prime ℓ invertible in $\mathcal{O}_S$.*

Then we have the same conclusion as in Theorem 12.4.3.

Proof. We only need to verify condition (iii) of Theorem 12.4.3. By Theorem 4.2.6 and Theorem 4.4.2, if $\mathrm{H}^2(X,\mathcal{O}_X)=0$, then $\mathrm{Br}(\overline{X})$ is finite and isomorphic to the direct sum $\oplus_\ell \mathrm{H}^3_{\text{ét}}(\overline{X},\mathbb{Z}_\ell)_{\text{tors}}$. In the surface case, Proposition 4.2.7 shows that $\mathrm{NS}(\overline{X})\{\ell\}\simeq \mathrm{H}^3_{\text{ét}}(\overline{X},\mathbb{Z}_\ell)_{\text{tors}}$. Thus under assumptions (ii) and (iii), $\mathrm{Br}(\overline{X})\{\ell\}=0$ for $\ell\notin S$. The group $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is a subgroup of $\mathrm{Br}(\overline{X})$. Thus hypothesis (iii) in Theorem 12.4.3 is satisfied. □

This corollary can be applied to rationally connected varieties. Indeed, over a field of characteristic 0 such a varietiy X is $\mathcal{O}_X$-acyclic, that is, $\mathrm{H}^i(X,\mathcal{O}_X)=0$ for all $i>0$, and is algebraically simply connected [Deb01, Cor. 4.18], hence $\mathrm{Pic}(\overline{X})_{\text{tors}}=\mathrm{NS}(\overline{X})_{\text{tors}}=0$.

12.5 Examples of Brauer–Manin obstruction

Reducible varieties

The following statements are Brauer group versions of the results of Stoll [Sto07, Lemma 5.10, Prop. 5.11, Prop. 5.12].

Lemma 12.5.1 *Let $X=P_1\sqcup P_2=Spec(k)\sqcup Spec(k)$. Then $X(k)=X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}$.*

Proof. Take any $(Q_v)\in X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}$. Let $S_1\subset\Omega\smallsetminus\Omega_{\mathbb{C}}$ consist of the places v such that $Q_v=P_1$. The complementary set $S_2=(\Omega\smallsetminus\Omega_{\mathbb{C}})\smallsetminus S_1$ consists of the places v such that $Q_v=P_2$. If $S_1=\Omega\smallsetminus\Omega_{\mathbb{C}}$, then $\{Q_v\}=P_1$, and if $S_2=\Omega\smallsetminus\Omega_{\mathbb{C}}$, then $\{Q_v\}=P_2$. We now suppose that we are not in one of these cases and deduce a contradiction. Choose $v_1\in S_1$ and $v_2\in S_2$. Since neither v_1 nor v_2 is a complex place, there is an $\alpha\in\mathrm{Br}(k)$ such that $\mathrm{inv}_{v_1}(\alpha_{v_1})=1/2$, $\mathrm{inv}_{v_2}(\alpha_{v_2})=1/2$ and $\alpha_v=0$ for $v\neq v_1,v_2$. Consider the element $\beta=(\alpha,0)\in\mathrm{Br}(X)=\mathrm{Br}(k)\oplus\mathrm{Br}(k)$. Then $\sum_{v\in\Omega\smallsetminus\Omega_{\mathbb{C}}}\mathrm{inv}_v(\beta(Q_v))=\mathrm{inv}_{v_1}(\alpha_{v_1})\neq 0$. □

Proposition 12.5.2 *Let $X=X_1\sqcup\cdots\sqcup X_n$ be a disjoint union of varieties over k. Then*

$$X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}=X_1(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}\sqcup\cdots\sqcup X_n(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}.$$

Proof. It is enough to prove the statement for $n=2$. By functoriality, the right hand side is included in the left hand side. Let $Y=\mathrm{Spec}(k)\sqcup\mathrm{Spec}(k)$. Consider the projection $p: X_1\sqcup X_2\to\mathrm{Spec}(k)\sqcup\mathrm{Spec}(k)$. By the functoriality of the Brauer–Manin set we have $p(X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}})\subset Y(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}$. Lemma 12.5.1 says that $Y(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}=Y(k)$. Thus $X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}\subset X_1(\mathbf{A}_k^{\mathbb{C}})\sqcup X_2(\mathbf{A}_k^{\mathbb{C}})$. Since $\mathrm{Br}(X)=\mathrm{Br}(X_1)\oplus\mathrm{Br}(X_2)$, we have $X(\mathbf{A}_k)^{\mathrm{Br}}\subset X_1(\mathbf{A}_k)^{\mathrm{Br}}\sqcup X_2(\mathbf{A}_k)^{\mathrm{Br}}$. □

As a special example, if $X=X_1\sqcup X_2$, $X_1(\mathbf{A}_k)=\emptyset$, $X_2(\mathbf{A}_k)=\emptyset$ and $X(\mathbf{A}_k)\neq\emptyset$ then $X(\mathbf{A}_k)^{\mathrm{Br}}=\emptyset$.

Corollary 12.5.3 *Let X be a finite k-scheme. Then $X(k)=X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}$.*

Proof. For X of dimension zero we have $\mathrm{Br}(X_{\text{red}})=\mathrm{Br}(X)$ (Prop. 7.2.4), so one may assume that X is reduced. By Proposition 12.5.2 it is enough to prove the statement when $X = \mathrm{Spec}(K)$, where K is a field. In this case it is a known consequence of Chebotarev's theorem (Theorem 12.1.5) that for a non-trivial extension of number fields K/k there are infinitely many places v such that k_v is not a direct summand of $K \otimes_k k_v$. For such places v we have $X(k_v) = \emptyset$, in particular $X(\mathbf{A}_k^{\mathbb{C}}) = \emptyset$. $\square$

Let us discuss the famous counter-example to the Hasse principle over $\mathbb{Q}$

$$(x^2 - 13)(x^2 - 17)(x^2 - 221) = 0$$

from another point of view. Let us think of x as a coordinate in $\mathbb{A}^1_{\mathbb{Q}}$ and define Z as the closed subset of $\mathbb{G}_{m,\mathbb{Q}}$ given by this polynomial. Consider

$$A = (x, 13) \in \mathrm{Br}(\mathbb{G}_{m,\mathbb{Q}}).$$

Take any $(M_v) \in Z(\mathbf{A}_{\mathbb{Q}})$. It is clear that $A(M_v) = 0$ when $\mathbb{Q}_v = \mathbb{R}$. Now let p be a prime. Let $v = v_p$ be the p-adic valuation; write p for v_p in the subscript. Let $x_p \in \mathbb{Q}_p$ be the coordinate of $M_p \in Z(\mathbb{Q}_p)$. If p is not one of 2, 13 or 17, then 13 and x_p are both in $\mathbb{Z}_p^*$, so the Hilbert symbol $(x_p, 13)_p = 1$ and thus $A(M_p) = 0$ in $\mathrm{Br}(\mathbb{Q}_p)$. Let $p = 13$. Then $x_{13}^2 = 17$ in $\mathbb{Q}_{13}$ and $x_{13} = \pm 2$ up to a square in $\mathbb{Q}_{13}^*$, hence the Hilbert symbol $(\pm 2, 13)_{13} = -1$ and so $A(M_{13}) \neq 0$. Let $p = 17$. Then $x_{17}^2 = 13$ in $\mathbb{Q}_{17}$, hence $x_{17} = \pm 8$ up to a square in $\mathbb{Q}_{17}^*$. Then $(\pm 8, 13)_{17} = 1$ and so $A(M_{17}) = 0$. Finally, let $p = 2$. Then $x_2^2 = 17$ in $\mathbb{Q}_2$, hence $x_2 = \pm 5$ up to a square in $\mathbb{Q}_2^*$. Then $(\pm 5, 13)_2 = 1$, as follows from the reciprocity law since $(\pm 5, 13)_5 = 1$ and $(\pm 5, 13)_{13} = 1$. We conclude $A(M_2) = 0$. This easy calculation shows that there is a Brauer–Manin obstruction attached to the class in $\mathrm{Br}(Z)$ which is the restriction of $A \in \mathrm{Br}(\mathbb{G}_{m,\mathbb{Q}})$.

Compare with Stoll [Sto07], Liu–Xu [LX15], Jahnel–Loughran [JL15].

Let us now describe some counter-examples to the Hasse principle on geometrically irreducible varieties.

Iskovskikh's counter-example to the Hasse principle

The following example was explored in [CTCS80, Exemple 5.4]. In a different guise, the case $c = 3$ is due to Iskovskikh [Isk71]. Let $U = U_c$ be the variety

$$y^2 + z^2 = (c - x^2)(x^2 - c + 1) \neq 0,$$

where $c \in \mathbb{N}$ is congruent to 3 modulo 4. Using Hensel's lemma, one easily checks that U has points in all completions of $\mathbb{Q}$.

Consider the Azumaya algebra on U defined by the quaternion algebra $A = (c - x^2, -1)$. Let $X = X_c$ be a smooth compactification of U_c. As proved in Example 5.2.10 the class of A comes from a class in $\mathrm{Br}(X) \subset \mathrm{Br}(U)$. By Proposition 9.5.2, for any place v of $\mathbb{Q}$, finite or infinite, the image of the evaluation map

$$\mathrm{ev}_A : X(\mathbb{Q}_v) \longrightarrow \mathrm{Br}(\mathbb{Q}_v) \subset \mathbb{Q}/\mathbb{Z}$$

coincides with the image of

$$\mathrm{ev}_A : U(\mathbb{Q}_v) \longrightarrow \mathrm{Br}(\mathbb{Q}_v) \subset \mathbb{Q}/\mathbb{Z}.$$

Let $K = \mathbb{Q}(\sqrt{-1})$. Let v be a place of $\mathbb{Q}$ and let w be a place of K over v. For $\rho_v \in \mathbb{Q}_v^*$ we have $(\rho_v, -1) = 0 \in \mathrm{Br}(\mathbb{Q}_v)$ if and only if ρ_v is a norm for the local extension $K_w/\mathbb{Q}_v$. We thus need to compute the images of the maps

$$\phi_v : U(\mathbb{Q}_v) \longrightarrow \mathbb{Q}_v^*/\mathrm{N}(K_w^*) \subset \mathbb{Z}/2$$

where ϕ_v sends $M_v = (x_v, y_v, z_v) \in U(\mathbb{Q}_v)$ to the class of $x_v^2 - c$.

If v splits in K, the target of ϕ_v is zero.

For $v = v_\infty$ we have $(\rho_v, -1) = 0 \in \mathrm{Br}(\mathbb{R})$ if and only if $\rho_v > 0$. The equation

$$y_\infty^2 + z_\infty^2 = (c - x_\infty^2)(x_\infty^2 - c + 1) \in \mathbb{R}^*$$

forces $c - x_\infty^2 > 0$, hence the image of ϕ_v is zero.

Suppose $v = p$ is a finite prime which is inert in K. We have $(\rho_p, -1) = 0$ if and only if $v(\rho_p)$ is even. If $v(x_v) < 0$, then $v(c - x_v^2)$ is even and thus $c - x_v^2$ is a norm. Suppose $v(x_v) \geq 0$. From the equality

$$(c - x_v^2) + (x_v^2 - c + 1) = 1$$

we deduce that at least one of $v(c - x_v^2)$ and $v(x_v^2 - c + 1)$ vanishes. From the equality

$$y_v^2 + z_v^2 = (c - x_v^2)(x_v^2 - c + 1) \in \mathbb{Q}_v^*,$$

we deduce that the sum of the valuations of $c - x_v^2$ and $x_v^2 - c + 1$ is even. Thus $v(c - x_v^2)$ is even, and the image of ϕ_v is zero.

For the unique ramified prime $v = 2$, an element $\rho_2 \in \mathbb{Q}_2^*$ is a sum of two squares if and only if it is the product of a power of 2 and a unit in $\mathbb{Z}_2^*$ which is congruent to 1 modulo 4. Write $x_2 = u/v$ with u and v in $\mathbb{Z}_2$, not both divisible by 2. Up to multiplication by a square, $c - x_2$ is equal to $cv^2 - u^2$, which by the hypothesis on c is congruent to $3v^2 - u^2$ modulo 4. Up to multiplication by a square, $x_2^2 - c + 1$ is equal to $u^2 - (c-1)v^2$ which by the hypothesis on c is congruent to $u^2 - 2v^2$ modulo 4. The possible values for (u^2, v^2) modulo 4 are $(0,1), (1,0), (1,1)$. In the first and second cases, $3v^2 - u^2$ is congruent to 3 modulo 4 hence is not a norm for $K_2/\mathbb{Q}_2$. In the third case, $u^2 - 2v^2$ is congruent to 3 modulo 4, hence is not a norm for $K_2/\mathbb{Q}_2$. Since $\mathbb{Q}_2^*/\mathrm{N}(K_2^*) \cong \mathbb{Z}/2$, and the product $(c - x_2^2)(x_2^2 - c + 1) = y_2^2 + z_2^2$ is a norm, we conclude that $c - x_2^2$ is never a norm for the extension $K_2/\mathbb{Q}_2$. Thus the image of ϕ_2 is $1 \in \mathbb{Z}/2$.

For any $(M_v) \in X(\mathbf{A}_\mathbb{Q})$, we thus have

$$\sum_{v \in \Omega} \mathrm{inv}_v A(M_v) = 1/2,$$

hence $X(\mathbb{A}_\mathbb{Q})^A = \emptyset$ implying $X(\mathbb{Q}) = \emptyset$.

Exercise 12.5.4 [CTCS80, Exemple 5.5], [San82a, §2] *Let $c \geq 2$ be an integer. Let X_c be a smooth projective variety over $\mathbb{Q}$ birationally equivalent to the affine surface with equation*

$$y^2 + 3z^2 = (c - x^2)(x^2 - c + 1).$$

Consider the unramified quaternion algebra $A = (c - x^2, -3)$ and prove that $X_c(\mathbf{A}_\mathbb{Q})^A = \emptyset$ if $c = 3^{2s+1}(3n - 1)$ for some integers $s \geq 0$ and $n \geq 1$.

Exercise 12.5.5 [CTCS80, Exemple 5.6] *Let $c \geq 3$ be an integer. Let X_c be a smooth projective variety over $\mathbb{Q}$ birationally equivalent to the affine surface with equation*

$$y^2 + z^2 = (c - x^2)(x^2 - c + 2).$$

Consider the unramified quaternion algebra $A = (c - x^2, -1)$ and prove that $X_c(\mathbf{A}_\mathbb{Q})^A = \emptyset$ if $c = 4^n(8m + 7)$ for some integers $n \geq 0$ and $m \geq 0$.

Swinnerton-Dyer's counter-example to weak approximation [SwD62]

Let U be the affine surface over $\mathbb{Q}$ defined by

$$y^2 + z^2 = (4x - 7)(x^2 - 2) \neq 0.$$

Let

$$A = (4x - 7, -1) \in \mathrm{Br}(U).$$

One shows that for any smooth compactification $U \subset X$ the class of A belongs to $\mathrm{Br}(X) \subset \mathrm{Br}(U)$. For any prime $p \neq 2$ a standard valuation argument based on the equality

$$(4x - 7)(4x + 7) - 16(x^2 - 2) = -17$$

shows that A vanishes on $U(\mathbb{Q}_p)$ and hence also on $X(\mathbb{Q}_p)$. For $p = 2$ one checks that A also vanishes on $U(\mathbb{Q}_2)$. The set of real points $U(\mathbb{R})$ has two connected components: the first one given by $-\sqrt{2} < x < \sqrt{2}$ and the second one given by $x > 7/4$. The connected components of $X(\mathbb{R})$ are obtained by taking the closure of the connected components of $U(\mathbb{R})$ in $X(\mathbb{R})$. It is clear that A takes the non-zero value in $\mathrm{Br}(\mathbb{R})$ on any point of the first component, and the zero value on any point of the second component. The reciprocity law then implies that all $\mathbb{Q}$-points of X are contained in the second component.

Principal homogeneous spaces under a specific torus

The following example is discussed in more detail in [CT14].

Let k be a number field, $a, b, c \in k^*$, and let U be the variety over k defined by the equation

$$(x^2 - ay^2)(z^2 - bt^2)(u^2 - abw^2) = c.$$

Let X be a smooth compactification of U. Computing residues, one easily checks that the class of the quaternion algebra $A = (x^2 - ay^2, b) \in \mathrm{Br}(U)$ lies in the subgroup $\mathrm{Br}(X)$, cf. Example 5.2.12.

Proposition 12.5.6 *Assume that $a, b, c \in k^*$ are such that for each place v of k the field extension $k_v(\sqrt{a}, \sqrt{b})$ is cyclic, hence of degree at most 2. Then*

(i) *The class A belongs to $\text{Б}(X)$.*

(ii) *For each $(M_v) \in X(\mathbf{A}_k)$ one has*

$$\sum_{v \in \Omega} \mathrm{inv}_v A(M_v) = \sum_{v \,|\, a \in k_v^{*2}} (c, b)_v = \sum_{v \,|\, a \notin k_v^{*2}} (c, b)_v \in \mathbb{Z}/2.$$

Proof. (i) Let F be a field extension of k. If a, b or ab is a square in F, then the left hand side of the equation of U has a linear factor. This easily implies that U_F is rational over F. Then the natural map $\mathrm{Br}(F) \to \mathrm{Br}(X_F)$ is an isomorphism by Corollary 5.2.6. This proves (i).

(ii) Let $v \in \Omega$ and let $M_v = (x_v, y_v, z_v, t_v, u_v, w_v)$ be a point of $U(k_v)$. Let us compute $(x_v^2 - a y_v^2, b)_v \in \mathrm{Br}(k_v)$. Assume that a is not a square in k_v. Then either b or ab is a square in k_v. In the first case $(x_v^2 - a y_v^2, b)_v = 0$, whereas in the second case $(x_v^2 - a y_v^2, b)_v = (x_v^2 - a y_v^2, a)_v = 0$. Now assume that a is a square in k_v. From the equation of U we obtain

$$(x_v^2 - a y_v^2, b)_v = (z_v^2 - b t_v^2, b)_v + (u_v^2 - ab w_v^2, b)_v + (c, b)_v.$$

Since $a \in k_v^{*2}$ we see that $(u_v^2 - ab w_v^2, b)_v = (u_v^2 - ab w_v^2, ab)_v = 0$. The first term of the right hand side is zero, hence $(x_v^2 - a y_v^2, b)_v = (c, b)_v$. By the continuity of ev_A this extends to any point of $X(k_v)$. □

Starting from this explicit formula, one easily produces counter-examples to the Hasse principle. For $k = \mathbb{Q}$ take $a = 17$, $b = 13$, $c = 5$.

The Reichardt–Lind counter-example to the Hasse principle

Let X be the smooth compactification of the smooth curve U over $\mathbb{Q}$ defined by

$$2y^2 = x^4 - 17 \neq 0.$$

One checks that $X(\mathbf{A}_\mathbb{Q}) \neq \emptyset$. (For the primes of good reduction this follows from the Hasse–Weil bound for the number of $\mathbb{F}_p$-points on a curve of genus 1 and the Hensel lemma.) In Example 5.2.11 we checked that the quaternion algebra $A = (y, 17)$ defines an element of $\mathrm{Br}(X) \subset \mathrm{Br}(U)$. It is obvious that $A(U(\mathbb{R})) = 0$, hence by the continuity of ev_A we have $A(X(\mathbb{R})) = 0$. One then checks that $A(U(\mathbb{Q}_p)) = 0$ for any prime $p \neq 17$. By the continuity of ev_A we obtain $A(X(\mathbb{Q}_p)) = 0$. Next, ev_A sends $U(\mathbb{Q}_{17})$ to one point $1/2 \in \mathbb{Q}/\mathbb{Z}$. This implies $\mathrm{inv}_{17} A(X(\mathbb{Q}_{17})) = 1/2$. Thus

$$\sum_{v \in \Omega} \mathrm{inv}_v A(M_v) = 1/2$$

for any $(M_v) \in X(\mathbf{A}_\mathbb{Q})$. We conclude that $X(\mathbb{Q}) = \emptyset$. (As a matter of fact, A is contained in the subgroup $\text{Б}(X) \subset \mathrm{Br}(X)$, as may be deduced from Corollary 9.5.4.)

For the history of this example we quote from Cassels' survey [Cas66, p. 284]: "... Lind [Lin40] in his dissertation gave examples of curves of genus 1 with points everywhere locally but not globally, including the example later given by Reichardt. We reproduce Lind's elegant argument, which has recently been rediscovered by Mordell, and which does not fall readily into the paradigm proposed in this paper. One has to prove that there are no solutions of

$$u^4 - 17v^4 = 2w^2 \tag{$*$}$$

in coprime integers u, v, w. We first show that w is a quadratic residue of 17. For if p is an odd prime divisor of w, it follows from $(*)$ that 17 is a quadratic residue of p, so p is a quadratic residue of 17 by the law of quadratic reciprocity and 2 is in any case a quadratic residue of 17. Hence u^4 and w^2 are both quartic residues of 17. Then $(*)$ implies that 2 is a quartic residue of 17, which is not the case." According to Cassels, Reichardt [Rei42] considered the curve over $\mathbb{Q}(\sqrt{2})$, computed its non-empty set of rational points over that field, then showed there are no Galois invariants.

Failure of weak approximation

It is delicate to exhibit counter-examples to the Hasse principle or to prove that for a given place v the set $X(k)$ is not dense in $X(k_v)$. It is much easier to give counter-examples to weak approximation at a finite set of places.

Proposition 12.5.7 *Let k be a number field and let X be a projective variety over k. Assume that $X(k) \neq \emptyset$ and that there exist an element $\alpha \in \mathrm{Br}(X)$ and a place w of k such that α takes at least two different values on $X(k_w)$. Then there exists a finite set S of places of k such that $X(k)$ is not dense in $\prod_{v\in S} X(k_v)$, so that weak approximation fails for X.*

Proof. By Proposition 12.3.1, there exists a finite set S of places of k such that α identically vanishes on each $X(k_v)$ for $v \notin S$. We thus have $w \in S$. Let $P \in X(k)$ be a rational point. For $v \in S$, $v \neq w$, let $N_v \in X(k_v)$ be the image of $P \in X(k) \subset X(k_v)$. Let $N_w \in X(k_w)$ be a point such that $\alpha(N_w) \in \mathrm{Br}(k_w)$ is not equal to $\mathrm{res}_{k_w/k}(\alpha(P)) \in \mathrm{Br}(k_w)$. By reciprocity (Theorem 12.1.8) and vanishing of α on $X(k_v)$ for $v \notin S$, we have

$$0 = \sum_{v\in S} \mathrm{inv}_v(\mathrm{res}_{k_v/k}(\alpha(P))).$$

We then have

$$\sum_{v\in S} \mathrm{inv}_v(\alpha(N_v)) = \mathrm{inv}_w(\alpha(N_w)) - \mathrm{inv}_w(\mathrm{res}_{k_w/k}(\alpha(P))) \neq 0 \in \mathbb{Q}/\mathbb{Z}.$$

This implies that for any choice of $N_v \in X(k_v)$ for $v \notin S$ we have

$$\sum_{v\in\Omega} \mathrm{inv}_v(\alpha(N_v)) \neq 0 \in \mathbb{Q}/\mathbb{Z}.$$

By reciprocity (Theorem 12.1.8) and continuity of the evaluation map on local points (Proposition 9.5.2), this implies that $\{N_v\} \in \prod_{v\in S} X(k_v)$ is not in the closure of $X(k)$. $\square$

Other examples

Many more examples have been constructed. In particular, counter-examples to the Hasse principle and weak approximation have been given for the following classes of varieties.

- Smooth projective curves of arbitrary genus $g \geq 1$.
- Smooth, projective, geometrically rational varieties of dimension at least 2, including smooth del Pezzo surfaces of degree d with $2 \leq d \leq 4$, in particular, smooth cubic surfaces (Swinnerton-Dyer; Cassels and Guy).
- Smooth compactifications of homogeneous spaces of connected linear algebraic groups.
- Surfaces with a pencil of curves of genus one.
- K3 surfaces, such as smooth quartics in $\mathbb{P}^3$.

  Precise references for such examples can be found in the surveys [VA13, VA17, Witt18].

12.6 Harari's formal lemma

This section is based on [CT03]. The following result is [Har94, Thm. 2.1.1, p. 226].

Theorem 12.6.1 (Harari) *Let X be a smooth integral variety over a number field k. Let S be a finite set of places of k and let $\mathcal{X}\to\mathrm{Spec}(\mathcal{O}_S)$ be a morphism of finite type with generic fibre X. Let $U \subset X$ be a non-empty open subset of X and let $\alpha \in \mathrm{Br}(U) \smallsetminus \mathrm{Br}(X)$. There exist infinitely many places v of k for which there is a point $M_v \in U(k_v) \cap \mathcal{X}(\mathcal{O}_v)$ with $\alpha(M_v) \neq 0$.*

Proof. The first part of the proof is a reduction to the case when X is a curve.

There is an irreducible divisor $Z \subset X$ such that the residue of α at the generic point of Z is a non-zero element $\partial_Z(\alpha) \in \mathrm{H}^1(k(Z), \mathbb{Q}/\mathbb{Z})$. Let n be the order of $\partial_Z(\alpha)$. Then $\partial_Z(\alpha)$ is an element of the subgroup $\mathrm{H}^1(k(Z), \mathbb{Z}/n)$ of $\mathrm{H}^1(k(Z), \mathbb{Q}/\mathbb{Z})$. After replacing X by an open subset we can assume that Z is smooth and $U = X \smallsetminus Z$. Then the exact sequence (3.16) shows that $\partial_Z(\alpha)$ comes from a non-zero element $\rho \in \mathrm{H}^1(Z, \mathbb{Z}/n)$. Let $Z_1\to Z$ be the finite cyclic cover defined by ρ, or, equivalently, the torsor over Z under $\mathbb{Z}/n$ with class ρ. The scheme Z_1 is connected. Indeed, the invariant subfield K of the kernel of the homomorphism $\mathrm{Gal}(\overline{k(Z)}/k(Z))\to\mathbb{Q}/\mathbb{Z}$ corresponding to $\partial_Z(\alpha)$ is a field extension of $k(Z)$ of degree n, so $\mathrm{Spec}(K)$ must be the generic point of Z_1.

Replacing X by an open subset, we may assume that there is a generically finite morphism $Z\to\mathbb{A}^d_k$. Since Z_1 is connected, Hilbert's irreducibility theorem shows the existence of infinitely many k-points M of $\mathbb{A}^d_k$ such that the fibre of the composite morphism $Z_1\to Z\to\mathbb{A}^d_k$ at M is integral. Let us pick one such point M. The inverse image of M under the morphism $Z\to\mathbb{A}^d_k$ is a closed point $P\in Z$ such that $\rho(P)\neq 0\in \mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z})$.

A local equation of $Z\subset X$ at P can be extended to a regular system of parameters of the regular local ring $\mathcal{O}_{X,P}$. One thus finds a closed integral curve $C\subset X$ containing P as a smooth closed point, which is transversal to Z at P. Shrinking X even more, we may assume that C is smooth and $Z\cap C=P$. Let $\alpha_C\in \mathrm{Br}(C\smallsetminus P)$ be the restriction of $\alpha\in\mathrm{Br}(X\smallsetminus Z)$. Since C and Z are transversal at P, by Theorem 3.7.4 the residue of α_C at P is

$$\partial_P(\alpha_C)=\rho(P)\in \mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z}),$$

thus $\partial_P(\alpha_C)\neq 0$. The embedding $C\subset X$ extends to an embedding of integral models over a suitable open subset of $\mathrm{Spec}(\mathcal{O}_S)$. Therefore, it is enough to prove the statement of the theorem for the smooth connected curve C.

So let C be a connected integral curve over k with a closed point P. Write $U=C\smallsetminus\{P\}$. Let $\alpha\in\mathrm{Br}(U)$ be an element with a non-zero residue

$$\chi=\partial_P(\alpha)\in \mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z})$$

of order n. Thus $\chi\in \mathrm{H}^1(k(P),\mathbb{Z}/n)\subset \mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z})$. Replacing C by an open set, we may assume that C is affine, $C=\mathrm{Spec}(A)$, and P is defined be the vanishing of some $f\in A$. Let A^{h} be the henselisation of A at P. The natural restriction map $\mathrm{H}^1(A^{\mathrm{h}},\mathbb{Z}/n)\to \mathrm{H}^1(k(P),\mathbb{Z}/n)$ is an isomorphism. Thus there exists a connected affine curve $D=\mathrm{Spec}(B)$ over k and an étale morphism $q:D\to C$ such that q induces an isomorphism $Q=q^{-1}(P)\xrightarrow{\sim}P$ and, moreover, χ is the restriction of some $\xi\in \mathrm{H}^1(D,\mathbb{Z}/n)$.

Let $V=D\smallsetminus Q$. Consider the cup-product $(f,\xi)\in\mathrm{Br}(V)$ of the class of f in $k[V]^*/k[V]^{*n}\subset \mathrm{H}^1(V,\mu_n)$ with $\xi\in \mathrm{H}^1(D,\mathbb{Z}/n)$. The difference $\beta=\alpha_D-(f,\xi)$ is an element of $\mathrm{Br}(V)$ with trivial residue at Q, hence $\beta\in\mathrm{Br}(D)$.

Replacing S by a larger finite set of places we can assume the existence of affine curves $\mathcal{C}$ and $\mathcal{D}$, each of them flat and of finite type over $\mathrm{Spec}(\mathcal{O})$, such that $D\to C$ extends to an $\mathcal{O}_S$-morphism $\mathcal{D}\to\mathcal{C}$. Let $\mathcal{P}$ be the Zariski closure of P in $\mathcal{D}$. By increasing S further we can ensure the following properties:

$f\in A$ comes from an element $f\in\mathcal{O}_S[\mathcal{C}]$;
$\xi\in \mathrm{H}^1(D,\mathbb{Z}/n)$ is the restriction of an element $\xi\in \mathrm{H}^1(\mathcal{D},\mathbb{Z}/n)$;
$\beta\in\mathrm{Br}(D)$ is the restriction of an element $\beta\in\mathrm{Br}(\mathcal{D})$;
the natural morphism $\mathcal{P}\to\mathrm{Spec}(\mathcal{O}_S)$ is finite and étale;
$\mathcal{P}$ is integral and maps isomorphically onto its image in $\mathcal{C}$.

By a version of Chebotarev's theorem (Theorem 12.1.6), there exist infinitely many places v of k for which there is a place w of $k(P)$ over v with $k_v\cong k(P)_w$ (i.e., w has degree 1 over v) and w is inert in the cyclic extension $k(P)(\chi)/k(P)$

defined by $\chi \in \mathrm{H}^1(k(P),\mathbb{Z}/n)$. For such a place v there exists a point $N_v^0 \in \mathcal{P}(\mathcal{O}_v)$ which maps isomorphically to a point $M_v^0 \in \mathcal{C}(\mathcal{O}_v)$.

Let $N_v \in \mathcal{D}(\mathcal{O}_v)$ be such that $f(N_v) \neq 0$ and let $M_v \in \mathcal{C}(\mathcal{O}_v)$ be the image of N_v. Then one has

$$\alpha(M_v) = \alpha(N_v) = \beta(N_v) + (f(N_v), \xi(N_v)) \in \mathrm{Br}(k_v) \tilde{\longrightarrow} \mathbb{Q}/\mathbb{Z}.$$

We have $\beta(N_v) \in \mathrm{Br}(\mathcal{O}_v) = 0$. The place w of $k(P)$ is inert in the cyclic extension $k(P)(\chi)$, so if N_v is close enough to N_v^0 for the v-adic topology on $\mathcal{D}(\mathcal{O}_v)$, the class $\xi(N_v) \in \mathrm{H}^1(k(P)_w, \mathbb{Z}/n) = \mathrm{H}^1(k_v, \mathbb{Z}/n)$ has order n. From the standard formula for the tame symbol we then get that $\alpha(M_v) \in \mathbb{Z}/n \subset \mathbb{Q}/\mathbb{Z}$ is equal to the class of the valuation $v(f(N_v))$ modulo n. The closed set $\mathcal{P} \times_{\mathcal{O}_S} \mathcal{O}_v \subset \mathcal{D} \times_{\mathcal{O}_S} \mathcal{O}_v$ contains the $\mathcal{O}_v$-section of $\mathcal{D} \times_{\mathcal{O}_S} \mathcal{O}_v \to \mathrm{Spec}(\mathcal{O}_v)$ defined by N_v^0, and is finite and étale over $\mathcal{O}_v$. Thus there exists $N_v \in \mathcal{D}(\mathcal{O}_v)$ arbitrarily close to N_v^0 such that $v(f(N_v)) \equiv 1 \mod n$, hence its image $M_v \in \mathcal{C}(\mathcal{O}_v) \subset C(k_v)$ satisfies $\alpha(M_v) \neq 0$. □

Remark 12.6.2 (i) Here is the simplest case at which the reader might want to look before reading the proof above. Let $X = \mathbb{A}^1_k = \mathrm{Spec}(k[t])$ and let $U = \mathbb{G}_{m,k}$ be given by $t \neq 0$. Let $a \in k^* \setminus k^{*2}$. Consider $\alpha = (a,t) \in \mathrm{Br}(U)$. There exist infinitely many places v for which there exists $t_v \in k_v^*$ with $(a, t_v) \neq 0 \in \mathrm{Br}(k_v)$.

(ii) In this example there also exist infinitely many places v such that α identically vanishes on $U(k_v)$. The analogous property holds more generally for any smooth connected curve X, but does not extend to higher dimension. Suppose X is a smooth and geometrically integral variety over k of dimension at least 2, $U \subset X$ is an open set, $\alpha \in \mathrm{Br}(U)$ and there exists a codimension 1 subvariety $Z \subset X$ such that $\partial(\alpha) \in \mathrm{H}^1(k(Z), \mathbb{Q}/\mathbb{Z})$ defines a cyclic extension $L/k(Z)$ with the property that k is algebraically closed in L. Then for almost all places v of k, the class α takes at least one non-zero value on $U(k_v)$.

Starting from Theorem 12.6.1, a combinatorial argument leads to the following extremely useful result. This version of D. Harari's "formal lemma" [Har94, Cor. 2.6.1, p. 233] was stated in [CTS00].

Theorem 12.6.3 (Harari) *Let X be a smooth, geometrically integral variety over a number field k. Let $U \subset X$ be a non-empty open set and let $B \subset \mathrm{Br}(U)$ be a finite subgroup. Let $(P_v) \in U(\mathbf{A}_k)^{B \cap \mathrm{Br}(X)}$. For any finite set S of places of k there exists an adelic point $(M_v) \in U(\mathbf{A}_k)$, where $M_v = P_v$ for $v \in S$, such that for any $\beta \in B$ we have*

$$\sum_{v \in \Omega} \mathrm{inv}_v\, \beta(M_v) = 0.$$

Proof. Replacing S by a bigger finite set of places we can find $\mathcal{O}_S$-schemes $\mathcal{X}$ and $\mathcal{U}$ of finite type, together with an open immersion of $\mathcal{O}_S$-schemes $\mathcal{U} \to \mathcal{X}$ which gives us back the open immersion $U \to X$ after restricting to the generic point $\mathrm{Spec}(k)$ of $\mathrm{Spec}(\mathcal{O}_S)$. In doing so we can ensure that $P_v \in \mathcal{U}(\mathcal{O}_v)$ for $v \notin S$.

Since B is finite, by increasing S further, we may assume that $B \subset \mathrm{Br}(\mathcal{U})$ and $B \cap \mathrm{Br}(X) \subset \mathrm{Br}(\mathcal{X})$. Since $\mathrm{Br}(\mathcal{O}_v) = 0$, this implies that $\beta(P_v) = 0$ for any $\beta \in B$ and any $v \notin S$. Likewise, $\beta(M_v) = 0$ for any $\beta \in B \cap \mathrm{Br}(X)$ and *any* point $M_v \in \mathcal{X}(\mathcal{O}_v)$, where $v \notin S$.

Let $\alpha \in B$, $\alpha \notin \mathrm{Br}(X)$. According to Theorem 12.6.1, there exist an infinite set T_α of places of k disjoint from S and a point $N_v \in U(k_v) \cap \mathcal{X}(\mathcal{O}_v)$ for each $v \in T_\alpha$ such that $\alpha(N_v) \neq 0$. The elements of $B \cap \mathrm{Br}(X)$ take the zero value on the points N_v, thus the evaluation of the elements of B at N_v defines a homomorphism

$$\varphi_{\alpha,v} : B/(B \cap \mathrm{Br}(X)) \longrightarrow \mathrm{Br}(k_v) \tilde{\longrightarrow} \mathbb{Q}/\mathbb{Z}$$

such that $\varphi_{\alpha,v}(\alpha) \neq 0$. Since $B/(B \cap \mathrm{Br}(X))$ is a finite group, the group $\mathrm{Hom}(B/(B \cap \mathrm{Br}(X)), \mathbb{Q}/\mathbb{Z})$ is finite too. There thus exists an infinite subset of T_α such that the attached homomorphisms $\varphi_{\alpha,v}$ are all equal. Replacing T_α by this subset, we may thus assume that there exists a homomorphism

$$\varphi_\alpha : B/(B \cap \mathrm{Br}(X)) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

with the property $\varphi_\alpha(\alpha) \neq 0$, such that for any $\beta \in B/(B \cap \mathrm{Br}(X))$ and any $v \in T_\alpha$ we have

$$\varphi_\alpha(\beta) = \beta(N_v) \in \mathbb{Q}/\mathbb{Z}. \tag{12.6}$$

Let C be the subgroup of $\mathrm{Hom}(B/(B \cap \mathrm{Br}(X)), \mathbb{Q}/\mathbb{Z})$ generated by the φ_α for $\alpha \in B$. Consider the natural bilinear pairing

$$B/(B \cap \mathrm{Br}(X)) \times C \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Since $\varphi_\alpha(\alpha) \neq 0$, the left kernel of this pairing is zero. We thus obtain an injective map of $B/(B \cap \mathrm{Br}(X))$ into $\mathrm{Hom}(C, \mathbb{Q}/\mathbb{Z})$. Comparing the orders of these finite groups we conclude that $C = \mathrm{Hom}(B/(B \cap \mathrm{Br}(X)), \mathbb{Q}/\mathbb{Z})$.

The assumption in the theorem ensures that the linear map $B \to \mathbb{Q}/\mathbb{Z}$ which sends β to $-\sum_{v \in S} \beta(P_v)$ descends to a linear map $B/(B \cap \mathrm{Br}(X)) \to \mathbb{Q}/\mathbb{Z}$. We have just seen that this map can be written as a sum of maps φ_α (possibly with repetitions). By (12.6) each of the φ_α involved in this sum can be written as $\beta \mapsto \beta(N_v)$, this time without repeating v – since for each α we have an infinite set of places T_α at our disposal. We have thus found a finite set T of places $v \notin S$ and points $N_v \in U(k_v) \cap \mathcal{X}(\mathcal{O}_v)$ for $v \in T$ such that

$$\sum_{v \in S} \mathrm{inv}_v \beta(P_v) + \sum_{v \in T} \mathrm{inv}_v \beta(N_v) = 0$$

for each $\beta \in B/(B \cap \mathrm{Br}(X))$. This implies

$$\sum_{v \in S} \mathrm{inv}_v \beta(P_v) + \sum_{v \in T} \mathrm{inv}_v \beta(N_v) = 0$$

for each $\beta \in B$. We then have

$$\sum_{v \in S} \mathrm{inv}_v \beta(P_v) + \sum_{v \in T} \mathrm{inv}_v \beta(N_v) + \sum_{v \notin S \cup T} \mathrm{inv}_v \beta(P_v) = 0$$

for each $\beta \in B$. This completes the proof once we take $S_1 = S \cup T$, $M_v = N_v$ for $v \in T$ and $M_v = P_v$ for $v \notin S \cup T$. □

Remark 12.6.4 Let $X \subset X_c$ be a smooth compactification. Theorem 12.6.3 for $U \subset X_c$ implies the same theorem for $U \subset X$ since $\mathrm{Br}(X_c) \subset \mathrm{Br}(X)$. Since $\mathrm{Br}(X_c) = \mathrm{Br}_{\mathrm{nr}}(k(U)/k)$, in the condition of Theorem 12.6.3 we can replace $B \cap \mathrm{Br}(X)$ by the smaller subgroup $B \cap \mathrm{Br}_{\mathrm{nr}}(k(U)/k)$, with the same conclusion.

Formal lemma for torsors under a torus

The following statement and its proof are taken from [BMS14, Prop. 3.1].

Theorem 12.6.5 *Let U be a smooth and geometrically integral variety over a number field k. Let T be a k-torus. Let $Y \to U$ be a torsor over U under T, and let $\theta \in \mathrm{H}^1(U,T)$ be its class. Let $B \subset \mathrm{Br}(U)$ be the finite subgroup consisting of cup-products $\theta \cup \gamma$, where γ is an element of the finite group $\mathrm{H}^1(k,\hat{T})$. Let $(M_v) \in U(\mathbf{A}_k)$ be a point orthogonal to $B \cap \mathrm{Br}_{\mathrm{nr}}(k(U)/k)$. Let $S \subset \Omega$ be a finite set of places. Then there exists an $\alpha \in \mathrm{H}^1(k,T)$ such that the twisted torsor Y^α has points in all completions of k and such that for each $v \in S$, the point M_v lies in the image of $Y^\alpha(k_v) \to U(k_v)$.*

Proof. By Theorem 12.6.3 and Remark 12.6.4 there exists an adelic point $(P_v) \in U(\mathbf{A}_k)$ with $M_v = P_v$ for $v \in S$ such that

$$\sum_{v \in \Omega} \mathrm{inv}_v \, [\theta(P_v) \cup \gamma] = 0 \in \mathbb{Q}/\mathbb{Z} \quad \text{for all} \quad \gamma \in \mathrm{H}^1(k,\hat{T}).$$

Thus $(\theta(P_v)) \in \oplus_{v \in \Omega} \mathrm{H}^1(k_v,T)$ is orthogonal to $\mathrm{H}^1(k,\hat{T})$. From the exact sequence (12.4) we obtain that $(\theta(P_v))$ is the image of an element $\alpha \in \mathrm{H}^1(k,T)$ under the diagonal map $\mathrm{H}^1(k,T) \to \oplus_{v \in \Omega} \mathrm{H}^1(k_v,T)$. Twisting Y by α gives a torsor over Y under T which has a k_v-point over P_v for each $v \in \Omega$. □

Remark 12.6.6 In the proof of [CTHS03, Thm. 3.1] there is a similar argument with a stronger hypothesis and a stronger conclusion. There we have an additional condition $\bar{k}[U]^* = \bar{k}^*$. Starting from an element of $X(\mathbf{A}_k)^{\mathrm{Br}}$, in the situation described in *loc. cit.* one produces an adelic point on a suitable twist Y^α with the additional property that this adelic point is orthogonal to (a suitable subgroup) of the unramified Brauer group of Y^α.

Chapter 13

Are rational points dense in the Brauer–Manin set?

Let X be a smooth, projective and geometrically integral variety over a number field k. When $X(k)$ is dense in $X(\mathbf{A}_k)$, weak approximation holds for X. In the previous chapter we have seen that this is impossible if the Brauer–Manin set $X(\mathbf{A}_k)^{\mathrm{Br}}$ is smaller than $X(\mathbf{A}_k)$. Thus a natural question is this: is $X(k)$ a dense subset of $X(\mathbf{A}_k)^{\mathrm{Br}}$? Write $X(k)^{cl}$ for the closure of $X(k)$ in $X(\mathbf{A}_k)$. If $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$ we shall say that *weak approximation holds for the Brauer–Manin set of* X. Roughly speaking, one asks whether the Brauer–Manin obstruction is the only obstruction to weak approximation – and, in particular, to the Hasse principle – in the sense that weak approximation holds for those adelic points which are not obstructed by the Brauer group. In particular, one would like to produce geometric classes of varieties such that weak approximation holds for their Brauer–Manin sets. One would also like to construct examples when this is not so.

In Section 13.1 we discuss Colliot-Thélène's conjecture that weak approximation holds for the Brauer–Manin set of rationally connected varieties. In Section 13.2 we look at Schinzel's Hypothesis (H), its consequences for rational points, and results of Green, Tao and Ziegler from additive combinatorics that in some cases can be used instead of Schinzel's Hypothesis. We state a conjecture of Harpaz and Wittenberg which allows one to establish Colliot-Thélène's conjecture for the total space of a fibration over $\mathbb{P}^1_k$ if this conjecture holds for the smooth k-fibres. We explain the idea in an important particular case. In Section 13.3 we give an overview of the theory of obstructions to the local-to-global principle, in other words, various canonically defined subsets of $X(\mathbf{A}_k)$ that contain $X(k)$. We discuss relations between them and give several examples to demonstrate insufficiency of these obstructions.

13.1 Rationally connected varieties: a conjecture

Rationally connected varieties

Let k be a field *of characteristic* 0. As usual, we denote an algebraic closure of k by $\bar{k}$. Unless otherwise mentioned, a *geometrically rational* variety X over k is a smooth, projective, geometrically integral variety such that $\overline{X} = X \times_k \bar{k}$ is birationally equivalent to the projective space of the same dimension.

Concrete examples of such varieties whose arithmetic is already difficult to understand are smooth projective models of affine hypersurfaces in $\mathbb{A}_k^{d+1}$ with equation

$$\mathrm{N}_{K/k}(\Xi) = P(t),$$

where K/k is a finite extension of number fields of degree $[K : k] = d$, the variable Ξ takes values in K (hence represents d variables with values in k) and $P(t) \in k[t]$ is a separable polynomial of degree at least 2.

Definition 13.1.1 *A* **rationally connected** *variety over a field k is a smooth, projective and geometrically integral variety X such that over any algebraically closed field K containing k, any two K-points of X are connected by a rational curve, i.e. lie in the image of a morphism $\mathbb{P}^1_K \to X_K$.*

Rationally connected varieties have been studied by Kollár, Miyaoka and Mori, and by Campana. They can be characterised by many equivalent properties. In particular, in the above definition one may simply assume that any two points are connected by a chain of rational curves, or even that two 'general' points are connected by such a chain. A standard reference is Kollár's book [Kol99] to which we refer for these equivalences and for the following properties.

(1) A rationally connected variety of dimension 1 is a smooth conic.

(2) A rationally connected variety of dimension 2 is a geometrically rational surface.

(3) Any geometrically unirational variety is rationally connected. (The converse is an open question.)

(4) By a theorem of Campana and Kollár–Miyaoka–Mori, any Fano variety (that is, a smooth projective variety with ample anticanonical bundle) is rationally connected. In particular, smooth hypersurfaces in $\mathbb{P}^n$ of degree $d \leq n$ are rationally connected.

(5) If $f : X \to C$ is a dominant morphism of smooth, projective, geometrically integral varieties such that C is a curve and the generic geometric fibre of f is rationally connected, then f has a section over $\bar{k}$. This is a deep theorem, proved by Graber, Harris and Starr [GHS03] in characteristic 0, and by de Jong and Starr in arbitrary characteristic [deJS03]. It implies that every closed fibre of f has an irreducible component of multiplicity 1. As a consequence of the existence of sections over rational curves one obtains that if $X \to Y$ is a dominant morphism of smooth, projective, geometrically integral varieties such that Y

and the generic geometric fibre are rationally connected, then X is rationally connected.

(6) If X is a rationally connected variety over k, then $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$ and $\mathrm{Pic}(\overline{X})$ is a free abelian group of finite type [Deb01, Cor. 4.18]. In particular, $\mathrm{Br}(\overline{X})$ is finite and $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite (Theorem 4.4.2).

(7) By a theorem of Enriques, Manin, Iskovskikh, and Mori, any geometrically rational surface is birationally equivalent to a surface of at least one of the following families:

(i) A smooth del Pezzo surface of degree d, where $1 \leq d \leq 9$.

(ii) A conic bundle over a conic (possibly with degenerate fibres).

A *del Pezzo surface* is a smooth, projective, geometrically integral surface X such that the anticanonical bundle ω_X^{-1} is ample. The integer $d = (\omega_X.\omega_X)$ is called the degree of X; it satisfies $1 \leq d \leq 9$. Del Pezzo surfaces of degree 4 are smooth complete intersections of two quadrics in $\mathbb{P}^4$, and del Pezzo surfaces of degree 3 are smooth cubic surfaces in $\mathbb{P}^3$, see [Man74] and [Kol99].

Geometrically rational surfaces X with $d = (\omega_X.\omega_X) \geq 5$ are arithmetically simple: they are rational over k when $X(k) \neq \emptyset$. If k is a number field, the property $X(k)^{cl} = X(\mathbf{A}_k)$ holds for such surfaces. In particular, they satisfy the Hasse principle.

Colliot-Thélène's conjecture

In the case of surfaces, the following conjecture was put forward as an open question by Colliot-Thélène and Sansuc in 1979, see [CTS80]. The general question was raised in Colliot-Thélène's lectures at the Institut Henri Poincaré in 1999 and mentioned again in [CT03].

Conjecture 13.1.2 *If X is a rationally connected variety over a number field k, then $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$.*

In other words, the Brauer–Manin set of a rationally connected variety is conjectured to satisfy weak approximation.

This conjecture is birationally invariant.

Since $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite when X is rationally connected, the closed set $X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)$ is open. In particular, if the conjecture holds, and $X(k) \neq \emptyset$, then *weak weak approximation* (Definition 12.2.4) holds for X.

A partial converse is due to Harari: if a smooth, projective and geometrically integral variety X over a number field k satisfies weak weak approximation over any finite extension of k, then the geometric fundamental group of X is trivial, see [Har00, Cor. 2.4] and the remark after it. This implies $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

Here are some of the consequences of conjectural weak weak approximation for rationally connected varieties. In particular, these are consequences of Conjecture 13.1.2.

(1) For any rationally connected variety X over a number field k with a k-point the set $X(k)$ is Zariski dense in X. Already in dimension 2, i.e. for geometrically rational surfaces, this is not known.

(2) Any finite group G is the Galois group of a Galois field extension of k (see [Eke90]). The case $k = \mathbb{Q}$ is the inverse Galois problem, a well known old open problem.

(3) Let X be a geometrically rational surface over a number field k such that the Brauer–Manin obstruction is the only obstruction to the existence of a rational point on X over any finite extension of k. Then X contains a point defined over some abelian extension of k [Kan87, Thm. 3, Remark].

There is theoretical evidence for Conjecture 13.1.2 for geometrically rational conic bundle surfaces. Indeed, in this case it follows from Schinzel's Hypothesis (H), see Section 13.2.1.

For conic bundles over the projective line with $r \leq 5$ geometric degenerate fibres, Conjecture 13.1.2 is known. The case $r \leq 3$ is easy: in this case the Hasse principle and weak approximation hold. For Châtelet surfaces, a particular kind of conic bundles with $r = 4$, the conjecture was proved by Colliot-Thélène, Sansuc and Swinnerton-Dyer [CTSS87]. The general case with $r = 4$ is due to Salberger (unpublished) and to Colliot-Thélène [CT90]. The case $r = 5$ is due to Salberger and Skorobogatov [SSk91]. Swinnerton-Dyer also discusses this case as well as some specific cases with $r = 6$. Short proofs of these results can be found in [Sko01, Ch. 7].

For del Pezzo surfaces of degree 4 with a k-point Conjecture 13.1.2 is known (Salberger and Skorobogatov [SSk91]). This is one case where theorems about zero-cycles ultimately lead to results on rational points. For general del Pezzo surfaces of degree 4, Wittenberg in his thesis [Witt07] develops a method of Swinnerton-Dyer [SwD95, CTSS98b] to produce strong evidence – conditional on Schinzel's Hypothesis (H) and the finiteness of Tate–Shafarevich groups of elliptic curves.

In higher dimension, the case of intersections of two quadrics has been much discussed (Mordell; Swinnerton-Dyer; Colliot-Thélène, Sansuc and Swinnerton-Dyer [CTSS87]; Heath-Brown [HB18]). Let us quote the results for arbitrary smooth complete intersections of two quadrics in $\mathbb{P}^n_k$. For $n \geq 5$, if there is a k-point, then weak approximation holds. For $n \geq 7$, the Hasse principle is known. For $n \geq 5$, this is also conjectured to hold, and is proved conditionally on Schinzel's Hypothesis (H) and the finiteness of Tate–Shafarevich groups of elliptic curves in [Witt07].

For diagonal cubic surfaces X over $\mathbb{Q}$, there is numerical evidence [CTKS87] that $X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}} \neq \emptyset$ implies $X(\mathbb{Q}) \neq \emptyset$. For diagonal cubic hypersurfaces of dimension at least 3 over $\mathbb{Q}$, Swinnerton-Dyer [SwD01] proves the Hasse principle conditionally on the finiteness of Tate–Shafarevich groups of elliptic curves over number fields.

When the number of variables is large with respect to the degree, the circle method can be applied. This method also gives good results in relatively low dimension for cubic hypersurfaces: smooth cubic hypersurfaces in $\mathbb{P}^n_\mathbb{Q}$ have rational points when $n \geq 9$ (Heath-Brown) and satisfy the Hasse principle for $n = 8$ (Hooley).

If X is birationally equivalent to a homogeneous space of a connected linear algebraic group with connected geometric stabilisers, then $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$. The case when the stabilisers are trivial is a theorem of Sansuc [San81, Cor. 8.7]. Using [San81, Lemme 6.1] (a special case of Harari's formal lemma) as in the proof of [San81, Cor. 8.7], one immediately deduces the case of connected stabilizers from a theorem of Borovoi [Bor96, Cor. 2.5]. For such X, one has the refined statement that $X(k)$ is not empty as soon as $X(\mathbb{A}_k)^{\text{Б}(X)} \neq \emptyset$, with $\text{Б}(X) \subset \mathrm{Br}(X)$ as defined after Remark 12.3.3.

13.2 Schinzel's Hypothesis (H) and additive number theory

13.2.1 Applications of Schinzel's hypothesis

Let us recall the statement of Schinzel's Hypothesis (H) (1958) which is an elaboration of qualitative conjectures of Bouniakowsky (1857) and Dickson (1904), and of quantitative conjectures of Hardy and Littlewood (1922) generalised by Bateman and Horn (1962).

Conjecture 13.2.1 Schinzel's Hypothesis (H) *Let $P_i(x) \in \mathbb{Z}[x]$, for $i = 1,\ldots,r$, be irreducible polynomials with positive leading coefficients. Assume that no prime divides all the numbers $\prod_{i=1}^n P_i(m)$, where $m \in \mathbb{Z}$. Then there exist infinitely many positive integers n such that each $P_i(n)$ is a prime number, for $i = 1,\ldots,r$.*

Note that only primes p with $p \leq \sum_i \deg(P_i)$ could divide all the numbers $\prod_{i=1}^n P_i(m)$. The only known case of this conjecture is the case of one polynomial of degree one: this is Dirichlet's theorem on primes in an arithmetic progression. That theorem was used by Hasse (1924) to prove the Hasse principle for zeros of quadratic forms in 4 variables once the case of 3 variables is known. In 1979, it was noticed [CTS82] that Hypothesis (H) can be used to give conditional proofs of the Hasse principle for other diophantine equations. Here is one of the simplest cases, taken directly from [CTS82, §5].

Theorem 13.2.2 (Colliot-Thélène–Sansuc) *Let $P(x) \in \mathbb{Q}[x]$ be an irreducible polynomial, and let $a \in \mathbb{Q}^*$. Assume Schinzel's Hypothesis (H). Then the Hasse principle and weak approximation hold for any smooth model of the affine variety*

$$y^2 - az^2 = P(x) \neq 0.$$

Proof. Let us denote this affine variety by U. It is enough to prove the theorem for U. We shall here make two simplifying hypotheses. We shall assume $a > 0$ and shall prove weak approximation only at the finite places. We refer the reader to [CTS82, §5] for the technical arguments required to handle the real place. (Such extra efforts are often needed when handling the archimedean places.)

Assume that we are given points

$$(y_p, z_p, x_p) \in U(\mathbb{Q}_p)$$

for all primes p. Let S be a finite set of primes containing $p = 2$, the primes p such that $v_p(a) \neq 0$, the primes p such that $P(x) \notin \mathbb{Z}_p[x]$, the primes for which the reduction of $P(x)$ modulo p has degree less than $\deg P(x)$ or is not separable, and the primes $p \leq \deg(P)$.

Using the irreducibility of $P(x)$, Hensel's lemma and Schinzel's Hypothesis (H), one finds $\lambda \in \mathbb{Q}$ very close to each $x_p \in \mathbb{Q}_p$ for $p \in S$ and such that

$$P(\lambda) = q \prod_{p \in S} p^{n_p} \in \mathbb{Q},$$

where $n_p \in \mathbb{Z}$ and q is a prime not in S ("the Schinzel prime"). A flexible version of this part of the argument was stated by Serre over any number field, see [CTS94, Prop. 4.1].

Then the rational number $P(\lambda) \neq 0$ is represented by the quadratic form $y^2 - az^2$ in each completion of $\mathbb{Q}$ (including the reals, since we assumed $a > 0$), except possibly in $\mathbb{Q}_q$. By Corollary 12.1.10, $P(\lambda)$ is represented by this form over $\mathbb{Q}_q$ and over $\mathbb{Q}$. Using weak approximation on the affine conic $y^2 - az^2 = P(\lambda)$ and the implicit function theorem (Theorem 9.5.1), one concludes that weak approximation away from the reals holds for U. □

In the above theorem, the hypothesis that the polynomial $P(t)$ is irreducible implies that for any smooth projective model X of U, we have $\mathrm{Br}(X)/\mathrm{Br}_0(X) = 0$ (see Proposition 10.2.3, ensuing exercises, and Remark 10.2.8). If one allows the separable polynomial $P(x)$ to be reducible, then one may have $\mathrm{Br}(X)/\mathrm{Br}_0(X) \neq 0$ and one may produce counter-examples to the Hasse principle, e.g. Iskovskikh counter-example (§12.5). Using a descent argument [CTS82] or Harari's formal lemma for elements of the Brauer group (Theorem 12.6.3) or for torsors under suitable tori (Theorem 12.6.5) on the open variety U, one may more generally prove that under Schinzel's hypothesis, $X(\mathbb{Q})$ is dense in $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$.

To prove general theorems along these lines, it is convenient to use the following Hypothesis (H_1). As noted by Serre, this general statement is a consequence of Hypothesis (H). The proof of this implication is given in [CTS94, Prop. 4.1].

Conjecture 13.2.3 Hypothesis (H_1) *Let k be a number field and let $P_i(t)$, for $i = 1, \cdots, n$, be irreducible polynomials in $k[t]$. Let S be a finite set of places of k containing all infinite places, all finite places v where the coefficients of some $P_i(t)$ are either all contained in the maximal ideal of $\mathcal{O}_v$ or one of the coefficients is not in $\mathcal{O}_v$, and all finite places above a prime p less than or equal to the degree of the polynomial $N_{k/\mathbf{Q}}(\prod_{i=1}^n P_i(t))$. Given $\lambda_v \in k_v$ for $v \in S$, one can find $\lambda \in k$, integral away from S, arbitrarily close to each λ_v in the v-adic topology for finite $v \in S$, arbitrarily big in the archimedean completions k_v, and such that for each $i = 1, \cdots, n$, $P_i(\lambda) \in k$ is a unit in k_w for all places $w \notin S$ except perhaps one place w_i, where it is a uniformising parameter.*

Using the formal lemma, one proves the following general result.

Theorem 13.2.4 *Let X be a smooth, projective and geometrically integral variety over a number field k and let $X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is geometrically integral and that each closed fibre $X_m/k(m)$ contains a component of multiplicity one Y such that the integral closure of $k(m)$ in the function field of Y is an* **abelian extension**. *Assuming Schinzel's Hypothesis (H), if $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k)} \neq \emptyset$, then there exists $c \in \mathbb{P}^1(k)$ such that X_c is smooth and $X_c(\mathbf{A}_k) \neq \emptyset$. Moreover, given a finite set S of places of k, and a point $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, one can find a point $c \in \mathbb{P}^1(k)$ such that X_c contain a k_v-point close to M_v for each $v \in S$.*

Proof. This is [CTSS98, Thm. 1.1]. □

Here the same reciprocity argument as in Hasse's proof is used: the abelian extensions mentioned in the theorem give rise to a cyclic extension L/K of number fields together with an element in K^* which is a local norm at all places of K except possibly one; then one concludes that the element is a global norm.

The following special case was proved in [CTS94].

Theorem 13.2.5 *Let X be a smooth, projective and geometrically integral variety over a number field k and let $X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is a smooth quadric of dimension 1 or 2. Then $\mathrm{Br}(X) = \mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k)$. Assuming Schinzel's hypothesis H, we have $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$.*

In relative dimension at least 3, that theorem holds unconditionally, and is easy to prove.

13.2.2 Enters additive combinatorics

Over the rationals, a breakthrough happened in 2010. Work of B. Green and T. Tao, followed by further work with T. Ziegler (2012), proves something which is essentially a two variable version of Schinzel's Hypothesis (H), when restricted to a system of polynomials with integral coefficients each of total degree one. The initial results of Green and Tao, together with further work by L. Matthiesen on additive combinatorics, first led to unconditional results in the spirit of "Schinzel implies Hasse". This is the work of Browning, Matthiesen and Skorobogatov [BMS14]. A typical result is the unconditional proof of weak approximation for the Brauer–Manin set of a conic bundle over $\mathbb{P}^1_{\mathbb{Q}}$ when all the singular fibres are above $\mathbb{Q}$-rational points of $\mathbb{P}^1_{\mathbb{Q}}$. They also prove a similar result for the total space of quadric bundles of relative dimension 2 over $\mathbb{P}^1_{\mathbb{Q}}$. Until then, for most such $\mathbb{Q}$-varieties, we did not know that existence of one rational point implies that rational points are Zariski dense – unless one was willing to accept Schinzel's Hypothesis (H).

The work of Green, Tao and Ziegler led to further progress. Here is the exact result used, reproduced from [HSW14].

Theorem 13.2.6 (Green–Tao–Ziegler) *Let $L_1(x,y), \ldots, L_r(x,y) \in \mathbb{Z}[x,y]$ be pairwise non-proportional linear forms, and let $c_1, \ldots, c_r \in \mathbb{Z}$. Assume that* *for each prime p there exists $(m,n)\in\mathbb{Z}^2$ such that p does not divide $L_i(m,n)+c_i$ for any $i=1,\ldots,r$. Let $K\subset\mathbb{R}^2$ be an open convex cone containing a point $(m,n)\in\mathbb{Z}^2$ such that $L_i(m,n)>0$ for $i=1,\ldots,r$. Then there exist infinitely many pairs $(m,n)\in K\cap\mathbb{Z}^2$ such that each $L_i(m,n)+c_i$ is a prime.*

From this theorem, Harpaz, Skorobogatov and Wittenberg [HSW14] deduced a number of results on weak approximation for the Brauer–Manin set. Let us describe the argument in a simple case. For simplicity, we do not consider approximation at the real place. We start with an easy consequence of Theorem 13.2.6. Since we are ignoring the real place, in Theorem 13.2.6, one may ignore the cone K. This shortens the proof of the following lemma.

Lemma 13.2.7 [HSW14, Prop. 1.2] *Let e_i, for $i=1,\ldots,m$, be distinct integers. Let S be a finite set of primes containing all the primes which divide some e_i-e_j for $i\neq j$. For each $p\in S$, suppose we are given $(u_p,v_p)\in\mathbb{Q}_p^2$ with $u_p-e_iv_p\neq 0$ for each $i=1,\ldots,m$. Let $\varepsilon>0$. Then there exist a pair $(u_0,v_0)\in\mathbb{Q}^2$ which is ε-close to each (u_p,v_p) for the p-adic topology, and distinct primes p_i outside of S such that for each i,*

$$u_0-e_iv_0=p_iq_i\in\mathbb{Q}^*,$$

where $q_i\in\mathbb{Q}^*$ is a unit outside of $S\cup\{p_i\}$. □

Theorem 13.2.8 *Let $k=\mathbb{Q}$. Let U be the surface*

$$y^2-az^2=b\prod_{i=1}^{2n}(t-e_i)\neq 0,$$

where $a,b\in\mathbb{Q}^*$ and $e_1,\ldots,e_{2n}$ are distinct elements of $\mathbb{Q}$. Assume $a>0$. Let X be a smooth projective variety containing U as a dense open subset and let $(M_p)\in X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}}$. Then there are $\mathbb{Q}$-points of U arbitrarily close to (M_p) at the finite primes. In particular, $\mathbb{Q}$-points are Zariski dense in U and weak weak approximation holds.

Proof. A linear change of variables allows us to assume $e_i\in\mathbb{Z}$ for each i. The argument used to prove Theorem 13.2.4 would lead to use of Schinzel's Hypothesis (H) for the system of polynomials $t-e_i$, and that is a wide open conjecture.

We shall instead use a simple but slightly mysterious trick to replace the unique variable t by two variables (u,v). We set $t=u/v$. Consider the variety V given by

$$Y^2-aZ^2=b\prod_{i=1}^{2n}(u-e_iv)\neq 0,\quad v\neq 0.$$

The formulas $y=Y/v^n$, $z=Z/v^n$, $t=u/v$ give an isomorphism $V\cong U\times\mathbb{G}_m$, where the coordinate on $\mathbb{G}_m$ is v. Let $V\subset V_c$ be a smooth compactification. Then V_c is birationally equivalent to $X\times\mathbb{P}^1_\mathbb{Q}$. By Corollary 5.2.6 the Brauer group is a stable birational invariant, so $\mathrm{Br}(X)\cong\mathrm{Br}(V_c)$.

Since X is geometrically rational, $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite (Cor. 4.4.4). Since any element of $\mathrm{Br}(X)$ vanishes on almost all $X(\mathbb{Q}_p)$, this allows us to move (M_p) in an adelic neighbourhood while staying in $X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}}$. We may thus assume that each M_p is in U and we may find an adelic point $(N_p) \in V_c(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}}$ such that for each place p the point N_p is in $V(\mathbb{Q}_p)$ and projects to $M_p \in U(\mathbb{Q}_p)$.

We are given a finite set S of places and a neighbourhood of $M_p \in U(\mathbb{Q}_p)$ for each $p \in S$. We produce neighbourhoods of the points N_p which map into the given neighbourhoods of M_p.

One then uses Harari's formal lemma (Theorem 12.6.3) for the finite family of quaternion algebras $\alpha_i = (a, u - e_i v) \in \mathrm{Br}(V)$; these classes need not belong to $\mathrm{Br}(V_c)$. This produces a new element $(P_p) \in V(\mathbf{A}_\mathbb{Q})$ with $P_p = N_p$ for $p \in S$, with coordinates Y_p, Z_p, u_p, v_p and such that $\sum_v \mathrm{inv}_v(\alpha_i(P_p)) = 0$. Let $K = \mathbb{Q}(\sqrt{a})$. From the exact sequence of class field theory (12.3)

$$1 \longrightarrow \mathbb{Q}^*/\mathrm{N}(K^*) \longrightarrow \bigoplus_p \mathbb{Q}_p^*/\mathrm{N}(K_p)^* \longrightarrow \mathbb{Z}/2,$$

we conclude that for each i there exists $c_i \in k^*$ such that for each place p, the map $\mathrm{Br}(\mathbb{Q}) \to \mathrm{Br}(\mathbb{Q}_p)$ sends the quaternion class (a, c_i) to $(a, u_p - e_i v_p)$. We thus have elements $c_i \in \mathbb{Q}^*$ and an adelic point (R_p) on the variety given by the system

$$\begin{cases} Y^2 - aZ^2 &= b\prod_{i=1}^{2n}(u - e_i v) \neq 0, \\ y_i^2 - az_i^2 &= c_i(u - e_i v) \neq 0, \quad i = 1, \ldots, 2n, \end{cases}$$

such that R_p projects to P_p. The variety given by this system is isomorphic to the product of the conic $Y^2 - aZ^2 = b\prod_{i=1}^{2n} c_i$ and the variety W given by

$$y_i^2 - az_i^2 = c_i(u - e_i v) \neq 0, \quad i = 1, \ldots, 2n.$$

The conic given by $Y^2 - aZ^2 = b\prod_{i=1}^{2n} c_i$ satisfies weak approximation. Now we apply Lemma 13.2.7 to a finite set S of primes containing the primes p dividing some $e_i - e_j$, $i \neq j$, the prime 2 and the primes p such that a or some c_i is not a unit in $\mathbb{Q}_p^*$. This produces a pair $(u_0, v_0) \in \mathbb{Q}^2$ close to each (u_p, v_p) at each place $p \in S$, such that each equation

$$y_i^2 - az_i^2 = c_i(u_0 - e_i v_0) \neq 0$$

has solutions in all completions of $\mathbb{Q}$, except possibly in $\mathbb{Q}_{p_i}$. Since each of these equations is the equation of a conic, it has a solution over $\mathbb{Q}$, and satisfies weak approximation. □

Theorem 13.2.8 is a special case of the following result from [HSW14].

Theorem 13.2.9 *Let X be a smooth, proper, geometrically integral variety over $\mathbb{Q}$ equipped with a dominant morphism $f : X \to \mathbb{P}^1_\mathbb{Q}$ such that*

(i) *the generic fibre of f is geometrically integral;*

(ii) *the only non-split fibres X_m are above $\mathbb{Q}$-rational points m of $\mathbb{P}^1_\mathbb{Q}$ and each such fibre contains a component Y of multiplicity one such that the integral closure of $\mathbb{Q}$ in $\mathbb{Q}(Y)$ is an abelian extension of $\mathbb{Q}$;*

(iii) *the Hasse principle and weak approximation hold for the smooth fibres. Then* $X(\mathbb{Q})^{cl} = X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$.

See Definition 9.1.3 for the definition of a split scheme over a field. The fibre X_m over the residue field $k(m)$ at a closed point m is split if and only if it contains a multiplicity one component which is geometrically integral.

Here are some concrete examples.

Corollary 13.2.10 *Let* $K_i/\mathbb{Q}$*, for* $i = 1, \ldots, r$*, be cyclic extensions. Let* $P_i(t)$*, for* $i = 1, \ldots, r$*, be non-zero polynomials that are products of linear factors over* $\mathbb{Q}$*. Let* X *be a smooth projective variety over* $\mathbb{Q}$ *that contains the variety given by the system of equations*

$$\mathrm{N}_{K_i/\mathbb{Q}}(\Xi_i) = P_i(t) \neq 0, \quad i = 1, \ldots, r,$$

as a dense open subset. Then $X(\mathbb{Q})^{cl} = X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$.

Corollary 13.2.11 *Let* $K_i/\mathbb{Q}$*, for* $i = 1, \ldots, r$*, be cyclic extensions. Let* $b_i \in \mathbb{Q}^*$ *and* $e_i \in \mathbb{Q}$*, for* $i = 1, \ldots, r$*. Then the variety over* $\mathbb{Q}$ *given by the system of equations*

$$\mathrm{N}_{K_i/\mathbb{Q}}(\Xi_i) = b_i(t - e_i) \neq 0, \quad i = 1, \ldots, r,$$

satisfies weak approximation.

To put this last result in perspective, here is what was known before 2010. Corollary 13.2.11 is obvious when $r = 1$. The case $r = 2$ and K_1 and K_2 both of degree 2 is easy, as it reduces to quadrics. An old result of Birch, Davenport and Lewis obtained by the circle method gives the statement for $r = 2$ and $K_1 = K_2$ of arbitrary degree over $\mathbb{Q}$. The case $r = 3$ and $K_1 = K_2 = K_3$ of degree 2 over $\mathbb{Q}$ was covered by Colliot-Thélène, Sansuc and Swinnerton-Dyer in [CTSS87]. Not much else was known.

The results above concern the total space of a 1-parameter family $X \to \mathbb{P}^1_k$ with the following properties:

(i) *The smooth fibres satisfy weak approximation.*

(ii) Each non-split fibre X_m over a closed point m contains a component of multiplicity one Y such that the algebraic closure of $k(m)$ in $k(Y)$ is *abelian.*

(iii) $k = \mathbb{Q}$ and the non-split fibres are over $\mathbb{Q}$-rational points (this last hypothesis is needed to use the results of Green, Tao and Ziegler).

It took time to get results where hypothesis (i) or (ii) could be relaxed. Unconditional results without the abelianity conditions were obtained under the stringent condition: all fibres of the morphism $X \to \mathbb{P}^1_k$ except one above a k-point contain a geometrically integral component of multiplicity one (Harari [Har94, Har97]).

A few, delicate results were obtained for birational models of varieties given by an equation

$$\mathrm{N}_{K/k}(\Xi) = P(t)$$

under no condition on the field extension K/k but under the condition that the polynomial $P(t)$ has at most two roots over $\overline{k}$ [HBS02, CTHS03, BH12, DSW12].

First interesting cases where the abelianity condition, was relaxed while allowing an arbitrary number of bad fibres were given by D. Wei [Wei12]. Under Schinzel's hypothesis H, he obtained results for varieties given by an equation

$$\mathrm{N}_{K/k}(\Xi) = P(t)$$

where K/k is an arbitrary field extension of degree 3.

In [HW15] Harpaz and Wittenberg propose a conjecture. Under this conjecture, they establish a result which requires neither condition (i) nor condition (ii). Work of Browning and Matthiesen has established this conjecture in situations analogous to those treated by Green, Tao and Ziegler.

13.2.3 Conjecture of Harpaz and Wittenberg

The following is [HW15, Conjecture 9.1].

Conjecture 13.2.12 (HW) *Let k be a number field. Let $n \geq 1$ be an integer and let $P_1(t), \ldots, P_n(t) \in k[t]$ be pairwise distinct irreducible monic polynomials. Write $k_i = k[t]/(P_i(t))$ and let $a_i \in k_i$ denote the class of t. Suppose that for each $i = 1, \ldots, n$ we are given a finite extension L_i of k_i and an element $b_i \in k_i^*$. Let S be a finite set of places of k containing the real places of k and the finite places above which, for some i, b_i is not a unit or L_i/k_i is ramified. Finally, for each $v \in S$, fix an element $t_v \in k_v$. Assume that for each $i = 1, \ldots, n$ and each $v \in S$, there exists $x_{i,v} \in (L_i \otimes_k k_v)^*$ such that*

$$t_v - a_i = b_i \mathrm{N}_{L_i \otimes_k k_v / k_i \otimes_k k_v}(x_{i,v}) \quad \in k_i \otimes_k k_v.$$

Then there exists $t_0 \in k$ satisfying the following conditions:

(1) t_0 is arbitrarily close to t_v for $v \in S$;

(2) for every $i = 1, ..., n$ and every finite place w of k_i with $w(t_0 - a_i) > 0$, either w lies above a place of S or the field L_i has a place of degree 1 over w.

It is convenient to introduce

$$\varepsilon = \sum_{i=1}^{n} [k_i : k].$$

For an arbitrary number field k, Conjecture HW is known in the following cases.

(i) $\varepsilon \leq 2$ (see [HW15, Thm. 9.11 (i)]). The essential ingredient is strong approximation. In the case $k_1 = k_2 = k$, one may also give a proof using Dirichlet's theorem in a suitable field extension of the ground field. If one wants to control the situation at the real places, this method requires the use of a theorem of Waldschmidt.

(ii) $\varepsilon = 3$ and $[L_i : k_i] = 2$ for each i (see [HW15, Thm. 9.11 (ii)]).

When $k = \mathbb{Q}$, Conjecture HW is also known in these cases:

(iii) Any $\varepsilon = n \geq 1$, $k_i = \mathbb{Q}$ for each $i = 1, \ldots, n$, and arbitrary number fields $L_1, \ldots, L_n$. This important case is due to L. Matthiesen [Mat18], who used the results by Green, Tao and Ziegler, as well as her joint work with T. Browning [BM17]. See [HW15, Theorem 9.14].

(iv) $\varepsilon = 3$, $n = 2$, $k_1 = \mathbb{Q}$ and $[k_2 : \mathbb{Q}] = 2$. This case was established by T. Browning and D. Schindler [BS] who used [Mat18] to strengthen the sieve method approach of T. Browning and R. Heath-Brown in [BH12].

(v) $\varepsilon = 3$, $n = 1$, $[k_1 : \mathbb{Q}] = 3$ and the extension L_1/k_1 is "almost abelian" [HW15, Definition 9.4], for example, L_1/k_1 is abelian or $[L_1 : k_1] = 3$. As noticed in [HW15, Remark 9.7], this follows from the work of Heath-Brown and Moroz on primes represented by cubic forms in two variables.

In [HW15, Prop. 9.9, Cor. 9.10], we find closely related conjectures which are possibly more appealing than Conjecture HW. The authors produce specific quasi-affine varieties W with the property that if strong approximation off any finite place v_0 holds for these varieties, then Conjecture HW holds. (Note that this hypothesis on strong approximation implies that these varieties satisfy the Hasse principle.) In the particular case where each $k_i = k$, each variety W is an open subset of a variety given by the system of equations

$$u - a_i v = b_i \mathrm{N}_{L_i/k}(\Xi_i), \quad i = 1 \ldots, r.$$

Here $a_i \in k$ and $b_i \in k^*$ are such that $a_i \neq a_j$ for $i \neq j$. The open set W is the complement to the union of the subset F_0 given by $u = v = 0$ and the subsets F_i, $i = 1, \ldots, n$, given by the condition that the projection to the coordinate Ξ_i belongs to the singular locus of $R_{L_i/k}(\mathbb{A}^1_{L_i}) \smallsetminus R_{L_i/k}(\mathbb{G}_{m,L_i})$ (which implies $u - a_i v = 0$).

Harpaz and Wittenberg prove the following main result [HW15, Thm. 9.17, Cor. 9.23].

Theorem 13.2.13 *Let X be a smooth, projective and geometrically integral variety over a number field k and let $X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is a rationally connected variety. Assuming Conjecture HW, if $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, then there exists $t_0 \in \mathbb{P}^1(k)$ with smooth fibre X_{t_0} such that $X_{t_0}(\mathbf{A}_k)^{\mathrm{Br}}$ is non-empty. Moreover, given a finite set S of places of k and a point $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$, one can choose t_0 such that X_{t_0} contains k_v-points close to M_v for $v \in S$.*

Using Borovoi's theorem [Bor96] quoted at the end of Section 13.1 we obtain

Corollary 13.2.14 *Let X be a smooth, projective and geometrically integral variety over a number field k and let $X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is birationally equivalent to a homogeneous space of a connected linear algebraic group over $k(\mathbb{P}^1)$ with connected geometric stabilizers. Assuming Conjecture HW, we have $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$.*

This result applies in particular to any smooth projective model X of a variety given by a system of equations

$$\mathrm{N}_{K_i/k}(\Xi_i) = P_i(t) \neq 0, \quad i = 1, \ldots, n.$$

Such systems have been considered in many special situations.

Remark 13.2.15 It is a non-trivial algebraic problem to decide when such a variety X satisfies $\mathrm{Br}(X) = \mathrm{Br}_0(X)$. For instance, if the polynomials $P_i(t)$ are all of degree 1 and no two of them are proportional, do we have $\mathrm{Br}(X) = \mathrm{Br}_0(X)$?

13.2.4 Main steps of the proof of Theorem 13.2.13

We denote the ring of integers of k_v by $\mathcal{O}_v$, the maximal ideal of $\mathcal{O}_v$ by $\mathfrak{m}_v$, and the residue field $\mathcal{O}_v/\mathfrak{m}_v$ by $\kappa(v)$.

As mentioned in [HW15, Remark 9.18 (i)], if one assumes Conjecture HW for arbitrary irreducible monic polynomials $P_1(t), \ldots, P_n(t)$, one can give a proof of Theorem 13.2.13 which is shorter than the proof in [HW15]. This is what we do here, under some simplifying assumptions. Unlike the proof in [HW15], the proof below uses Severi–Brauer schemes.

Two important steps are Theorems 13.2.16 and 13.2.20, leading to the proof of Theorem 13.2.13. The first of them addresses the following question: given a dominant morphism $X \to \mathbb{P}^1_k$ for which the vertical Brauer–Manin obstruction is trivial, is there a k-point in $\mathbb{P}^1_k$ such that the fibre over this point is smooth and everywhere locally soluble?

Theorem 13.2.16 *Let X be a smooth, projective and geometrically integral variety over a number field k and let $X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is geometrically integral and that every closed geometric fibre contains a component of multiplicity 1. Assuming Conjecture HW, if $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$, then there exists a $t_0 \in k = \mathbb{A}^1_k(k)$ such that X_{t_0} is smooth and has points in all completions of k. Moreover, given a finite set S of places of k, and a point $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, one can choose t_0 such that X_{t_0} contains k_v-points close to M_v for $v \in S$.*

Proof. For simplicity of notation *let us only consider the case where all the non-split fibres are above k-points of $\mathbb{A}^1_k = \mathrm{Spec}(k[t]) \subset \mathbb{P}^1_k$*. Let $a_1 \ldots, a_n$ be the coordinates of these points. So we assume that all the other fibres, including the fibre at infinity, are smooth and geometrically integral. We concentrate on the existence of a k-point with everywhere locally soluble fibre and omit the proof of the last claim of the theorem.

Let E_i be an irreducible component of multiplicity 1 in the fibre above a_i and let L_i be the integral closure of k in the function field $k(E_i)$. Let $U \subset X$ be the complement to the union of the fibre at infinity and the fibres above a_i. Consider the product of corresponding norm 1 tori $T = \prod_{i=1}^n R^1_{L_i/k}\mathbb{G}_{m,L_i}$ and the torsor over U under T given by the equations

$$t - a_i = \mathrm{N}_{L_i/k}(\Xi_i) \neq 0, \quad i = 1, \ldots, n.$$

This torsor over $U \subset X$ is the inverse image of a torsor under T over the complement to $\{a_1, \ldots, a_n\}$ in $\mathbb{P}^1_k$. Thus in this particular case the group B introduced in the proof of the formal lemma for torsors (Theorem 12.6.5) consists of elements coming from $\mathrm{Br}(k(\mathbb{P}^1))$, which are therefore vertical elements. Applying this theorem, under the hypothesis $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$, we find elements $b_i \in k^*$ and a family $(M_v) \in U(\mathbf{A}_k)$, with projections $t_v \in k_v$ for $v \in \Omega$, such that for each $v \in \Omega$ the system

$$0 \neq t_v - a_i = b_i \mathrm{N}_{L_i/k}(\Xi_i), \quad i = 1, \ldots, n,$$

has solutions over k_v.

Choose a *finite* set S of places of k, large enough for various purposes. Firstly, we include into S all archimedean places and all non-archimedean places v for which $v(a_i) < 0$ for some i. Next, we require that

each L_i/k is unramified at any $v \notin S$;

each b_i is a unit at any $v \notin S$;

the fibre at infinity X_∞ has good reduction at any $v \notin S$ and has points in all k_v for $v \notin S$ (this is possible by the Lang–Weil–Nisnevich inequality [LW54], [Po18, Thm. 7.7.1] as this fibre is smooth and geometrically integral);

each $E_{i,\mathrm{smooth}}$ (which is geometrically integral over L_i) has points in all completions $L_{i,w}$, where w is a place of L_i not lying above a place of S.

We fix a connected, regular proper model $\mathcal{X}/\mathbb{P}^1_{\mathcal{O}_S}$ of $X/\mathbb{P}^1_k$. Given a place $v \notin S$ and a point in $t_v \in k_v$ one can then consider the reduction of X_{t_v} at v. Namely, t_v extends to a unique point of $\mathbb{P}^1(\mathcal{O}_v)$, we consider the restriction of $\mathcal{X}/\mathbb{P}^1_{\mathcal{O}_S}$ to $\mathcal{O}_v$, and then the reduction modulo $\mathfrak{m}_v$.

Now we appeal to Conjecture HW. It produces a point $t_0 \in k$ very close to t_v for $v \in S$, and such that for any i and any $v \notin S$ either $v(t_0 - a_i) \leq 0$ or there exists a place of L_i of degree 1 over v.

Claim: $X_{t_0}(\mathbf{A}_k) \neq \emptyset$.

For $v \in S$ this is a consequence of the implicit function theorem (Theorem 9.5.1).

If v is not in S and $v(t_0 - a_i) < 0$ for some i, then $v(t_0) < 0$ and so the fibre X_{t_0} reduces modulo $\mathfrak{m}_v$ to the same smooth $\kappa(v)$-variety as the fibre X_∞, hence has k_v-points.

If v is not in S and $v(t_0 - a_i) = 0$ for each i, then t_0 does not reduce to the same point as any of the a_i. Thus, provided S has been chosen big enough at the beginning, the fibre X_{t_0} reduces to a smooth and geometrically integral variety over the finite field $\kappa(v)$ with a fixed Hilbert polynomial. This allows one to apply the Lang–Weil–Nisnevich inequality [LW54], [Po18, Thm. 7.7.1] which guarantees that the reduction of X_{t_0} modulo $\mathfrak{m}_v$ has a $\kappa(v)$-point. By Hensel's lemma, X_{t_0} has a k_v-point.

Finally, suppose that $v \notin S$ is such that $v(t_0 - a_i) > 0$ for some i. On the one hand, this implies that X_{t_0} reduces to the same variety over $\kappa(v)$ as X_{a_i}. On the other hand, by Conjecture HW, this implies that there is a place w of L_i of degree 1 over v such that $E_i \times_{L_i} L_{i,w}$ is geometrically integral over

$L_{i,w}$. Again, provided S was chosen big enough, the reduction of $E_i \times_{L_i} L_{i,w}$ over the field $\kappa(w) = \kappa(v)$ is geometrically integral and by Lang–Weil–Nisnevich [LW54], [Po18, Thm. 7.7.1]. has a smooth $\kappa(v)$-point. Using Hensel's lemma, we conclude that X_{t_0} contains a k_v-point. □

Assume that the smooth fibres of $X \to \mathbb{P}^1_k$ satisfy the Hasse principle and weak approximation. Then, assuming Conjecture HW, the proof of Theorem 13.2.16 easily implies that $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, hence also in $X(\mathbf{A}_k)^{\mathrm{Br}}$ – which then coincides with $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$. Such a general result was out of reach of the theorems based on Hypothesis (H).

The following (unconditional) corollary was originally obtained by the descent method, see [CTS00, Theorem A] which is an improvement of an earlier result [CTSS98, §2.2].

Corollary 13.2.17 *Let k be a number field and let X be a smooth, projective and geometrically integral variety over k. Let $f : X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is geometrically integral and that each closed fibre contains an irreducible component of multiplicity one. Assume that the sum of the degrees of the closed points of $\mathbb{P}^1_k$ with a non-split fibre is at most 2. If $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$, then there exists a $t_0 \in k = \mathbb{A}^1(k)$ such that X_{t_0} is smooth and has points in all completions of k. Moreover, given a finite set S of places of k, and a point $\{M_v\} \in X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, one can find a t_0 such that X_{t_0} contains k_v-points close to M_v for $v \in S$.*

Proof. The case when the non-split fibres are above two k-points corresponds to the case $n \leq 2$ in the proof of Theorem 13.2.16. In general, we have $\varepsilon \leq 2$, where ε is defined after the statement of Conjecture HW in Section 13.2.3. As recalled there, Conjecture HW is known for $\varepsilon \leq 2$. □

In what follows we use resolution of singularities in characteric 0 (Hironaka's theorem) without further mention.

Lemma 13.2.18 *Let X be a smooth, projective and geometrically integral variety over k. Let Y and Z be smooth projective varieties with dominant morphisms $Y \to X$ and $Z \to X$. Assume that the generic fibre of each of these morphisms is a Severi–Brauer variety, with associated classes $\alpha_Y \in \mathrm{Br}(k(X))$ and $\alpha_Z \in \mathrm{Br}(k(X))$. Suppose that $\alpha_Y - \alpha_Z \in \mathrm{Br}(k(X))$ is the image of some $\rho \in \mathrm{Br}(k)$. For a class $\zeta \in \mathrm{Br}(k(X))$, the following statements are equivalent:*

(i) *The image of ζ in $\mathrm{Br}(k(Y))$ belongs to $\mathrm{Br}(Y)$;*

(ii) *The image of ζ in $\mathrm{Br}(k(Z))$ belongs to $\mathrm{Br}(Z)$.*

Proof. Recall that if U is a Severi–Brauer variety over a field F, then the class of U in $\mathrm{Br}(F)$ goes to zero under the natural map $\mathrm{Br}(F) \to \mathrm{Br}(F(U))$.

Let P be a Severi–Brauer variety over k with class $\rho \in \mathrm{Br}(k)$.

Let W be a desingularisation of $Y \times_X Z$, that is, a smooth, projective and geometrically integral variety equipped with a birational morphism $W \to Y \times_X Z$. We can choose W so that the generic fibre of $W \to Z$ is isomorphic to the pullback of the generic fibre of $Y \to X$. We thus have a commutative diagram of smooth, projective and geometrically integral varieties

$$\begin{array}{ccccc} P \times_k W & \longrightarrow & W & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ P \times_k Z & \longrightarrow & Z & \longrightarrow & X \end{array}$$

The generic fibre $Y_{k(X)}$ of $Y \to X$ is a Severi–Brauer variety over $k(X)$ with class α_Y. Extending the ground field from $k(X)$ to $k(Z)$ we obtain a Severi–Brauer variety $Y_{k(Z)}$ over $k(Z)$. Since $\alpha_Y - \alpha_Z \in \mathrm{Br}(k(X))$ is the image of $\rho \in \mathrm{Br}(k)$, the class of $Y_{k(Z)}$ in $\mathrm{Br}(k(Z))$ is equal to the image of ρ. Thus the generic fibre of each of the morphisms $P \times_k W \to W$ and $P \times_k W \to P \times_k Z$ is a projective space. This implies that the natural map $\mathrm{Br}(k(P \times_k Z)) \to \mathrm{Br}(k(P \times_k W))$ induces an isomorphism $\mathrm{Br}(P \times_k Z) \tilde{\longrightarrow} \mathrm{Br}(P \times_k W)$.

Suppose that (i) holds. Then the image of $\zeta \in \mathrm{Br}(k(X))$ in $\mathrm{Br}(k(Y))$ belongs to the subgroup $\mathrm{Br}(Y)$. Hence the image of ζ in $\mathrm{Br}(k(P \times_k W))$ belongs to $\mathrm{Br}(P \times_k W)$. Then the image of ζ in $\mathrm{Br}(k(P \times_k Z))$ lies in the subgroup $\mathrm{Br}(P \times_k Z)$.

But then already the image of ζ in $\mathrm{Br}(k(Z))$ is contained in $\mathrm{Br}(Z)$. Indeed, by the functoriality of residues (Theorem 3.7.4) this follows from the fact that all fibres of the projection $P \times_k Z \to Z$ are geometrically integral. □

We actually need a more general lemma.

Lemma 13.2.19 *Let X be a smooth, projective and geometrically integral variety over k. Let Y_i and Z_i be smooth projective varieties with dominant morphisms $Y_i \to X$ and $Z_i \to X$, for $i = 1, \ldots, n$, whose generic fibres are Severi–Brauer varieties with classes $\alpha_i \in \mathrm{Br}(k(X))$ and $\beta_i \in \mathrm{Br}(k(X))$, respectively. Suppose that for $i = 1, \ldots, n$ the difference $\alpha_i - \beta_i$ is in the image of the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(k(X))$. Let Y/X be the fibred product of the $Y_i \to X$ over X and let Z/X be the fibred product of the $Z_i \to X$ over X. For a class $\zeta \in \mathrm{Br}(k(X))$, the following statements are equivalent:*

(i) *the image of ζ in $\mathrm{Br}(k(Y))$ belongs to $\mathrm{Br}_{\mathrm{nr}}(k(Y))$;*

(ii) *the image of ζ in $\mathrm{Br}(k(Z))$ belongs to $\mathrm{Br}_{\mathrm{nr}}(k(Z))$.*

Proof. Let W be a good desingularisation of $Y \times_X Z$ as above. For each i, let P_i be a Severi–Brauer variety over k with class $\rho_i \in \mathrm{Br}(k)$ such that $\alpha_i - \beta_i$ is the image of ρ_i in $\mathrm{Br}(k(X))$. Let $P = \prod_i P_i$. For any field F such that each $\rho_{i,F} \in \mathrm{Br}(F)$ vanishes, P_F is F-isomorphic to a product of projective spaces. The above proof extends *mutatis mutandis*. □

Theorem 13.2.20 *Let X be a smooth, projective and geometrically integral variety over a number field k and let $f : X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is geometrically integral and that each closed fibre contains a component of multiplicity one. Let $U \subset \mathbb{P}^1_k$ be a non-empty open set such that $X_U = f^{-1}(U) \to U$ is a smooth morphism. Let $B \subset \mathrm{Br}(X_U)$ be a finite subgroup and let $(M_v) \in X(\mathbf{A}_k)$ be an adelic point orthogonal to the* *intersection of* $\mathrm{Br}(X)$ *with* $B + f^*(\mathrm{Br}(k(\mathbb{P}^1))) \subset \mathrm{Br}(k(X))$. *Then, assuming Conjecture HW, for any finite set of places* S *there exists* $t_0 \in U(k)$ *such that* $X_{t_0}(\mathbf{A}_k)^B$ *is non-empty and contains an adelic point* (P_v), *where* P_v *is close to* M_v *for each* $v \in S$.

Proof. By Gabber's theorem (Theorem 3.3.2) the cohomological Brauer–Grothendieck group of a smooth variety coincides with its Brauer–Azumaya group. Thus we can assume that $B \subset \mathrm{Br}(X_U)$ is generated by the classes of Azumaya algebras A_i over X_U, for $i = 1, \ldots, n$. Let $Y_U \to X_U$ be the fibre product of the corresponding Severi–Brauer schemes. Using resolution of singularities, the morphism $Y_U \to X_U$ can be completed to a morphism $g : Y \to X$, where Y is a smooth and projective variety over k.

Let us prove that *each closed fibre of the composition* $h : Y \to X \to \mathbb{P}^1_k$ *contains an irreducible component of multiplicity* 1. Indeed, this condition is equivalent to the condition that h is locally split for étale topology on $\mathbb{P}^1_k$. To check it we can assume $k = \bar{k}$. Since the morphism $X \to \mathbb{P}^1_k$ is locally split by assumption, for any closed point $m \in \mathbb{P}^1_k$ there exists a connected étale neighbourhood $V \to \mathbb{P}^1_k$ whose image contains m and such that $X_V \to V$ has a section $V \to X_V$. The image of this section is an integral curve $W \subset X_V$ which is étale over $\mathbb{P}^1_k$. The morphism $Y \to X$ gives rise to a morphism $Y_V \to X_V$. Let $Y_W \to W$ be the restriction of $Y_V \to X_V$ to the curve W. The generic fibre of $Y_W \to W$ is a product of Severi–Brauer varieties. Applying Tsen's theorem (Theorem 1.2.12 (i)) to $k(W)$ shows that $Y_W \to W$ has a rational section, which must be a morphism since W is a regular curve and $Y_W \to W$ is proper. This proves that h is locally split for the étale topology.

We now go back to the case of a number field k.

Since each closed fibre of $h : Y \to X \to \mathbb{P}^1_k$ contains an irreducible component of multiplicity 1, from Remark 10.1.6 (see also Theorem 3.7.4) we see that the subgroup of $\mathrm{Br}(U)$, which consists of the elements α such that $f^*(\alpha) \in \mathrm{Br}(Y_U)$ belongs to the subgroup $\mathrm{Br}(Y) \subset \mathrm{Br}(Y_U)$, is finite modulo $\mathrm{Br}(k)$. Let $\gamma_1, \ldots, \gamma_m \in \mathrm{Br}(U)$ be elements generating this group modulo $\mathrm{Br}(k)$.

Write B' for the intersection of $\mathrm{Br}(X)$ with $B + f^*(\mathrm{Br}(k(\mathbb{P}^1))) \subset \mathrm{Br}(k(X))$. Since B is a finite group, an application of Theorem 3.7.4 shows that B' is also finite modulo $\mathrm{Br}(k)$.

By assumption, we have an adelic point $(M_v) \in X(\mathbf{A}_k)^{B'}$. Since B' is finite modulo $\mathrm{Br}(k)$, there is an adelic point $(M'_v) \in X_U(\mathbf{A}_k)^{B'}$ such that M'_v is very close to M_v for each $v \in S$. We now rename M'_v and call it M_v.

By Harari's formal lemma (Theorem 12.6.3) we may assume that

$$\sum_{v \in \Omega} \mathrm{inv}_v A_i(M_v) = \sum_{v \in \Omega} \mathrm{inv}_v \gamma_j(M_v) = 0,$$

for each $i = 1, \ldots, n$ and $j = 1, \ldots, m$. By class field theory (Theorem 12.1.8 (iii)) it follows that there exists $\rho_i \in \mathrm{Br}(k)$ whose image in $\mathrm{Br}(k_v)$ is $A_i(M_v)$ for each place $v \in \Omega$.

Let $A'_i = A_i - \rho_i \in \mathrm{Br}(X_U)$, for $i = 1, \ldots, n$. We can choose Azumaya algebras over X_U representing these classes and consider the associated Severi–

Brauer schemes. Let Y'_U be the fibred product of these schemes over X_U. As above, we extend $Y'_U \to X_U$ to a morphism $Y' \to X$, where Y' is a smooth, projective and geometrically integral variety over k.

Since $A'_i(M_v) = 0$, there is a k_v-point N_v in the fibre of $Y'_U \to X_U$ above M_v. Thus we have an adelic point $(N_v) \in Y'_U(\mathbf{A}_k)$ above $(M_v) \in X_U(\mathbf{A}_k)$.

We claim that $(N_v) \in Y'_U(\mathbf{A}_k)$ is orthogonal to $\mathrm{Br}_{\mathrm{vert}}(Y')$, where the vertical part of the Brauer group is taken with respect to the morphism $Y' \to \mathbb{P}^1_k$. Indeed, $\mathrm{Br}_{\mathrm{vert}}(Y')$ consists of the images of the elements of $\mathrm{Br}(U)$ which become unramified on Y'. By Lemma 13.2.19, these elements of $\mathrm{Br}(U)$ are exactly those which become unramified on Y. Modulo $\mathrm{Br}(k)$, this group is spanned by the classes γ_j, for $j = 1, \ldots, m$, and we have

$$\sum_{v \in \Omega} \mathrm{inv}_v \gamma_j(N_v) = 0,$$

since N_v is over M_v.

If we now assume Conjecture HW and apply[1] Theorem 13.2.16 to the fibration $Y' \to \mathbb{P}^1_k$, we find that there exists a $t_0 \in U(k)$ such that the fibre Y'_{t_0} has an adelic point (R_v), with R_v close to N_v for $v \in S$. Let $Q_v \in X_{t_0}(k_v)$ be the image of R_v under the morphism $Y'_{t_0} \to X_{t_0}$. For each $i = 1, \ldots, n$ and each $v \in \Omega$ we have $A'_i(Q_v) = 0$, hence $A_i(Q_v)$ is the image of ρ_i in $\mathrm{Br}(k_v)$. Thus

$$\sum_{v \in \Omega} \mathrm{inv}_v A_i(Q_v) = 0,$$

with Q_v close to M_v for $v \in S$. □

Now we are ready to sketch the proof of the main theorem of Harpaz and Wittenberg.

Proof of Theorem 13.2.13. (Sketch) The generic fibre is is a rationally connected variety. By the theorem of Graber–Harris–Starr [GHS03] it implies that each special fibre of $f : X \to \mathbb{P}^1_k$ contains an irreducible component of multiplicity one. By Corollary 4.4.4, it also implies that the group $\mathrm{Br}(X_\eta)$ is finite modulo the image of $\mathrm{Br}(k(t))$. Thus we can choose an open subset $U \subset \mathbb{P}^1_k$ such that $X_U \to U$ is smooth and there is a finite group $B \subset \mathrm{Br}(X_U)$ that spans $\mathrm{Br}(X_\eta)$ modulo the image of $\mathrm{Br}(k(t))$. Then one looks for a t_0 as in the previous theorem, with the extra condition that the image of B spans the finite group $\mathrm{Br}(X_{t_0})/\mathrm{Br}(k)$. By Harari's specialisation result ([Har94, §3] and [Har97, Thm. 2.3.1], see also [HW15, Prop. 4.1]), the set of k-points such that the last condition is fulfilled is a Hilbert set. The question is thus to show that in the previous theorem one may require t_0 to lie in a Hilbert set. We refer here to [HW15, Thm. 9.22] (see also [Sme15, Prop. 6.1]. □

Building upon the results in additive combinatorics one then obtains the following *unconditional* statement, first proved by Skorobogatov [Sko13]. His

[1] Here we cannot restrict to the case where we have the simplifying assumption (non-split fibres only at rational points) made in our proof of Theorem 13.2.16.

proof (of a slightly more general statement) uses the result of Browning and Matthiesen [BM17] on systems of equations

$$u - a_i v = b_i \mathrm{N}_{L_i/k}(\Xi_i), \quad i = 1\dots, r,$$

obtained using additive combinatorics, but his argument looks somewhat different as it uses descent and universal torsors [CTS87a, Sko01]. In the proof we give here, descent has been replaced by the use of the formal lemma for torsors in the proof of Theorem 13.2.16.

Theorem 13.2.21 (Skorobogatov) *Let $U \subset \mathbb{A}^1_{\mathbb{Q}}$ be the open subset given by $P_1(t)\dots P_n(t) \neq 0$, where each polynomial $P_i(t)$ is a product of linear factors over $\mathbb{Q}$. Let X_0 be the smooth quasi-affine variety over $\mathbb{Q}$ defined by*

$$\mathrm{N}_{K_i/\mathbb{Q}}(\Xi_i) = P_i(t) \neq 0, \quad i = 1, \dots, n,$$

where $K_1, \dots, K_n$ are arbitrary number fields, and let $g : X_0 \to U$ be the projection to the coordinate t. Let $X_0 \subset X$ be an open embedding into a smooth, projective and geometrically integral variety over $\mathbb{Q}$ equipped with a dominant morphism $f : X \to \mathbb{P}^1_{\mathbb{Q}}$ extending the map $X_0 \to U$. Then $X(\mathbb{Q})^{cl} = X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$. In particular, if $X(\mathbb{Q})$ is not empty, then $X(\mathbb{Q})$ is Zariski dense in X and weak weak approximation holds for X.

Proof. The statement of the theorem does not depend on the choice of X. A convenient way to construct X is as follows. Let T be the product of norm 1 tori given by

$$\mathrm{N}_{K_i/\mathbb{Q}}(\Xi_i) = 1, \quad i = 1, \dots, n.$$

Choose a smooth T-equivariant compactification $T \subset Y$, which exists as proven in [CTHS03]. The contracted product $X_0 \times^T Y$ has a natural proper morphism to U such that all fibres are smooth compactifications of torsors under T, in particular, they are geometrically integral. Extending $X_0 \times^T Y \to U$ we produce a smooth, projective and geometrically integral variety X over $\mathbb{Q}$ together with a proper morphism $f : X \to \mathbb{P}^1_{\mathbb{Q}}$ such that $X_U \to U$ is smooth and $X_0 \subset X_U$ is an open subset.

Let $m \in U$ be a closed point and let X_m be the closed fibre at m. We have arranged that X_m is smooth and geometrically integral; moreover, X_m is geometrically rational. The residue of $\beta \in \mathrm{Br}(X_\eta)$ at the generic point of X_m lies in

$$\mathrm{H}^1_{\text{ét}}(X_m, \mathbb{Q}/\mathbb{Z}) \subset \mathrm{H}^1(k(m)(X_m), \mathbb{Q}/\mathbb{Z}).$$

Using the fact that smooth, projective, rational varieties over an algebraically closed field of characteristic 0 have no non-trivial finite étale covers, one shows that the natural map

$$\mathrm{H}^1(k(m), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X_m, \mathbb{Q}/\mathbb{Z})$$

is an isomorphism. Thus the above residue is an element of $\mathrm{H}^1(k(m), \mathbb{Q}/\mathbb{Z})$.

Using the Faddeev exact sequence (Theorem 1.25) one sees that

$$\mathrm{Br}(X_\eta) \subset f^*\mathrm{Br}(\mathbb{Q}(\mathbb{P}^1)) + \mathrm{Br}(X_U).$$

From this one deduces that there exists a finite subgroup $B \subset \mathrm{Br}(X_U)$ which surjects onto the (finite) group $\mathrm{Br}(X_\eta)/f^*\mathrm{Br}(\mathbb{Q}(\mathbb{P}^1))$. Then one proceeds as in the proofs of Theorems 13.2.20 and 13.2.13. The composite fibration $Y'_U \to X_U \to U$ is smooth, the complement of U consists of $\mathbb{Q}$-points. Since $k = \mathbb{Q}$, Matthiesen's theorem (see Section 13.2.3) guarantees the validity of Conjecture HW in the present situation. The proof of Theorem 13.2.13, via Theorem 13.2.20, thus specialises to an unconditional proof in the present case. □

Harpaz and Wittenberg [HW15], using more elaborate arguments, some of them coming from Harari's thesis [Har94], actually prove the following more general, unconditional result. (For this, just like Harari in [Har94], they have to discuss what happens when representatives of $\mathrm{Br}(X_\eta)$ have non-trivial residues at split fibres.)

Theorem 13.2.22 *Let X be a smooth, projective, geometrically integral variety over $\mathbb{Q}$ and let $X \to \mathbb{P}^1_{\mathbb{Q}}$ be a morphism with rationally connected generic fibre. Assume that all non-split fibres are above $\mathbb{Q}$-points of $\mathbb{P}^1_{\mathbb{Q}}$. If $X_P(\mathbb{Q})$ is dense in $X_P(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}(X_P)}$ for smooth fibres X_P over rational points of $\mathbb{P}^1_{\mathbb{Q}}$, then $X(\mathbb{Q})$ is dense in $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$.*

The theorem also holds if $X \to \mathbb{P}^1_{\mathbb{Q}}$ has exactly exactly two non-split fibres, one above a $\mathbb{Q}$-point and another one above a closed point of degree 2. Indeed, in this case Conjecture HW was proved by Browning and Schindler in [BS]. This subsumes the earlier result of Derenthal–Smeets–Wei [DSW12] for the equation

$$\mathrm{N}_{K/\mathbb{Q}}(\Xi) = P(t),$$

where $P(t)$ is irreducible of degree 2 and K is an arbitrary number field. Both proofs are based on the work of Browning and Heath-Brown [BH12] who used sieve methods.

13.2.5 Fibrations with two non-split fibres and ramified descent

A very special example for Corollary 13.2.17 is the Hasse principle for quadratic forms in four variables. Consider the system

$$0 \neq t = b_1(x_1^2 - a_1 y_1^2) = b_2(x_2^2 - a_2 y_2^2), \tag{13.1}$$

where a_1, a_2, b_1, b_2 are non-zero elements of a number field k, and assume that it has solutions everywhere locally. Let S be the set of places of k containing the infinite places, the places above 2, and the primes where at least one of a_1, a_2, b_1, b_2 is not a unit. Hasse's method is to apply Dirichlet's theorem on primes in an arithmetic progression to find a $t_0 \in k$ which is a unit away from

$S \cup \{v_0\}$, where v_0 is a finite place where t_0 has valuation 1, and such that t_0 is close to the t-coordinate of given k_v-points for $v \in S$. Then each conic $t_0 = b_i(x_i^2 - a_i y_i^2)$, $i = 1, 2$, has points in all completions of k except possibly in k_{v_0}. The reciprocity law (Corollary 12.1.10) then implies that it has a solution also in k_{v_0} and in k. The proof of this result given here is *different*. The argument based on the Tate–Nakayama duality and the formal lemma for torsors directly produces a point t_0 such that each of the two equations $t_0 = b_i(x_i^2 - a_i y_i^2)$ has solutions in *all* completions.

The proof of Theorem 13.2.16 given above is arranged in such a way that the fibre at infinity is smooth. For the equation (13.1) the fibre at infinity is not smooth. Let us give a simple direct argument instead of referring to Theorem 13.2.16. Let U be the quasi-affine variety given by (13.1). Assume that U is everywhere locally soluble. Let $L = k(\sqrt{a_1}, \sqrt{a_2})$. The equation

$$0 \neq t = \mathrm{N}_{L/k}(\Xi)$$

defines a torsor over $\mathbb{G}_{m,k}$ under the norm torus $T = R^1_{L/k}(\mathbb{G}_{m,L})$. Let $Y \to U$ be the torsor obtained by pulling it back to U via the projection $U \to \mathbb{G}_{m,k}$ given by the coordinate t. We have $\mathrm{Br}_{\mathrm{nr}}(k(U)/k) = \mathrm{Im}(\mathrm{Br}(k))$, since U is birationally equivalent to the product of $\mathbb{P}^1_k$ and a quadric, hence the Brauer group of a smooth projective model of U is reduced to the image of $\mathrm{Br}(k)$. The formal lemma for torsors (Theorem 12.6.5) now gives an element $c \in k^*$ such that the system

$$0 \neq t = b_1(x_1^2 - a_1 y_1^2) = b_2(x_2^2 - a_2 y_2^2) = c\mathrm{N}_{L/k}(\Xi)$$

is everywhere locally soluble. This implies that the system

$$b_1(x_1^2 - a_1 y_1^2) = c = b_2(x_2^2 - a_2 y_2^2)$$

is everywhere locally soluble too. In other words, the fibre of $U \to \mathbb{G}_{m,k}$ over $c \in k^*$ is everywhere locally soluble. It is the product of two conics, so we use the Hasse principle for conics to conclude that $U(k) \neq \emptyset$.

If one considers a system of equations

$$0 \neq t = b_i \mathrm{N}_{k_i/k}(\Xi_i), \quad i = 1, \ldots, r,$$

with arbitrary number fields $k_1, \ldots, k_r$, and with no vertical Brauer–Manin obstruction to the existence of a rational point, the same argument will produce a $c \in k^*$ such that the system

$$c = b_i \mathrm{N}_{k_i/k}(\Xi_i), \quad i = 1, \ldots, r,$$

is everywhere locally soluble. However, in the case of arbitrary number fields $k_1, \ldots, k_r$ one cannot ensure that this system has solutions in k: here the obstruction coming from the vertical Brauer group is not enough. As in Theorem 13.2.13, the whole Brauer group of (a smooth projective model) of the variety has to be taken into account.

Note that in Theorem 13.2.16 there is no geometric assumption on the generic fibre of $X\to\mathbb{P}^1_k$ other that it is geometrically integral. This theorem can be applied to the problem of lifting an adelic point to some twist of a ramified cyclic cover, as discussed in Section 10.4.

Theorem 13.2.23 *Let k be a number field. Let X and Y be smooth, projective and geometrically integral varieties over k. Assume that μ_n acts on X, and Y is birationally equivalent to X/μ_n. Let $F\in k(Y)^*$ be a rational function such that the generic fibre of $X\to X/\mu_n$ is given by $t^n=F$. Write $\mathrm{div}(F)=\sum m_D D$, where D are irreducible divisors in Y. Let k_D be the algebraic closure of k in the function field $k(D)$. Assume that $(n,m_D)=1$ for some D. Let $B\subset \mathrm{Br}(Y)$ be the subgroup consisting of the classes (χ,F), where χ is an element of the finite group*

$$\bigcap_D \mathrm{Ker}[m_D\,\mathrm{res}_{k_D/k}:\mathrm{H}^1(k,\mathbb{Z}/n)\to\mathrm{H}^1(k_D,\mathbb{Z}/n)].$$

If $Y(\mathbf{A}_k)^B\neq\emptyset$, then there exists a $c\in k^*$ such that the twisted cover X_c with generic fibre $ct^n=F$ has points in all completions of k. Moreover, given a finite set S of places of k and a point $\{M_v\}\in Y(\mathbf{A}_k)^B$, where $M_v\notin \mathrm{Supp}(\mathrm{div}(F))$ for $v\in S$, one can choose c close to $F(M_v)$ for $v\in S$.

Proof. In the notation of Section 10.4 consider the morphism $W\to\mathbb{P}^1_k$. Recall that W is stably birationally equivalent to Y and that at most two closed fibres of $W\to\mathbb{P}^1_k$, namely the k-fibres above 0 and ∞, are non-split. The k-fibres of $W\to\mathbb{P}^1_k$ other than the fibres at 0 and ∞ are cyclic twists of X. By Proposition 10.4.1 (i) the vertical Brauer group $\mathrm{Br}_{\mathrm{vert}}(W/\mathbb{P}^1_k)$ is generated by B modulo the image of $\mathrm{Br}(k)$. By Proposition 10.4.1 (iii), the assumption $(n,m_D)=1$ implies that each closed fibre of $W\to\mathbb{P}^1_k$ contains an irreducible component of multiplicity 1. It remains to apply Corollary 13.2.17. □

Let n be a positive integer. Let $a,b,c,d\in k^*$ and let $S\subset\mathbb{P}^3_k$ be the smooth surface given by

$$ax^n+by^n=cz^n+dw^n.$$

We assume that S is everywhere locally soluble and there is no Brauer–Manin obstruction with respect to the finite subgroup $B\subset\mathrm{Br}(S)$ consisting of the classes $(a(x/y)^n+b,\chi)$. Here χ belongs to the kernel of the restriction map

$$\mathrm{H}^1(k,\mathbb{Z}/n)\longrightarrow\mathrm{H}^1(L,\mathbb{Z}/n),$$

where L is the étale k-algebra

$$L=\big(k[t]/(t^n+b/a)\big)\otimes_k\big(k[t]/(t^n+d/c)\big).$$

Then, by Theorem 13.2.23, there exists a $\rho\in k^*$ such that each of the smooth plane curves

$$ax^n+by^n=\rho v^n,\quad cz^n+dw^n=\rho v^n \tag{13.2}$$

is everywhere locally soluble. When $n = p$ is a prime, we have $B = 0$, hence the relevant vertical Brauer group is reduced to the image of $\mathrm{Br}(k)$ (Proposition 10.4.2).

For $n = 3$ this statement is a starting point in Swinnerton-Dyer's paper [SwD01, Lemma 2, p. 901], see also a similar situation in [SkS05] and [HS16]. The challenge here, assuming $S(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, is to produce a ρ such that for each of the two curves (13.2) there is no Brauer–Manin obstruction to the existence of a k-point, or at least to the existence of a 0-cycle of degree 1.

13.3 Beyond the Brauer–Manin obstruction

13.3.1 Insufficiency of the Brauer–Manin obstruction

For $n \geq 4$ any smooth hypersurface $X \subset \mathbb{P}^n$ satisfies $\mathrm{Br}(k) = \mathrm{Br}(X)$ (Corollary 4.4.5). Thus $X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k)$. If $d > n$, where d is the degree of X, then the Bombieri–Lang conjecture states that $X(k)$ is not Zariski dense in X. A refinement of this conjecture predicts that a 'hyperbolic' hypersurface has finitely many rational points. Thus for any such X with $X(k) \neq \emptyset$, the set $X(k)$ cannot be dense in $X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k)$. It is also very unlikely that the Hasse principle holds for rational points of smooth, projective hypersurfaces of dimension at least 3 of arbitrary degree. Conditional examples with $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ and $X(k) = \emptyset$ can be found in [SW95, Po01].

In 1999, Skorobogatov [Sko99] gave the first unconditional example of a smooth, geometrically connected, projective variety X over a number field k such that $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ but nevertheless $X(k) = \emptyset$. In this example $k = \mathbb{Q}$ and the variety X is a surface of Kodaira dimension 0, which geometrically is a bielliptic surface. See also [Sko01, Ch. 8].

The following, more elementary example was constructed in [CTPS16]. The idea to use a curve with a unique rational point is due to Poonen [Po10].

Let C be a smooth, projective, geometrically integral curve over a number field $k \subset \mathbb{R}$ such that $C(k)$ consists of just one point, $C(k) = \{P\}$. (Poonen showed that such a curve exists for any number field k; moreover, Mazur and Rubin proved that C can be chosen to be an elliptic curve.) Let us write v_0 for the given real place $k \subset \mathbb{R}$. Let $\Pi \subset C(\mathbb{R})$ be an open interval containing P. Let $f : C \to \mathbb{P}^1_k$ be a surjective morphism unramified at P. Choose a coordinate function t on $\mathbb{A}^1_k = \mathbb{P}^1_k \smallsetminus f(P)$ such that f is unramified above $t = 0$. We have $f(P) = \infty$. Take any $a > 0$ in k such that a is an interior point of the interval $f(\Pi)$ and f is unramified above $t = a$.

Let $v \neq v_0$ be a place of k. There exists a quadratic form $Q(x_0, x_1, x_2)$ over k of rank 3 that represents zero in all completions of k other than k_v and k_{v_0}, but not in k_v or k_{v_0}. We can assume that Q is positive definite at v_0. Choose $n \in k$ with $n > 0$ and $-nQ(1,0,0) \in k_v^{*2}$. Let $Y_1 \subset \mathbb{P}^3_k \times \mathbb{A}^1_k$ be given by $Q(x_0, x_1, x_2) + nt(t-a)x_3^2 = 0$, and let $Y_2 \subset \mathbb{P}^3_k \times \mathbb{A}^1_k$ be given by $Q(X_0, X_1, X_2) + n(1 - aT)X_3^2 = 0$. We glue Y_1 and Y_2 by identifying $T = t^{-1}$, $X_3 = tx_3$, and $X_i = x_i$ for $i = 0, 1, 2$. This produces a quadric bundle $Y \to \mathbb{P}^1_k$

with exactly two degenerate fibres (over $t = a$ and $t = 0$), each given by the quadratic form $Q(x_0, x_1, x_2)$ of rank 3. It is straightforward to check that Y is smooth over k.

Define $X = Y \times_{\mathbb{P}^1_k} C$. This is a flat, surjective, proper morphism $X \to C$ whose fibres are geometrically integral quadrics. The assumption that f is unramified at $t = 0$ and $t = a$ implies that X is also smooth.

For example, we can take $k = \mathbb{Q}$, $\mathbb{Q}_v = \mathbb{Q}_2$ and consider Y defined by

$$x_0^2 + x_1^2 + x_2^2 + 7t(t-a)x_3^2 = 0.$$

Proposition 13.3.1 *In the above notation we have* $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ *whereas* $X(k) = \emptyset$.

Proof. Since $C(k) = \{P\}$ we have $X(k) \subset X_P$. The fibre X_P is the smooth quadric $Q(x_0, x_1, x_2) + nx_3^2 = 0$. This quadratic form is positive definite thus X_P has no points in $k_{v_0} = \mathbb{R}$ and so $X(k) = \emptyset$. By assumption X_P has local points in all completions of k other than k_v and k_{v_0}. The condition $-nQ(1, 0, 0) \in k_v^{*2}$ implies that X_P contains k_v-points, so X_P has local points in all completions of k but one. Choose $N_u \in X_P(k_u)$ for each place $u \neq v_0$. Let $M \in \Pi$ be such that $f(M) = a$. Then the singular point of the real fibre X_M (the vertex of the quadratic cone) is a smooth real point of X. Take it as the v_0-component of the adelic point (N_u) of X.

We claim that $(N_u) \in X(\mathbf{A}_k)^{\mathrm{Br}}$.

Indeed, the fibres of $X \to C$ are geometrically integral. By Proposition 10.2.7 the natural map $\mathrm{Br}(C) \to \mathrm{Br}(X)$ is surjective. Thus it is enough to show that the adelic point on C such that its components at all places other than v_0 are equal to P and its component at v_0 is M, is orthogonal to $\mathrm{Br}(C)$. The real point M is path-connected to P, so this adelic point is in the connected component of the diagonal image of the k-point P in $C(\mathbf{A}_k)$. By the continuity of the real evaluation map it is contained in $C(\mathbf{A}_k)^{\mathrm{Br}}$, so the proposition follows. □

13.3.2 Distinguished subsets of the adelic space

Let G be a linear k-group scheme. Let X and Y be varieties over k and let $f : Y \to X$ be a G-torsor. Such torsors are classified by the pointed set $\mathrm{H}^1(X, G)$. For a number field k, we have a natural map

$$\theta : \mathrm{H}^1(k, G) \longrightarrow \prod_v \mathrm{H}^1(k_v, G).$$

By a theorem of Borel and Serre, the fibres of θ are finite, see [SerCG, III.4.6].

For any ring R containing k, the pullback of $f : Y \to X$ induces a map

$$X(R) \longrightarrow \mathrm{H}^1(R, G).$$

When R is the ring of adèles $\mathbf{A}_k$ we obtain a map

$$X(\mathbf{A}_k) \longrightarrow \prod_v \mathrm{H}^1(k_v, G).$$

Define $X(\mathbf{A}_k)^f \subset X(\mathbf{A}_k)$ as the inverse image of $\theta(\mathrm{H}^1(k,G))$ under this map.

To a 1-cocycle $\sigma : \Gamma = \mathrm{Gal}(\bar{k}/k) \to G(\bar{k})$ one associates a twisted group G_σ defined with respect to the action of G on itself by conjugations. The isomorphism class of G_σ depends only on the class $[\sigma] \in \mathrm{H}^1(k,G)$. Twisting the G-torsor $f : Y \to X$ one obtains a G_σ-torsor $f_\sigma : Y^\sigma \to X$; its isomorphism class depends only on $[\sigma]$. See [Sko01, Ch. 2] for more details.

The class $[\sigma]$ is the image of a k-point $P \in X(k)$ under the map $X(k) \to \mathrm{H}^1(k,G)$ if and only if there exists a k-point $M \in Y^\sigma(k)$ such that $f_\sigma(M) = P$. This implies

$$X(k) = \coprod_{[\sigma]\in \mathrm{H}^1(k,G)} f_\sigma(Y^\sigma(k)).$$

Similarly, by the definition of $X(\mathbf{A}_k)^f$ we have

$$X(\mathbf{A}_k)^f = \bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)).$$

Combining these formulae shows that the diagonal map $X(k) \hookrightarrow X(\mathbf{A}_k)$ gives rise to an embedding $X(k) \subset X(\mathbf{A}_k)^f$.

Proposition 13.3.2 *Let X be a variety over a number field k. Then $X(\mathbf{A}_k)^f$ is a closed subset of $X(\mathbf{A}_k)$.*

Proof. See [Sko09, Cor. 2.7] in the case when X is proper, and [CDX, Prop. 6.4] in general. Let us sketch this proof.

Let S_0 be a finite subset of the set of places Ω of k containing all the archimedean places, and let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}_{k,S_0}$ with generic fibre X. For finite subsets $S \subset \Omega$ containing S_0 the sets $\mathcal{X}(\mathbf{A}_{k,S})$ form an open covering of $X(\mathbf{A}_k)$, see Section 12.1.3. Thus it is enough to check that $X(\mathbf{A}_k)^f \cap \mathcal{X}(\mathbf{A}_{k,S})$ is closed in $\mathcal{X}(\mathbf{A}_{k,S})$. Using Lang's theorem about the triviality of torsors for a connected group over a finite field, and Hensel's lemma, one shows

$$f(Y(\mathbf{A}_k)) = X(\mathbf{A}_k) \cap \prod_{v\in\Omega} f(Y(k_v)) \subset \prod_{v\in\Omega} X(k_v).$$

As $f(Y(k_v))$ is closed in $X(k_v)$ for every $v \in \Omega$, one concludes that $f(Y(\mathbf{A}_k))$ is a closed subset of $X(\mathbf{A}_k)$. In particular, $f(Y(\mathbf{A}_k)) \cap \mathcal{X}(\mathbf{A}_{k,S})$ is closed in $\mathcal{X}(\mathbf{A}_{k,S})$. Finally, using the Borel–Serre theorem, one shows that there are only finitely many twisted forms $f_\sigma : Y^\sigma \to X$ (up to isomorphism) such that $f_\sigma(Y^\sigma(\mathbf{A}_k))$ meets $\mathcal{X}(\mathbf{A}_{k,S})$. It follows that $X(\mathbf{A}_k)^f \cap \mathcal{X}(\mathbf{A}_{k,S})$ is a union of finitely many closed subsets of $\mathcal{X}(\mathbf{A}_{k,S})$ and so is closed. □

Let us define

$$X(\mathbf{A}_k)^{\mathrm{H}^1(X,G)} = \bigcap_f X(\mathbf{A}_k)^f,$$

where f ranges over all G-torsors $f : Y \to X$.

When G is a torus, a close relation between $X(\mathbf{A}_k)^{\mathrm{H}^1(X,G)}$ and $X(\mathbf{A}_k)^{\mathrm{Br}}$ was established by Colliot-Thélène and Sansuc [CTS87a]. This was extended by Skorobogatov [Sko99] to groups of multiplicative type, see [Sko01, §6.1]. The set $X(\mathbf{A}_k)^f$ attached to a torsor $f: Y \to X$ for a finite, non-commutative group k-scheme was used by Harari [Har00] to construct examples where weak approximation fails and this failure is not accounted for by the Brauer–Manin obstruction. The general definition of $X(\mathbf{A}_k)^f$ was spelled out in [HS02, §4], see also [Sko01, §5.3].

For various classes of linear groups we define closed subsets of $X(\mathbf{A}_k)$ containing $X(k)$, as follows:

$$X(\mathbf{A}_k)^{\mathrm{desc}} = \bigcap_{\text{linear } G} X(\mathbf{A}_k)^{\mathrm{H}^1(X,G)},$$

$$X(\mathbf{A}_k)^{\text{ét}} = \bigcap_{\text{finite } G} X(\mathbf{A}_k)^{\mathrm{H}^1(X,G)},$$

$$X(\mathbf{A}_k)^{\mathrm{conn}} = \bigcap_{\text{connected linear } G} X(\mathbf{A}_k)^{\mathrm{H}^1(X,G)}.$$

If X is a quasi-projective variety over a field k, then $\mathrm{Br}_{\mathrm{Az}}(X) = \mathrm{Br}(X)$ by Gabber's theorem (Theorem 3.3.2). So, if X is smooth and quasi-projective over a number field k, the connection between torsors for PGL_n and Azumaya algebras, gives $X(\mathbf{A}_k)^{\mathrm{Br}} = \bigcap_n X(\mathbf{A}_k)^{\mathrm{H}^1(X,\mathrm{PGL}_n)}$, see [HS02, Thm. 4.10]. One concludes that

$$X(\mathbf{A}_k)^{\mathrm{desc}} \subset X(\mathbf{A}_k)^{\mathrm{Br}}. \tag{13.3}$$

A theorem of Harari [Har02] gives

$$X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)^{\mathrm{conn}}$$

for any geometrically integral X.

Allowing $f: Y \to X$ to be a torsor for any *finite* group k-scheme G, we define more subsets of $X(\mathbf{A}_k)$ containing $X(k)$:

$$X(\mathbf{A}_k)^{\text{ét,Br}} = \bigcap_f \bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}});$$

$$X(\mathbf{A}_k)^{\text{ét,desc}} = \bigcap_f \bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{desc}}).$$

Next, allowing $f: Y \to X$ to be a torsor for any linear group k-scheme G, define

$$X(\mathbf{A}_k)^{\mathrm{desc,desc}} = \bigcap_f \bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{desc}}).$$

Skorobogatov's example was first interpreted in [Sko99] as an example of a smooth and projective variety such that $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ but $X(\mathbf{A}_k)^{\text{ét,Br}} = \emptyset$. It was then interpreted in [HS02] as an example with $X(\mathbf{A}_k)^{\mathrm{H}^1(X,G)} = \emptyset$ for a suitable finite k-group G, hence with $X(\mathbf{A}_k)^{\mathrm{desc}} = \emptyset$. This raised the question of the relation between these various obstructions.

Theorem 13.3.3 *Let k be a number field. Let X be a smooth, quasi-projective, geometrically integral variety over k. Then*

$$X(\mathbf{A}_k)^{\mathrm{desc}} = X(\mathbf{A}_k)^{\text{ét},\mathrm{desc}} = X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}.$$

In particular, the following two conditions are equivalent:

(i) *any G-torsor $Y \to X$, where G is a linear group k-scheme, has a twisted form $Y^\sigma \to X$ such that $Y^\sigma(\mathbf{A}_k) \neq \emptyset$;*

(ii) *any G-torsor $Y \to X$, where G is a finite group k-scheme, has a twisted form $Y^\sigma \to X$ such that $Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$.*

This theorem is a consequence of the following inclusions

$$X(\mathbf{A}_k)^{\mathrm{desc}} \subset X(\mathbf{A}_k)^{\text{ét},\mathrm{desc}} \subset X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}} \subset X(\mathbf{A}_k)^{\mathrm{desc}}.$$

The second inclusion is (13.3). In the case when X is projective, the first inclusion is a theorem of Skorobogatov [Sko09, Thm. 1.1], who extends results of Stoll [Sto07], and the third inclusion is a theorem of Demarche [Dem09], who extends results of Harari [Har02]. The general case of quasi-projective varieties is due to Cao, Demarche, and Xu, see [CDX, Thm. 7.5]. Among the ingredients of their proof is the result that with suitable modifications and isotropy conditions at the archimedean places, the Brauer–Manin obstruction to strong approximation for homogeneous spaces of connected linear algebraic groups with connected stabilisers is the only obstruction (Borovoi–Demarche [BD13], after work of Colliot-Thélène–Xu [CTX09], Harari [Har08], Demarche [Dem11]).

This theorem is complemented by the following result of Y. Cao [Cao], which answers a question that was asked by Poonen in the case when X is projective.

Theorem 13.3.4 (Y. Cao) *Let X be a smooth, quasi-projective, geometrically integral variety over a number field k. Then*

$$X(\mathbf{A}_k)^{\mathrm{desc},\mathrm{desc}} = X(\mathbf{A}_k)^{\mathrm{desc}},$$

hence

$$X(\mathbf{A}_k)^{\mathrm{desc},\mathrm{desc}} = X(\mathbf{A}_k)^{\mathrm{desc}} = X(\mathbf{A}_k)^{\text{ét},\mathrm{desc}} = X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}.$$

As an immediate corollary, we obtain $X(\mathbf{A}_k)^{\mathrm{desc},\ldots,\mathrm{desc}} = X(\mathbf{A}_k)^{\mathrm{desc}}$.

Y. Harpaz and T. Schlank [HSc13] used étale homotopy theory of Artin and Mazur to produce a subset $X(\mathbf{A}_k)^h \subset X(\mathbf{A}_k)$ which contains $X(k)$. (See also [Pál15].) For any smooth and geometrically connected variety X (not necessarily proper) they prove that $X(\mathbf{A}_k)^h = X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$. In the case of curves, there are interesting connections with Grothendieck's section conjecture, for which we refer to the book by J. Stix [Stix].

The above results raise the question: is the étale Brauer–Manin obstruction the only obstruction to the existsence of rational points, that is, if the set $X(k)$ is empty, then is the set $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$ empty too? This seems to be unlikely already in the case of smooth hypersurfaces of dimension at least 3, which have trivial geometric fundamental group and trivial Brauer group $\mathrm{Br}(X) = \mathrm{Br}_0(X)$.

Unconditional examples of smooth, projective, geometrically integral varieties X over a number field k with $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}} \neq \emptyset$ but with $X(k) = \emptyset$ have been found. Poonen [Po10] uses a threefold with a dominant morphism to a curve with finitely many rational points such that the generic fibre is a Châtelet surface. Over $\bar{k}$ such a variety becomes birationally equivalent to the product of a curve and a projective space. Harpaz and Skorobogatov [HS14] construct surfaces with a dominant morphism to a curve with finitely many rational points such that the fibres over rational points are singular unions of curves of genus 0. Colliot-Thélène, Pál and Skorobogatov [CTPS16] construct such varieties X with a dominant map to a curve with finitely many rational points, such that the generic fibre is a quadric of dimension $d \geq 1$. Such varieties are geometrically birationally equivalent to the product of a curve (of genus at least one) and a projective space.

The proof uses the following lemma.

Lemma 13.3.5 *Let $f : X \to B$ be a surjective flat morphism of smooth, proper, geometrically integral varieties over a field k of characteristic 0, the generic fibre of which is a smooth quadric of dimension at least 1 and all geometric fibres are reduced. Then for any torsor $X' \to X$ for a finite k-group scheme G there exists a G-torsor $B' \to B$ such that there is an isomorphism $X' \cong X \times_B B'$ of G-torsors over X.*

Proof. The geometric fibres of f are connected and reduced, and the generic geometric fibre of f is simply connected. By [SGA1, X, Cor. 2.4] this implies that each geometric fibre is simply connected. The result then follows from [SGA1, IX, Cor. 6.8]. □

It is easy to construct examples over $k = \mathbb{Q}$ with $d \geq 2$: in fact, the threefold from Section 13.3.1 is such an example (also reproduced in [Po18, §8.6.2]).

Proposition 13.3.6 *The threefold X considered in Proposition 13.3.1 is such that $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}} \neq \emptyset$ whereas $X(k) = \emptyset$.*

Proof. We keep the notation of the proof of Proposition 13.3.1. There we constructed an adelic point $(N_u) \in X(\mathbf{A}_k)^{\mathrm{Br}}$. Let us show that $(N_u) \in X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$.

Let G be a finite k-group scheme. Lemma 13.3.5 implies that any G-torsor X'/X is isomorphic to $X \times_C C' \to X$ for some G-torsor C'/C. Let $\sigma \in Z^1(k, G)$ be a 1-cocycle defining the k-torsor which is the fibre of $C' \to C$ at P. Twisting X'/X and C'/C by σ and replacing the group G by the twisted group G^σ and changing notation, we can assume that C' contains a k-point P' that maps to P in C. The irreducible component C'' of C' that contains P' is a geometrically integral curve over k. Let $X'' \subset X'$ denote the inverse image of C'' in X'. The fibres of the morphism $X \to C$ are geometrically integral, hence such are also the fibres of $X' \to C'$ and $X'' \to C''$. Thus X'' is a geometrically integral variety over k.

There are natural isomorphisms $X''_{P'} \cong X'_{P'} \cong X_P$, so we can define $N'_u \in X''(k_u)$ as the point that maps to $N_u \in X(k_u)$ for each $u \neq v$. The map $C'' \to C$ is finite and étale. The image of $C''(\mathbb{R})$ in $C(\mathbb{R})$ is thus closed and open. The image of the connected component of $P' \in C''(\mathbb{R})$ is the whole connected component of $P \in C(\mathbb{R})$, hence contains Π. The inverse image of the interval Π in $C''(\mathbb{R})$ is a disjoint union of intervals, one of which contains P' and maps bijectively onto Π. Let us call this interval Π'. Let M' be the unique point of Π' over M. Let $N'_v \in X''_{M'}(\mathbb{R})$ be the point that maps to $N_v \in X_M(\mathbb{R})$. Thus the adelic point $(N'_u) \in X''(\mathbf{A}_k) \subset X'(\mathbf{A}_k)$ projects to the adelic point $(N_u) \in X(\mathbf{A}_k)$. By the definition of the étale Brauer–Manin obstruction, to prove that $(N_u) \in X(\mathbf{A}_k)^{\text{ét,Br}}$ it suffices to show that $(N'_u) \in X'(\mathbf{A}_k)^{\text{Br}}$. This follows by the argument in the last paragraph of Proposition 13.3.1. □

It is more delicate to give examples with $d = 1$, that is, conic bundles.

Theorem 13.3.7 *There exist a real quadratic field k, an elliptic curve E and a smooth, projective and geometrically integral surface X over k with a surjective morphism $f : X \to E$ satisfying the following properties:*

(i) *the fibres of $f : X \to E$ are conics;*

(ii) *there exists a closed point $P \in E$ such that the field $k(P)$ is a totally real biquadratic extension of $\mathbb{Q}$ and the restriction $X \setminus f^{-1}(P) \to E \setminus P$ is a smooth morphism;*

(iii) $X(\mathbf{A}_k)^{\text{ét,Br}} \neq \emptyset$ *and* $X(k) = \emptyset$.

Here one can take $k = \mathbb{Q}(\sqrt{10})$ and take E to be the elliptic curve

$$y^2 + y = x^3 + x^2 - 12x - 21$$

of conductor 67 and discriminant -67. We refer to [CTPS16, §5] for the construction of the conic bundle $f : X \to E$ and the proof of Theorem 13.3.7. Note that in this theorem we cannot take the ground field to be $\mathbb{Q}$, see Proposition 13.3.10. The construction given in [CTPS16] works over a number field with at least two real places, and requires good control of the Galois representation on the torsion points of E over k.

In all these unconditional examples, the varieties have a non-constant map to a curve of genus at least one, hence have a non-trivial Albanese variety. A. Smeets [Sme17] has given examples with trivial Albanese varieties. Under the *abc* conjecture, he even produces examples with trivial geometric fundamental group.

13.3.3 Open questions about the closure of $X(k)$

Let X be a smooth, projective, geometrically integral variety. Define the topological space $X(\mathbf{A}_k)_{\bullet}^{\text{Br}}$ by replacing $X(k_v)$ for each archimedean place v by $\pi_0(X(k_v))$, i.e., by the set of connected components of $X(k_v)$.

Here are some complements to Proposition 13.3.6 and Theorem 13.3.7.

Proposition 13.3.8 *Let E be an elliptic curve over a number field k such that the Tate–Shafarevich group $Ш(E)$ is finite. Let $f: X \to E$ be a Severi–Brauer scheme over E. Then $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$. Moreover, $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}}_{\bullet}$.*

Proof. Since $f: X \to E$ is a projective morphism with smooth geometrically integral fibres, there exists a finite set of places Σ such that $E(k_v) = f(X(k_v))$ for $v \notin \Sigma$. We may assume that Σ contains the archimedean places of k. At an arbitrary place v the set $f(X(k_v))$ is open and closed in $E(k_v)$. Let $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$. By functoriality we then have $(f(M_v)) \in E(\mathbf{A}_k)^{\mathrm{Br}}$. The finiteness of $Ш(E)$ implies [Sko01, Prop. 6.2.4] the exactness of the Cassels–Tate dual sequence

$$0 \longrightarrow E(k) \otimes \hat{\mathbb{Z}} \longrightarrow \prod E(k_v)_{\bullet} \longrightarrow \mathrm{Hom}(\mathrm{Br}(E), \mathbb{Q}/\mathbb{Z}), \tag{13.4}$$

where $E(k_v)_{\bullet} = E(k_v)$ if v is a finite place of k, and $E(k_v)_{\bullet} = \pi_0(E(k_v))$ if v is an archimedean place. By a theorem of Serre, the image of $E(k) \otimes \hat{\mathbb{Z}}$ in that product coincides with the topological closure of $E(k)$, see [Ser64], [Wan96]. Approximating at the places of Σ, we find a k-point $M \in E(k)$ such that the fibre $X_M = f^{-1}(M)$ is a Severi–Brauer variety with points in all k_v for $v \in \Sigma$, hence also for all places v. Since X_M is a Severi–Brauer variety over k, it contains a k-point, hence $X(k) \neq \emptyset$. For the last statement of the theorem we include into Σ the places where we want to approximate. If $k_v \simeq \mathbb{R}$, each connected component $X(k_v)$ maps surjectively onto a connected component of $E(k_v)$. The Severi–Brauer varieties satisfy the Hasse principle and weak approximation, so an application of the implicit function theorem finishes the proof. □

Remark 13.3.9 The same argument works more generally for any projective morphism $f: X \to E$ with split fibres, provided that the smooth k-fibres satisfy the Hasse principle. For the last statement to hold, the smooth k-fibres also need to satisfy weak approximation.

The following proposition explains why a counterexample similar to that of Theorem 13.3.7 cannot be constructed over $\mathbb{Q}$.

Proposition 13.3.10 *Let E be an elliptic curve over a number field k such that both $E(k)$ and $Ш(E)$ are finite. Let $f: X \to E$ be a conic bundle. Suppose that there exists a real place v_0 of k such that for each real place $v \neq v_0$ no singular fibre of $f: X \to E$ is over a k_v-point of E. Then $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$.*

Proof. If a k-fibre of f is not smooth, then this fibre contains a k-point. We may thus assume that the fibres above $E(k)$ are smooth. Let $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$. Then $(f(M_v)) \in E(\mathbf{A}_k)^{\mathrm{Br}}$. Set $N_v = f(M_v)$ for each place v. The finiteness of $Ш(E)$ implies the exactness of (13.4). Hence there exists $N \in E(k)$ such that $N = N_v$ for each finite place v and such that N lies in the same connected component as N_v for v archimedean. The fibre X_N is a smooth conic with points in all finite completions of k. For an archimedean place $v \neq v_0$, the map $X(k_v)\to E(k_v)$ sends each connected component of $X(k_v)$ onto a connected component of $E(k_v)$. Since N and N_v are in the same connected component of $E(k_v)$, this implies $X_N(k_v) \neq \emptyset$. Thus the conic X_N has points in all completions of k except possibly k_{v_0}. By the reciprocity law it has points in all completions of k and hence in k. □

Let us finally mention some open problems concerning $X(k)^{cl}$.

Curves

When X is a curve, it is an open question whether the image of $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}}_\bullet$. If the genus of X is 1, this is the case if the Tate–Shafarevich group of the Jacobian of X is finite [Sko01, Thm. 6.2.3, Cor. 6.2.4]. These results hold more generally when X is a torsor for an abelian variety. If X is a curve of higher genus with Jacobian J such that $Ш(J)$ is finite (which is expected to be always true) and also $J(k)$ is finite, it is a theorem of Scharashkin and (independently) Skorobogatov [Sko01, Cor. 6.2.6] that $X(k) = X(\mathbf{A}_k)^{\mathrm{Br}}_\bullet$. Stoll has shown that the same statement remains true under the weaker assumption that J is isogenous to an abelian variety which has a direct factor A of positive dimension such that $Ш(A)$ and $A(k)$ are both finite [Sto07].

K3 surfaces

For K3 surfaces Skorobogatov conjectured that $X(k)$ should be dense in $X(\mathbf{A}_k)^{\mathrm{Br}}$. There are conditional results in this direction, particularly, but not exclusively, for surfaces which are geometrically Kummer. See [CTSS98b, SkS05, HS16].

Enriques surfaces

Enriques surfaces with interesting $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$ were studied by Harari and Skorobogatov in [HS05], where the following example was constructed. Let Y be the Kummer surface over $\mathbb{Q}$ with affine equation

$$z^2 = (x^2 - a)(x^2 - ab^2)(y^2 - a)(y^2 - ac^2),$$

where $a = 5$, $b = 13$, $c = 2$. Let X be the quotient of Y by the involution that changes the signes of all the coordinates. Then X is an Enriques surface such that $X(\mathbb{Q})$ is not dense in $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$. Following work of Várilly-Alvarado and Viray [VV11], an Enriques surface such that $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}} = \emptyset$, hence $X(k) = \emptyset$, whereas $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, was constructed in [BBMPV].

One may ask if $X(k)^{cl} = X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$ for any Enriques surface X. See [Sko09, §3] for a discussion of this question for arbitrary surfaces of Kodaira dimension 0.

Remark 13.3.11 The property "$X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$" is preserved by birational equivalence of smooth projective varieties over k. If $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite, then the property $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$ is also preserved by birational equivalence of smooth projective varieties over k. When $\mathrm{Br}(X)/\mathrm{Br}(k)$ is infinite, the situation with weak approximation is not clear, but a similar statement concerning the density of $X(k)$ in $X(\mathbf{A}_k)_\bullet^{\mathrm{Br}}$ is wrong [CTPS16, Remark 6.2 (2)].

Chapter 14

The Brauer–Manin obstruction for zero-cycles

The Brauer–Manin obstruction for rational points has an analogue for zero-cycles, which conjecturally governs the local-to-global principle for zero-cycles on an arbitrary smooth projective variety X – unlike the original version for rational points! For example, one expects that if X has a family of local 0-cycles of degree 1 for each completion of k, which is orthogonal to $\mathrm{Br}(X)$ with respect to the Brauer–Manin pairing, then X has a global 0-cycle of degree 1. This is the subject of Section 14.1. In Section 14.2 we discuss the simplest case of Salberger's trick which sometimes allows one to prove these conjectures; this trick can be interpreted as an accessible analogue of Schinzel's Hypothesis (H). If one knows that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for rational points, then in some cases one can conclude that the Brauer–Manin obstruction controls the existence of 0-cycles of degree 1 as well. This work of Y. Liang is presented in Section 14.3. Finally, in Section 14.4 we explain a fibration theorem of Harpaz and Wittenberg, which says that the Brauer–Manin obstruction controls the existence of 0-cycles of degree 1 on a variety fibred over $\mathbb{P}^1_k$ if this property is known for the smooth fibres.

14.1 Local-to-global principles for zero-cycles

The Brauer–Manin pairing for zero-cycles

Let k be a number field. We denote by Ω the set of places of k. For a place $v \in \Omega$ we always identify $\mathrm{Br}(k_v)$ with a subgroup of $\mathbb{Q}/\mathbb{Z}$ via the local invariant inv_v, see Definition 12.1.7.

Let X be a smooth, projective, geometrically integral variety over k. For each place $v \in \Omega$, we have the pairing from Chapter 5.3:

$$\mathrm{CH}_0(X_{k_v}) \times \mathrm{Br}(X_{k_v}) \longrightarrow \mathrm{Br}(k_v) \subset \mathbb{Q}/\mathbb{Z}. \tag{14.1}$$

For an archimedean place v, this pairing vanishes on $\mathrm{N}_v(\mathrm{CH}_0(X_{k'_v}))$, where k'_v is an algebraic closure of k_v and N_v is the natural norm map $\mathrm{CH}_0(X_{k'_v})\to\mathrm{CH}_0(X_{k_v})$. Define

$$\mathrm{CH}'_0(X_{k_v}) = \mathrm{CH}_0(X_{k_v})/\mathrm{N}_v(\mathrm{CH}_0(X_{k'_v}))$$

if v is archimedean, and $\mathrm{CH}'_0(X_{k_v}) = \mathrm{CH}_0(X_{k_v})$ otherwise.

Given an element $\alpha \in \mathrm{Br}(X)$, there exists a finite set $S \subset \Omega$ and a smooth projective model $\mathcal{X}$ over the ring of S-integers $\mathcal{O}_S \subset k$ such that α belongs to the image of $\mathrm{Br}(\mathcal{X})\to\mathrm{Br}(X)$, see Proposition 12.3.1. Let $v \notin S$ be a non-archimedean place. Let M_v be a closed point of X_{k_v} with residue field $L = k_v(M_v)$, and let $\mathcal{O}_L$ be the ring of integers of the local field L. Since $\mathcal{X}/\mathcal{O}_S$ is projective, the map $\mathcal{X}(\mathcal{O}_L)\to X(L)$ is a bijection by the valuative criterion of properness. Let $\tilde{M}_v$ be the point of $\mathcal{X}(\mathcal{O}_L)$ whose image in $X(L)$ is M_v. Then $\alpha(M_v) \in \mathrm{Br}(L)$ is equal to the image of $\alpha(\tilde{M}_v) \in \mathrm{Br}(\mathcal{O}_L)$. But $\mathrm{Br}(\mathcal{O}_L) = 0$ by Theorem 3.4.2 (ii). Thus each $\alpha \in \mathrm{Br}(X)$ pairs trivially with the local Chow groups $\mathrm{CH}_0(X_{k_v})$ for almost all places $v \in \Omega$. Therefore we have a well-defined map

$$\prod_{v\in\Omega} \mathrm{CH}'_0(X_{k_v}) \longrightarrow \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}).$$

Class field theory gives an exact sequence (12.1)

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \bigoplus_{v\in\Omega} \mathrm{Br}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

The mere fact that this is a complex at the middle term implies that the following sequence is a complex too:

$$\mathrm{CH}_0(X) \longrightarrow \prod_{v\in\Omega} \mathrm{CH}'_0(X_{k_v}) \longrightarrow \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}).$$

Here the first map is the diagonal map and the second map is induced by the local pairings (14.1).

For an abelian group A, write $\widehat{A} = \varprojlim A/n$. Since $\mathrm{Br}(X_{k_v})$ is a torsion group, the local pairing (14.1) gives rise to a pairing

$$\widehat{\mathrm{CH}_0(X_{k_v})} \times \mathrm{Br}(X_{k_v}) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

From this we obtain a complex

$$\widehat{\mathrm{CH}_0(X)} \longrightarrow \prod_{v\in\Omega} \widehat{\mathrm{CH}'_0(X_{k_v})} \longrightarrow \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}). \tag{14.2}$$

By [CT95b, Thm. 1.3 (b)], if v is an archimedean place, then $\mathrm{N}_v(A_0(X_{k'_v}))$ is the divisible subgroup of $A_0(X_{k_v})$. Thus restricting (14.2) to the degree 0 subgroups we obtain a complex

$$\widehat{A_0(X)} \longrightarrow \prod_{v\in\Omega} \widehat{A_0(X_{k_v})} \longrightarrow \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}). \tag{14.3}$$

Conjectures

Work of Cassels and Tate on elliptic curves (the Cassels–Tate dual exact sequence), of Colliot-Thélène and Sansuc on geometrically rational surfaces [CTS81], and of Kato and Saito [KS86] on higher class field theory has led to the following general conjecture, which encompasses a number of its predecessors.

Conjecture 14.1.1 (E) *For any smooth, projective variety X over a number field k, the complex (14.2) is exact.*

It subsumes the following two conjectures.

Conjecture 14.1.2 ($\mathbf{E}_1$) *For any smooth, projective variety X over a number field k, if there exists a family $\{z_v\}$ of local 0-cycles of degree 1 on X such that, for all $A \in \mathrm{Br}(X)$, we have*

$$\sum_{v\in\Omega} \mathrm{inv}_v A(z_v) = 0 \in \mathbb{Q}/\mathbb{Z},$$

then there exists a 0-cycle of degree 1 on X.

Conjecture 14.1.3 ($\mathbf{E}_0$) *For any smooth, projective variety X over a number field k, the complex (14.3) is exact.*

For the history of these conjectures, see [CTS81], [KS86, p. 303], [Sai89, §8], [CT95b], [CT99], [vHa03], and the introduction to [Witt12]. Note that these conjectures are about *all* smooth, projective, geometrically connected varieties over number fields. The groups involved are rather mysterious. Indeed, it is a conjecture of Bloch and Beilinson that for a smooth and projective variety X over a number field k, the group $\mathrm{CH}_0(X)$ is finitely generated. The Chow group $\mathrm{CH}_0(X_{k_v})$ over a local field k_v, for $v \in \Omega$, is often a huge group.

In the case $X = \mathrm{Spec}(k)$, conjecture (E) follows from the exact sequence (12.1) of class field theory.

For curves, classical results of Cassels and Tate imply the conjecture – modulo finiteness of Tate–Shafarevich groups. See [Man66] (curves of genus 1), [Sai89], [CT99], [Witt12, Remark 1.1 (iv), p. 2121].

For Châtelet surfaces, i.e. smooth projective models X of surfaces given by an affine equation $y^2 - az^2 = P(x)$, where $a \in k^*$ and $P(x) \in k[x]$ is a separable polynomial of degree 3 or 4, conjectures (E_0) and (E_1) were proved in [CTSS87] by reduction to the theorem $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$, also proved there. Indeed, these surfaces have the very special property that any 0-cycle of degree 1 is rationally equivalent to a k-point. Then Salberger [Salb88], by a very innovative method, to be discussed in Section 14.2, proved the conjectures for arbitrary conic bundles over $\mathbb{P}^1_k$. For varieties fibred over $\mathbb{P}^1_k$, with generic fibre a Severi–Brauer variety, further progress was achieved in papers by Colliot-Thélène, Swinnerton-Dyer, Skorobogatov [CTS94, CTSS98], and Salberger [Salb03].

A series of papers by Colliot-Thélène [CT00], Frossard [Fro03], van Hamel [vHa03], Wittenberg [Witt12] and Y. Liang [Lia12, Lia13a, Lia13b, Lia14, Lia15] established cases of conjecture (E) for fibrations over an arbitrary curve C, when the Tate–Shavarevich group of the Jacobian of C is finite, and the generic fibre is birationally equivalent to a Severi–Brauer variety or to a homogeneous space of a connected linear algebraic group, with various restrictions.

Most of these results are now covered by the work of Harpaz and Wittenberg [HW15].

The smooth projective surface X over $\mathbb{Q}$ with the property $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq \emptyset$ but $X(\mathbb{Q}) = \emptyset$ discovered by Skorobogatov in [Sko99] does not belong to the class of varieties handled in [HW15] (it is not geometrically uniruled). B. Creutz has recently shown that it contains a zero-cycle of degree one [Cr17], as predicted by the conjecture.

14.2 Salberger's trick

Here is a simple case of Salberger's argument [Salb88], as streamlined in [CTS94] and in [CTSS98].

Theorem 14.2.1 *Let k be a number field, $a \in k^*$, $c \in k^*$, and let $P(t) \in k[t]$ be a monic irreducible polynomial of degree d. Assume that the equation*

$$y^2 - az^2 = c\,P(t) \neq 0$$

is solvable in k_v for all $v \in \Omega$. Then we have the following statements.

(i) *For any integer $N \geq d$ the equation has a solution in a field extension of k of degree N.*

(ii) *There exists a zero-cycle of degree 1 on X.*

Proof. Statement (ii) follows from (i) by considering N and $N+1$. Let us prove (i). For simplicity, we shall here assume that a is totally positive, i.e. positive in each real completion of k. Let U be the k-variety defined by

$$y^2 - az^2 = c\,P(t) \neq 0.$$

We can find a finite set of places S containing all the non-archimedean and dyadic places of k, such that at a place $w \notin S$ we have the following properties: $c \in \mathcal{O}_w^*$; the coefficients of $P(t)$ are in $\mathcal{O}_w$; the reduction of P modulo the maximal ideal of $\mathcal{O}_w$ is a separable polynomial.

For each $v \in S$ pick a monic separable polynomial $G_v(t) \in k_v[t]$ of degree N with all its roots in k_v, each of them corresponding to the image of a point of $U(k_v)$. In particular, $G_v(t)$ is coprime to $P(t)$.

Pick a place $v_0 \notin S$ such that a is a square in k_{v_0}. Then choose a monic irreducible polynomial $G_{v_0}(t) \in k_{v_0}[t]$ of degree N over k_{v_0}.

For each $v \in S \cup \{v_0\}$ Euclid's algorithm gives polynomials $Q_v(t)$ and $R_v(t)$ in $k_v[t]$, with $\deg(R_v(t)) < d \leq N$ such that

$$G_v(t) = P(t)Q_v(t) + R_v(t).$$

Hence each $Q_v(t)$ is monic and $\deg(Q_v(t)) = N - d$. Each $R_v[t]$ is coprime to $P(t)$ since $G_v(t)$ is coprime to $P(t)$.

Let K be the field $k[t]/P(t)$. Let $\xi_v \in (K \otimes_k k_v)^*$ be the image of $R_v(t)$.

Dirichlet's theorem 12.1.1 for the field K implies that there is an element $\xi \in K^*$ which is arbitrarily close to each ξ_v for $v \in S \cup \{v_0\}$ and such that its prime decomposition in K involves only primes above the primes in $S \cup \{v_0\}$ and a prime w such that $w(\xi) = 1$ and w has degree 1 over k. The element $\xi \in k[t]/P(t)$ lifts to a unique polynomial $R(t)$ of $k[t]$ such that $\deg(R(t)) < d$.

Choose a place $v_1 \notin S \cup \{v_0\}$ such that a is a square at v_1. If $N > d$ we use strong approximation in k away from v_1 to produce a monic polynomial $Q(t) \in k[t]$ whose coefficients are integral away from v_1 and very close to respective coefficients of $Q_v(t)$ for $v \in S \cup \{v_0\}$. If $N = d$, take $Q(t) = 1$.

One then defines

$$G(t) := P(t)Q(t) + R(t).$$

By Krasner's lemma [Po18, Prop. 3.5.74] this polynomial is irreducible, since it is close to the irreducible polynomial $G_{v_0}(t)$. It is monic and has integral coefficients away from $S \cup \{v_0, v_1\}$.

Let $L = k[t]/G(t)$. This is a field extension of degree N of k. Let $\theta \in L$ be the class of t. The element θ is integral outside $S \cup \{v_0, v_1\}$.

Claim: *The conic over L with equation $y^2 - az^2 = c\,P(\theta)$ has an L-point.*

If w is a place of L above S, then the conic has an L_w-point because $G(t)$ is very close to $G_v(t)$. If w is a place above v_0, then the conic has an L_w-point because a is a square in k_{v_0}. The same applies to v_1. The same also holds for the archimedean places of L by our simplifying assumption.

The formula for the resultant of two polynomials shows that the product of the conjugates of $P(\theta)$ is an element of k^* which is equal, up to sign, to the product of the conjugates of $G(\alpha)$, where α is the class of t in $K = k[t]/P(t)$. Since $P(\alpha) = 0$, the definition of $G(t)$ implies that $G(\alpha) = R(\alpha) = \xi$. The degree 1 condition on the Dirichlet prime w implies that $\mathrm{N}_{K/k}(\xi) \in k^*$, away from $S \cup \{v_0\}$, has in its factorisation only one prime, and that its valuation at this prime is 1. Since $P(\theta) \in L$ is integral away from $S \cup \{v_0, v_1\}$, this implies that the prime decomposition of $P(\theta) \in L$ involves only one prime w' of L not dividing a prime of $S \cup \{v_0, v_1\}$. Thus our conic has points in all completions of L except possibly at the prime w'. Corollary 12.1.10 then implies that it has an L-point. $\square$

Remark 14.2.2 In the above theorem we assumed $U(\mathbf{A}_k) \neq \emptyset$ but we did not assume the existence of an adelic point orthogonal to the unramified Brauer group of U. But this is automatic. Indeed, the hypothesis that $P(t)$ is irreducible implies that the Brauer group of a smooth projective model of $y^2 - az^2 = P(t)$ is reduced to the image of $\mathrm{Br}(k)$, see Section 10.2.

Remark 14.2.3 Salberger's trick may be interpreted as a successful substitute for Schinzel's hypothesis H. Given an irreducible polynomial $P(t)$ over a number field k, it is hard to find an almost integral element $\alpha \in k$ such that $P(\alpha)$ is almost a prime. However, for any $N \geq \deg(P(t))$ one may produce a field extension L of k of degree N and an almost integral element $\beta \in L$ such that $P(\beta)$ is almost a prime in L. There is a similar comparison in the case of a finite set of polynomials (see [CT98, Prop. 17] for the example of twin primes). The above proof then becomes parallel to the proof of Theorem 13.2.2.

14.3 From rational points to zero-cycles

The following proposition is a baby case.

Proposition 14.3.1 [Lia13b, Prop. 3.2.3] *Let k be a number field and let X be a smooth, proper, geometrically integral variety over k. Assume that for any finite field extension K/k, the Hasse principle holds for rational points of X_K. Then the Hasse principle holds for 0-cycles of degree 1 on X.*

Proof. By the Lang–Weil–Nisnevich estimates, there exists a finite set S of places of k such that for any place $v \notin S$, one has $X(k_v) \neq \emptyset$. Fix a closed point m of some degree N over X. For each $v \in S$, let $z_v = z_v^+ - z_v^-$ be a local 0-cycle of degree 1 where z_v^+ and z_v^- are effective 0-cycles. Let $z_v^1 = z_v^+ + (N-1)z_v^-$. This is an effective 0-cycle of degree congruent to 1 modulo N. Since S is finite, we can add to each z_v^1 a suitable positive multiple $n_v m$ of the closed point m and ensure that all the effective cycles $z_v^2 = z_v^1 + n_v m$, for $v \in S$, have the same common degree d congruent to 1 modulo N.

Here comes the basic trick. Let $Y = X \times_k \mathbb{P}^1_k$ and let $f : Y \to \mathbb{P}^1_k$ be the natural projection. Fix a rational point $q \in \mathbb{P}^1(k)$. On Y we have the effective 0-cycles $z_v^2 \times q$ of degree d.

A moving lemma based on the implicit function theorem (Theorem 9.5.1, see also [Po18, Prop. 3.5.73]) ensures that there exists an effective 0-cycle z_v^3 on Y very close to $z_v^2 \times q$ and such that z_v^3 and $f_*(z_v^3)$ are "reduced". This means that $z_v^3 = \sum_j R_j$ with distinct closed points R_j on Y_{k_v} and $f : R_j \to f(R_j)$ is an isomorphism for each j. We may assume that for each $v \in S$, the support of the 0-cycle $f_*(z_v^3)$ lies in $\operatorname{Spec} k[t] = \mathbb{A}^1_k \subset \mathbb{P}^1_k$. Each $f_*(z_v)$ is defined by a separable monic polynomial $P_v(t) \in k_v[t]$. We pick a finite place v_0 outside S and an arbitrary monic irreducible polynomial $P_{v_0}(t) \in k_{v_0}[t]$. By weak approximation on the coefficients, we then approximate the $P_v(t)$, for $v \in S \cup v_0$, by a monic polynomial $P(t) \in k[t]$. The polynomial $P(t)$ is irreducible, hence it defines a closed point $M \in \mathbb{A}^1_k$ of degree d.

If the approximation is close enough, Krasner's lemma [Po18, Prop. 3.5.74] and the implicit function theorem (Theorem 9.5.1, [Po18, Prop. 3.5.73]) imply that the fibre $f^{-1}(M) = X \times_k k(M)$ has points in all completions of $k(M)$ at the places above $v \in S$. By the choice of S, $X \times_k k(M)$ has points in all the other completions. By assumption, $X \times_k k(M)$ satisfies the Hasse principle over $k(M)$, hence it has a $k(M)$-point. Thus X has a point in an extension of degree d. As d is congruent to 1 mod N, and the closed point m has degree N, we conclude that the k-variety X has a 0-cycle of degree 1. □

Theorem 14.3.2 (Y. Liang) *Let k be a number field and X a smooth, projective, geometrically integral variety over k. Assume that $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$ and that the geometric Picard group $\mathrm{Pic}(\overline{X})$ is torsion-free. For any finite field extension K of k, assume that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for rational points of X_K. Then the Brauer–Manin obstruction to the existence of a 0-cycle of degree 1 on X is the only obstruction: conjecture (E_1) holds for X.*

Proof. Over any field k of characteristic 0, the assumptions on the geometry of X imply that $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite (Theorem 4.4.2). Let $A_1, \ldots, A_n$ be elements of $\mathrm{Br}(X)$ whose images generate $\mathrm{Br}(X)/\mathrm{Br}_0(X)$.

Let S be a finite set of places containing the archimedean places, such that X has a good projective model $\mathcal{X}$ over the ring $\mathcal{O}_S$ of S-integers, the elements A_i extend to elements of the Brauer group of $\mathcal{X}$, and $X(k_v) \neq \emptyset$ for $v \notin S$. In particular, for each $v \notin S$, each A_i vanishes when evaluated on any 0-cycle of X_{k_v}.

Suppose that we have a family of 0-cycles z_v of degree 1 on X_{k_v}, where $v \in \Omega$, which is orthogonal to $\mathrm{Br}(X)$ with respect to the Brauer–Manin pairing. This is equivalent to the condition

$$\sum_{v \in \Omega} \mathrm{inv}_v A_i(z_v) = 0 \in \mathbb{Q}/\mathbb{Z}, \quad \text{for all} \quad i = 1, \ldots, n.$$

Let N be an integer which is a multiple of the degree of a closed point of X and also annihilates each $A_i \in \mathrm{Br}(X)$.

Let $Y = X \times_k \mathbb{P}^1_k$ and $f : Y \to \mathbb{P}^1_k$ be the projection. Proceeding as in the previous proof, we replace the original 0-cycles z_v, $v \in S$, by reduced effective 0-cycles z'_v on Y each of the same degree d congruent to 1 modulo N, with the property that $f_*(z'_v)$ is reduced. We may choose coordinates so that the support of $f_*(z'_v)$ lies in $\mathrm{Spec}\, k[t] = \mathbb{A}^1_k \subset \mathbb{P}^1_k$. They are then defined by the vanishing of separable, monic polynomials $P_v(t)$ of degree d. One then approximates the $P_v(t)$ for $v \in S$ and an irreducible monic polynomial $P_{v_0} \in k_{v_0}[t]$ at another place v_0 by a monic polynomial $P(t) \in k[t]$. Just as before, $P(t)$ defines a closed point $M \in \mathbb{P}^1_k$.

For each place $v \in S$, we have the effective zero-cycle z'_v close to z_v on $X_M \otimes_k k_v$. This gives rise to $k(M)_w$-rational points R_w of the $k(M)$-variety $X_{k(M)}$ over the completions of $k(M)$ at the places w above the places in S.

At each place w of $k(M)$ above a place $v \notin S$, we take an arbitrary $k(M)_w$-point, for instance, a point coming from a k_v-point on X.

Then we have

$$\sum_{w \in \Omega_{k(M)}} \mathrm{inv}_w A_i(R_w) = 0 \in \mathbb{Q}/\mathbb{Z}, \quad \text{for all} \quad i = 1, \ldots, n.$$

This is enough to ensure that the adelic point $\{R_w\} \in X_{k(M)}(\mathbf{A}_{k(M)})$ is orthogonal to $\mathrm{Br}(X_{k(M)})$ provided we can choose the point M, i.e. the polynomial $P(t)$, in such a way that the map

$$\mathrm{Br}(X)/\mathrm{Br}(k) \longrightarrow \mathrm{Br}(X_{k(M)})/\mathrm{Br}(k(M))$$

is surjective. By [Lia13b, Prop. 3.1.1] (an easy special case of a more general theorem of Harari [Har97, Thm. 2.3.1]), the geometric assumptions on X imply that there exists a finite Galois extension L of k such that the above surjectivity holds for any closed point M as long as the tensor product $L \otimes_k k(M)$ is a field. But this last condition is easy to ensure. Indeed, it is enough to require from the very beginning that N is also a multiple of $[L : k]$. Then $d = [k(M) : k]$, being congruent to 1 modulo N, is prime to $[L : k]$. □

Liang [Lia13b, Thm. A and Thm. B] proves the following general result.

Theorem 14.3.3 *Let k be a number field. Let X be a smooth, projective, geometrically integral, rationally connected variety over k. Assume that for any finite field extension K of k, the set $X(K)$ is dense in $X(\mathbf{A}_K)^{\mathrm{Br}}$. Then conjecture (E) holds for X.*

To prove this, Liang first proves a version of the previous theorem for 0-cycles of degree 1, keeping track of "approximation" modulo a positive integer. Here z is said to be close to z_v modulo n if z and z_v have the same image in $\mathrm{CH}_0(X_{k_v})/n$. This uses work of Wittenberg [Witt12]. Using that X is a rationally connected variety, one then proceeds from this statement to the exact sequence (E). This uses results of Kato–Saito and Saito–Sato on the Chow groups of 0-cycles of rationally connected varieties over local fields in the good reduction case.

The results of Liang thus establish Conjecture (E) for smooth projective varieties which are birationally equivalent to a homogeneous space of a connected linear algebraic group with connected stabilisers, since the conjecture $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$ is known for such varieties (Sansuc [San81] when the stabilisers are trivial, Borovoi [Bor96] in general). Until [Lia13b] the validity of (E) was unknown even for smooth compactifications of 3-dimensional tori.

14.4 Fibration theorem for zero-cycles

The various papers quoted at the very end of Section 14.1 were inspired by Salberger's paper [Salb88]. In a manner parallel to the case of rational points, Salberger's method allowed one to obtain unconditional results for 0-cycles on fibrations $X \to \mathbb{P}^1_k$ under the following assumptions:

- for any closed point $m \in \mathbb{P}^1_k$, the fibre X_m contains an irreducible component Y of multiplicity 1 such that the integral closure of $k(m)$ in $k(Y)$ is *abelian*;
- the Hasse principle and weak approximation hold for the smooth closed fibres of $X \to \mathbb{P}^1_k$.

If all fibres over closed points of $\mathbb{A}^1_k \subset \mathbb{P}^1_k$ are split, the second assumption can be weakened [CT10, Lia12] using arguments similar to those of Harari [Har94, Har07] in the case of rational points.

These restrictions on the algebra and arithmetic of fibres have now been removed. In fact, we have the following unconditional result [HW15].

Theorem 14.4.1 (Harpaz–Wittenberg) *Let X be a smooth, projective, geometrically integral variety over a number field k, and let $f : X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the geometric generic fibre is a rationally connected variety. If the smooth fibres satisfy Conjecture* (E)*, then X satisfies Conjecture* (E)*.*

Corollary 14.4.2 *Let X be a smooth, projective, geometrically integral variety over a number field k, and let $f : X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is birationally equivalent to a homogeneous space of a connected linear algebraic group over $k(\mathbb{P}^1)$ with connected geometric stabilizers. Then Conjecture* (E) *holds for X.*

This corollary can be applied to smooth projective models of varieties given by a system of equations

$$\mathrm{N}_{K_i/k}(\Xi_i) = P_i(t), \quad i = 1, \ldots, n,$$

where K_i is a finite étale k-algebra (for example, a finite field extension) and $P_i(t) \in k[t]$, for each $i = 1, \ldots, n$.

Harpaz and Wittenberg actually prove their result for varieties fibred over a smooth projective curve C of arbitrary genus, under the assumption that Conjecture (E) holds for C, for instance when the Tate–Shafarevich group of the Jacobian of C is finite.

We shall only describe one idea in the proof of Theorem 14.4.1. This is a 0-cycle analogue of Theorem 13.2.16.

Theorem 14.4.3 *Let X be a smooth, projective, geometrically integral variety over a number field k, and let $f : X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that all non-split fibres of f are above k-points of $\mathbb{A}^1_k$, say given by $t = e_i \in k$, where $i = 1, \ldots, n$ and t is a coordinate in $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$. Assume that each non-split fibre contains an irreducible component of multiplicity* 1*. If there exists an adelic point $\{P_v\} \in X(\mathbf{A}_k)$ which is orthogonal to $\mathrm{Br}_{\mathrm{vert}}(X)$, then for any integer $N \geq n$ there exists a closed point $m \in \mathbb{P}^1_k$ of degree N such that the fibre X_m has points in all the completions of $k(m)$.*

Proof. Write $P(t) = \prod_{i=1}^n (t - e_i)$. Let $U \subset \mathbb{A}^1_k$ be the open set given by $P(t) \neq 0$, and let $V = f^{-1}(U)$. Fix an irreducible component $E_i \subset X_{e_i}$ of multiplicity 1, and let k_i be the integral closure of k in $k(E_i)$. The smooth locus $E_{i,\mathrm{smooth}}$ is a geometrically integral variety over k_i.

Since $X(\mathbf{A}_k) \neq \emptyset$, the natural map $\mathrm{Br}(k) \to \mathrm{Br}_{\mathrm{vert}}(X) \subset \mathrm{Br}(X)$ is injective. By Corollary 10.1.5, the assumption on the multiplicity of E_i implies that $\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}(k)$ is a finite group. Using this, by a small deformation argument we can assume that $\{P_v\} \in V(\mathbf{A}_k)$.

Let T be the product of norm 1 k-tori attached to the extensions k_i/k, for $i = 1, \ldots, n$. Consider the T-torsor over U given by the system of equations

$$t - e_i = \mathrm{N}_{k_i/k}(\Xi_i), \quad i = 1, \ldots, n.$$

Its pullback to V is a T-torsor over V. Since there is no vertical Brauer–Manin obstruction for rational points, by the formal lemma for torsors and rational points (Theorem 12.6.5) applied to this torsor over V, there exist $b_i \in k^*$, for $i = 1, \ldots, n$, and for each $v \in \Omega$, $\alpha_v \in f(V(k_v)) \subset U(k_v) \subset k_v$, such that the system of equations

$$\alpha_v - e_i = b_i \mathrm{N}_{k_i/k}(\Xi_i) \neq 0, \quad i = 1, \ldots, n,$$

has solutions with $\Xi_i \in (k_i \otimes_k k_v)^*$. Let $S \subset \Omega$ be a finite set of places containing the infinite places, the primes where at least one of the extensions k_i/k is ramified, the primes of bad reduction for X, the primes where at least one e_i is not integral, then the primes v dividing some $e_i - e_j$, where $i \neq j$, and the primes where b_i is not a unit. Then $f : X \to \mathbb{P}^1_k$ extends to a dominant morphism $\mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_S}$, where $\mathcal{X}$ is proper over $\mathcal{O}_S$. Using the Lang–Weil–Nisnevich estimates, we arrange that for any closed point $s \in \mathbb{P}^1_{\mathcal{O}_S}$ such that the fibre $\mathcal{X}_s$ is split, $\mathcal{X}_s$ has a smooth rational point over the residue field of s. This is achieved by including in S enough places with small residue characteristic.

By Chebotarev's theorem (Theorem 12.1.3), there are infinitely many primes outside S that completely split in each of the extensions k_i/k. Let v_0 and v_1 be such primes.

By the implicit function theorem (Theorem 9.5.1, [Po18, Prop. 3.5.73]) for each $v \in S$ we can find pairwise distinct elements $\alpha_r^v \in f(V(k_v)) \subset U(k_v) \subset k_v$, for $r = 1, \ldots, N$, that are close to α_v. In particular, we can arrange that for any $v \in S$ and $i = 1, \ldots, n$ we have

$$b_i^{-1}(\alpha_r^v - e_i) \in \mathrm{N}_{k_i/k}((k_i \otimes_k k_v)^*) \subset k_v^*.$$

For each place $v \in S$, we define

$$G_v(t) = \prod_{r=1}^{N} (t - \alpha_r^v).$$

We note that $G_v(e_i)$ is the product of a global element $(-1)^N b_i^N \in k$ and an element of $\mathrm{N}_{k_i/k}((k_i \otimes_k k_v)^*) \subset k_v^*$. Let $G_{v_1}(t) \in \mathcal{O}_{v_1}[t]$ be a monic irreducible polynomial of degree N with integral coefficients. Dividing $G_v(t)$ by $P(t)$ in $k_v[t]$, for $v \in S \cup \{v_1\}$, and using Lagrange interpolation, we obtain

$$G_v(t) = P(t)Q_v(t) + \sum_{i=1}^{n} G_v(e_i) \frac{\prod_{j \neq i}(t - e_j)}{\prod_{j \neq i}(e_i - e_j)},$$

where the polynomials $Q_v(t)$ are monic of degree $N - n$.

Applying Proposition 12.1.4, for each $i = 1, \ldots, n$ we find an element $c_i \in k$ close to $G_v(e_i)$ for $v \in S \cup \{v_1\}$ and such that for any $v \notin S \cup \{v_1\}$ either c_i is a unit at v, or k_i has a place of degree 1 over v. Moreover, we choose c_i integral away from $S \cup \{v_0, v_1\}$.

Using strong approximation in k away from v_0 for the coefficients of polynomials, we find $Q(t) \in k[t]$ with coefficients integral away from $S \cup \{v_0, v_1\}$ and close to each $Q_v(t)$ for $v \in S \cup \{v_1\}$ coefficient-wise. Consider the polynomial

$$G(t) = P(t)Q(t) + \sum_{i=1}^{n} c_i \frac{\prod_{j\neq i}(t - e_j)}{\prod_{j\neq i}(e_i - e_j)}.$$

By construction, $G(e_i) = c_i$ for $i = 1, \ldots, n$. Also, the coefficients of $G(t)$ are integral away from $S \cup \{v_0, v_1\}$. Moreover, in the v-adic topology, where $v \in S \cup \{v_1\}$, the element c_i is close to $G_v(e_i)$ and $Q(t)$ is close to $Q_v(t)$, hence $G(t)$ is close to $G_v(t)$. Since $G_{v_1}(t)$ is irreducible in $k_{v_1}[t]$, we see that $G(t)$ is irreducible in $k[t]$.

Write $F = k[t]/(G(t))$, so that $m = \mathrm{Spec}(F)$ is the closed point of $U \subset \mathbb{A}^1_k$ defined by $G(t) = 0$. We claim that X_m has points in all completions of F.

If w is a place of F over $v \in S$, then $G(t)$ is v-adically close to $G_v(t) = \prod_{r=1}^{N}(t - \alpha_r^v)$. But each $\alpha_r^v \in k_v$ lifts to $V(k_v)$, proving the claim for such w.

The primes of F not above S are closed points of $\mathbb{P}^1_{\mathcal{O}_S}$. We only need to consider the finitely many closed points in $\mathbb{P}^1_{\mathcal{O}_S}$ where the closure of m in $\mathbb{P}^1_{\mathcal{O}_S}$ meets the closure of one of the e_i's. Indeed, S was chosen big enough so that the fibre of $\mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_S}$ above any other closed point is split, and then by the Lang–Weil–Nisnevich estimates has a smooth rational point over the residue field.

Let w be a closed point of $\mathbb{P}^1_{\mathcal{O}_S}$ contained in the closure of m and in the closure of e_i. The degree of w is 1 since the degree of e_i is 1. This closed point w lies above a prime $v \notin S$ dividing $c_i = G(e_i)$. Let us first consider the case $v \neq v_0, v_1$. By the construction of c_i, the field k_i has a place of degree 1 over v. This implies that the fibre $\mathcal{X}_w$ is split. By the choice of S, we have that $\mathcal{X}_w$ has a smooth rational point over the residue field. As $\mathcal{X}_w$ is the reduction of X_m at the place w of $F = k(m)$, we can apply Hensel's lemma to deduce that X_m has a point in the completion F_w.

It remains to deal with v_0 and v_1. Recall that these primes are split in all extensions k_i/k. This implies that all the fibres of $X \times_k k_{v_0} \to \mathbb{P}^1_{v_0}$ are split. The same argument works for v_1. $\square$

Remark 14.4.4 The argument in Theorem 14.2.1 only shows the existence of a fibre X_M over a closed point M which has points in all completions of $k(M)$ except, possibly, one. In the above theorem, whose proof uses Proposition 12.1.4, we get a fibre with points in *all completions* of $k(M)$.

Chapter 15

Abelian varieties and K3 surfaces

Let k be a field *finitely generated* over its prime subfield and let X be a smooth and proper variety over k. Recall that $\Gamma = \mathrm{Gal}(k_\mathrm{s}/k)$. This section is motivated by the question: is the group $\mathrm{Br}(X^\mathrm{s})^\Gamma$ finite? A more accessible question concerns the finiteness of the subgroup of $\mathrm{Br}(X^\mathrm{s})^\Gamma$ formed by the elements of order prime to $\mathrm{char}(k)$. The first main result of this chapter says that this holds when X is an abelian variety. This is proved in Section 15.3, after a preliminary discussion of the Galois action on cohomology in Section 15.1 and an observation in Section 15.2 that the ℓ-adic Tate conjecture for divisors is equivalent to the finiteness of $\mathrm{Br}(X^\mathrm{s})\{\ell\}^\Gamma$. In Section 15.4 we discuss the Brauer group of a product of varieties, and deduce finiteness results for the Brauer group of a variety dominated by a product of curves. We recall the basic properties of K3 surfaces such as their Hodge structure and the period map in Section 15.5. Section 15.6 introduces the original Kuga–Satake construction [KS67] and its interpretation by Deligne [Del72]. A modern incarnation of this construction in terms of Shimura varieties carrying universal families is given in Section 15.7. Finally, the last section contains the proofs of the main results in the case of K3 surfaces: the Tate conjecture for divisors and the finiteness of the subgroup of $\mathrm{Br}(X^\mathrm{s})^\Gamma$ consisting of the elements of order prime to $\mathrm{char}(k)$.

15.1 The Tate module of the Brauer group as a Galois representation

Let k be a field with separable closure k_s and absolute Galois group $\Gamma = \mathrm{Gal}(k_\mathrm{s}/k)$. Let X be a smooth, proper, geometrically integral variety over k. Let ℓ be a prime not equal to the characteristic of k. From Theorem 4.2.6 we know that $\mathrm{Br}(X^\mathrm{s})\{\ell\}$ is an abelian group of cofinite type. More precisely, it is an extension of a finite abelian group by the divisible subgroup

$$\mathrm{Br}^0(X^{\mathrm{s}})\{\ell\} = T_\ell(\mathrm{Br}(X^{\mathrm{s}})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2-\rho}.$$

Let $V_\ell(\mathrm{Br}(X^{\mathrm{s}})) = T_\ell(\mathrm{Br}(X^{\mathrm{s}})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. This is a vector space over $\mathbb{Q}_\ell$ of dimension $b_2 - \rho$. Write cl_ℓ for the ℓ-adic cycle class map $\mathrm{NS}(X^{\mathrm{s}}) \to \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$. Tensoring the terms of (4.6) with $\mathbb{Q}_\ell$ we obtain an exact sequence of Γ-modules

$$0 \longrightarrow \mathrm{NS}(X^{\mathrm{s}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \xrightarrow{\mathrm{cl}_\ell} \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell(1)) \longrightarrow V_\ell(\mathrm{Br}(X^{\mathrm{s}})) \longrightarrow 0. \qquad (15.1)$$

We write $\nu = \mathrm{NS}(X^{\mathrm{s}})_{\mathrm{tors}}$ and write ν_ℓ for the ℓ-primary subgroup of ν. As was pointed out in the discussion following (4.6) we have canonical isomorphisms

$$\nu_\ell = \mathrm{NS}(X^{\mathrm{s}})_{\mathrm{tors}} \otimes \mathbb{Z}_\ell = \mathrm{NS}(X^{\mathrm{s}})\{\ell\} = \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}}.$$

Proposition 15.1.1 *Let X be a smooth, projective, geometrically integral variety over a field k. For a prime $\ell \neq \mathrm{char}(k)$ we have the following statements.*

(i) *The exact sequence of Γ-modules (15.1) splits, and so gives a direct sum decomposition of Γ-modules*

$$\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell(1)) \cong (\mathrm{NS}(X^{\mathrm{s}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell) \oplus V_\ell(\mathrm{Br}(X^{\mathrm{s}})). \qquad (15.2)$$

(ii) *For almost all primes ℓ the Γ-module $\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell$ is a direct summand of $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$.*

(iii) *For almost all primes ℓ and all positive integers n the Γ-module $\mathrm{NS}(X^{\mathrm{s}})/\ell^n$ is a direct summand of $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^n})$.*

(iv) *For almost all primes ℓ and all positive integers n there is an exact sequence of abelian groups*

$$0 \longrightarrow (\mathrm{NS}(X^{\mathrm{s}})/\ell^n)^\Gamma \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^n})^\Gamma \longrightarrow \mathrm{Br}(X^{\mathrm{s}})[\ell^n]^\Gamma \longrightarrow 0. \qquad (15.3)$$

The proof is based on the following general fact.

Lemma 15.1.2 *Let X be a smooth, projective, geometrically integral variety over a field k such that $d = \dim(X) \geq 2$. Let $L \in \mathrm{NS}(X^{\mathrm{s}})$ be the class of a very ample line bundle on X. Then the integral symmetric bilinear form on $\mathrm{NS}(X^{\mathrm{s}})/\nu$ given by*

$$(x, y) \mapsto x \cup y \cup L^{d-2}$$

is non-degenerate, i.e. its kernel is trivial.

Proof. Let $\mathrm{NS}(X^{\mathrm{s}})^L \subset \mathrm{NS}(X^{\mathrm{s}})$ be the kernel of the map $\mathrm{NS}(X^{\mathrm{s}}) \to \mathbb{Z}$ given by $x \mapsto x \cup L^{d-1}$. It is clear that $\nu \subset \mathrm{NS}(X^{\mathrm{s}})^L$. Since $L^d > 0$ and $\mathrm{NS}(X^{\mathrm{s}})^L$ is the orthogonal complement to L, it is enough to show that the restriction of our form to $\mathrm{NS}(X^{\mathrm{s}})^L/\nu$ is negative definite. If $d = 2$ this statement is a consequence of the Hodge index theorem when $\mathrm{char}(k) = 0$, but is actually true in all characteristics [Gro58].

The case $d \geq 3$ is reduced to the case $d = 2$ as follows. Consider the symmetric bilinear form on $\mathrm{H}^2_{\text{ét}}(X^{\text{s}}, \mathbb{Z}_\ell(1))$ with values in $\mathbb{Z}_\ell$ defined by

$$(x, y) \mapsto x \cup y \cup \mathrm{cl}_\ell(L)^{d-2}.$$

This form is compatible with the form $x \cup y \cup L^{d-2}$ on $\mathrm{NS}(X^{\text{s}})$ under cl_ℓ.

The field k_{s} is infinite, so by the Bertini theorem there is a smooth hyperplane section $Y \subset X^{\text{s}}$ defined over k^{s}. By the hyperplane (weak) Lefschetz theorem, the restriction map

$$r : \mathrm{H}^2_{\text{ét}}(X^{\text{s}}, \mathbb{Z}_\ell(1)) \longrightarrow \mathrm{H}^2_{\text{ét}}(Y, \mathbb{Z}_\ell(1))$$

is an isomorphism for $d \geq 4$ and an injection for $d = 3$, see [Kat04, Thm. B.4]. For any $x, y \in \mathrm{H}^2_{\text{ét}}(X^{\text{s}}, \mathbb{Z}_\ell(1))$ we have

$$x \cup y \cup \mathrm{cl}_\ell(L)^{d-2} = r(x) \cup r(y) \cup \mathrm{cl}_\ell(L|_Y)^{d-3} \ \in \ \mathbb{Z}_\ell.$$

Similarly, for $x, y \in \mathrm{NS}(X^{\text{s}})$ we have

$$x \cup y \cup L^{d-2} = r(x) \cup r(y) \cup (L|_Y)^{d-3} \ \in \ \mathbb{Z}.$$

The natural restriction map $\mathrm{NS}(X^{\text{s}}) \otimes \mathbb{Z}_\ell \to \mathrm{NS}(Y) \otimes \mathbb{Z}_\ell$ is identified with the map $r : \mathrm{cl}_\ell(\mathrm{NS}(X^{\text{s}})) \otimes \mathbb{Z}_\ell \to \mathrm{cl}_\ell(\mathrm{NS}(Y)) \otimes \mathbb{Z}_\ell$, which is injective since $d \geq 3$. Applying this argument $d-2$ times we obtain a smooth k^{s}-surface $S \subset X^{\text{s}}$ such that the natural map $\mathrm{NS}(X^{\text{s}}) \otimes \mathbb{Z}_\ell \subset \mathrm{NS}(S) \otimes \mathbb{Z}_\ell$ is injective. This gives rise to an injective map $\mathrm{NS}(X^{\text{s}})/\nu \subset \mathrm{NS}(S)/\mathrm{NS}(S)_{\text{tors}}$. Moreover, the restriction of the integral bilinear form $x \cup y$ on $\mathrm{NS}(S)/\mathrm{NS}(S)_{\text{tors}}$ to $\mathrm{NS}(X^{\text{s}})/\nu$ is our original form $x \cup y \cup L^{d-2}$.

Define $\mathrm{NS}(S)^L \subset \mathrm{NS}(S)$ as the orthogonal complement to the restriction of L to S with respect to $x \cup y$. Then $\mathrm{NS}(X^{\text{s}})^L/\nu \subset \mathrm{NS}(S)^L/\mathrm{NS}(S)_{\text{tors}}$. The form $x \cup y$ is negative definite on $\mathrm{NS}(S)^L/\mathrm{NS}(S)_{\text{tors}}$, hence the form $x \cup y \cup L^{d-2}$ is negative definite, hence non-degenerate, on $\mathrm{NS}(X^{\text{s}})^L/\nu$. □

Proof of Proposition 15.1.1. Let $d = \dim(X)$. For $d = 1$ all statements are trivial, so we can assume $d \geq 2$. Let $L \in \mathrm{NS}(X^{\text{s}})$ be the class of a very ample divisor defined over k. Thus $L \in \mathrm{NS}(X^{\text{s}})^\Gamma$.

Let us prove (i). Deligne's hard Lefschetz theorem [Del80] (valid in all characteristics) says that the map

$$\mathrm{H}^2_{\text{ét}}(X^{\text{s}}, \mathbb{Q}_\ell(1)) \longrightarrow \mathrm{H}^{2d-2}_{\text{ét}}(X^{\text{s}}, \mathbb{Q}_\ell(d-1))$$

sending x to $x \cup \mathrm{cl}_\ell(L)^{d-1}$ is an isomorphism. Now Poincaré duality implies that the symmetric bilinear form $x \cup y \cup \mathrm{cl}_\ell(L)^{d-2}$ on $\mathrm{H}^2_{\text{ét}}(X^{\text{s}}, \mathbb{Q}_\ell(1))$ is non-degenerate. Since L is Γ-invariant, the above form is Γ-invariant too. Its restriction to $\mathrm{NS}(X^{\text{s}}) \otimes \mathbb{Q}_\ell$ is non-degenerate by Lemma 15.1.2. Hence $\mathrm{H}^2_{\text{ét}}(X^{\text{s}}, \mathbb{Q}_\ell(1))$ is the direct sum of the Γ-module $\mathrm{NS}(X^{\text{s}}) \otimes \mathbb{Q}_\ell$ and its orthogonal complement with respect to $x \cup y \cup \mathrm{cl}_\ell(L)^{d-2}$, which is isomorphic to the Γ-module $V_\ell(\mathrm{Br}(X^{\text{s}}))$. This gives (15.2).

Let us prove (ii). Let $\delta \in \mathbb{Z}$ be the discriminant of the integral symmetric bilinear form $x \cup y \cup L^{d-2}$ on $\mathrm{NS}(X^{\mathrm{s}})/\nu$. By Lemma 15.1.2 we have $\delta \neq 0$. Suppose that ℓ does not divide the order of ν, then $\nu_\ell = 0$ and both $\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell$ and $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$ are free $\mathbb{Z}_\ell$-modules. Suppose also that $(\ell, \delta) = 1$. The form $x \cup y \cup L^{d-2}$ extends to the form $x \cup y \cup \mathrm{cl}_\ell(L)^{d-2}$ on $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$. Since ℓ does not divide δ, we see that $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$ is the direct sum of $\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell$ and its orthogonal complement. The form $x \cup y \cup \mathrm{cl}_\ell(L)^{d-2}$ is Γ-invariant since L is Γ-invariant, hence we obtain a direct sum of Γ-modules, so (ii) is proved.

Now we prove (iii). By part (ii) we know that the Γ-module $\mathrm{NS}(X^{\mathrm{s}})/\ell^n$ is a direct summand of the Γ-module $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))/\ell^n$ for almost all ℓ. We have an exact sequence

$$0 \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))/\ell^n \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^n}) \longrightarrow \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))[\ell^n] \longrightarrow 0.$$

By a theorem of Gabber [Ga83], the $\mathbb{Z}_\ell$-module $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell)$ has no torsion for almost all ℓ. Since $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell)$ and $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$ are isomorphic as abelian groups, for almost all ℓ we have $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))[\ell] = 0$, hence $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^n}) = \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))/\ell^n$. This proves (iii).

Finally, (iv) follows directly from (iii). □

15.2 Tate conjecture for divisors

When k is big (for example, $k = \bar{k}$) and X does not satisfy the conditions of Theorem 4.4.2, the Brauer group $\mathrm{Br}(X)$ can well be infinite. When k is not too big, there are reasons to hope for some kind of finiteness for the transcendental Brauer group of X. The fields that are "not too big" include number fields, but also fields that are finitely generated over a prime subfield. So here is the motivating question of this chapter.

Question. Let X be a smooth, proper and geometrically integral variety over a field k that is *finitely generated* over a prime subfield. Is $\mathrm{Br}(X^{\mathrm{s}})^\Gamma$ finite?

Recall that the *Tate conjecture for divisors* says that if k is a field finitely generated over a prime subfield, then for any prime $\ell \neq \mathrm{char}(k)$ the natural inclusion

$$(\mathrm{NS}(X^{\mathrm{s}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)^\Gamma \hookrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell(1))^\Gamma$$

should be an isomorphism.

Theorem 15.2.1 *Let X be a smooth, projective and geometrically integral variety over a field k which is finitely generated over a prime subfield. Let ℓ be a prime not equal to the characteristic of k. The ℓ-adic Tate conjecture for divisors holds for X if and only if* $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma$ *is finite.*

Proof. In Proposition 15.1.1 we proved that there is a direct sum decomposition of Γ-modules (15.2):

$$\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell(1)) \cong (\mathrm{NS}(X^{\mathrm{s}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell) \oplus V_\ell(\mathrm{Br}(X^{\mathrm{s}})).$$

Therefore the ℓ-adic Tate conjecture is false for X if and only if $V_\ell(\mathrm{Br}(X^{\mathrm{s}}))$ contains a non-zero Γ-invariant element. Thus it is enough to prove that $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma$ is infinite if and only if $V_\ell(\mathrm{Br}(X^{\mathrm{s}}))^\Gamma \neq 0$.

Let $S(n)$ be the set of Γ-invariant elements of order ℓ^n in $\mathrm{Br}(X^{\mathrm{s}})$. If there is a positive integer m such that $S(m) = \emptyset$, then $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma$ is contained in the finite group $\mathrm{Br}(X^{\mathrm{s}})[\ell^{m-1}]$. So if $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma$ is infinite, then $S(n) \neq \emptyset$ for each $n \geq 1$. The projective limit of finite non-empty sets is non-empty, hence $\varprojlim S(n)$ is a non-empty subset of $T_\ell(\mathrm{Br}(X^{\mathrm{s}}))^\Gamma$ which does not contain 0. This implies that $T_\ell(\mathrm{Br}(X^{\mathrm{s}}))^\Gamma \neq 0$ and hence $V_\ell(\mathrm{Br}(X^{\mathrm{s}}))^\Gamma \neq 0$.

Conversely, if $V_\ell(\mathrm{Br}(X^{\mathrm{s}}))^\Gamma \neq 0$, then $T_\ell(\mathrm{Br}(X^{\mathrm{s}}))^\Gamma$ contains a copy of $\mathbb{Z}_\ell$. But then

$$\big(T_\ell(\mathrm{Br}(X^{\mathrm{s}})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell\big)^\Gamma = \mathrm{Br}^0(X^{\mathrm{s}})\{\ell\}^\Gamma \subset \mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma$$

contains a copy of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. □

Corollary 15.2.2 *Let A be an abelian variety over a field k which is finitely generated over its prime subfield. Then* $\mathrm{Br}(A^{\mathrm{s}})\{\ell\}^\Gamma$ *is finite for all primes ℓ not equal to* $\mathrm{char}(k)$.

Proof. Theorem 15.2.1 applies because the Tate conjecture for divisors is known for abelian varieties: it was proved by Zarhin in characteristic $p > 2$ [Zar75, Zar76], by Faltings in characteristic zero [Fal83, Fal86], and by Mori in characteristic 2, see [Mor85]. □

In the next sections we will show how to go beyond Theorem 15.2.1 to obtain a positive answer to our question for abelian varieties and K3 surfaces.

15.3 Abelian varieties

In this section A is an abelian variety over a field k finitely generated over its prime subfield. By Theorem 15.2.1 we know that $\mathrm{Br}(A^{\mathrm{s}})\{\ell\}^\Gamma$ is finite for all prime numbers ℓ different from $\mathrm{char}(k)$. Our aim now is to prove that this group is actually zero for almost all ℓ. For this it is enough to prove that the ℓ-torsion subgroup $\mathrm{Br}(A^{\mathrm{s}})[\ell]$ has no non-zero Γ-invariant elements for almost all ℓ. By Proposition 15.1.1 for almost all ℓ we have an exact sequence of Γ-modules (15.3):

$$0 \longrightarrow (\mathrm{NS}(A^{\mathrm{s}})/\ell)^\Gamma \longrightarrow \mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}}, \mu_\ell)^\Gamma \longrightarrow \mathrm{Br}(A^{\mathrm{s}})[\ell]^\Gamma \longrightarrow 0,$$

so our task is to prove that for almost all ℓ each Γ-invariant class in $\mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}}, \mu_\ell)$ comes from a divisor on A.

For an abelian variety A the cohomology group $\mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}}, \mu_\ell)$ has a nice interpretation in terms of the torsion subgroup $A[\ell]$. Namely, for each $n \geq 1$ we have a canonical isomorphism

$$\mathrm{H}^n_{\text{ét}}(A^{\mathrm{s}}, \mathbb{Z}/\ell) = \wedge^n_{\mathbb{Z}/\ell} \mathrm{H}^1_{\text{ét}}(A^{\mathrm{s}}, \mathbb{Z}/\ell).$$

The Kummer sequence gives a canonical isomorphism

$$\mathrm{H}^1_{\text{ét}}(A^{\text{s}}, \mu_\ell) = \mathrm{Pic}(A^{\text{s}})[\ell] = A^\vee[\ell],$$

where $A^\vee$ is the dual abelian variety of A. We have $(A^\vee)^\vee = A$ ([Lan83, Ch. V, §2, Prop. 9], [Mum74, p. 132]). The ℓ-torsion subgroups of A and $A^\vee$ are related by the Weil pairing

$$e_{\ell,A} : A[\ell] \times A^\vee[\ell] \longrightarrow \mu_\ell,$$

which is a perfect Γ-invariant pairing. Thus we obtain a canonical isomorphism of Γ-modules

$$\mathrm{H}^1_{\text{ét}}(A^{\text{s}}, \mu_\ell) = \mathrm{Hom}(A[\ell], \mu_\ell),$$

which gives a canonical isomorphism and an injection of Γ-modules

$$\mathrm{H}^2_{\text{ét}}(A^{\text{s}}, \mu_\ell) = \mathrm{Hom}(\wedge^2_{\mathbb{Z}/\ell} A[\ell], \mu_\ell) \hookrightarrow \mathrm{Hom}(A[\ell], A^\vee[\ell]).$$

Definition 15.3.1 *A homomorphism* $\phi : A[\ell] \to A^\vee[\ell]$ *is called symmetric if* $e_{\ell,A}(x, \phi y) = e_{\ell,A^\vee}(\phi x, y)$ *for any* $x, y \in A[\ell]$.

Lemma 15.3.2 *For* $\ell \neq 2$ *the injective image of* $\mathrm{H}^2_{\text{ét}}(A^{\text{s}}, \mu_\ell)$ *in* $\mathrm{Hom}(A[\ell], A^\vee[\ell])$ *is the subgroup of symmetric homomorphisms* $\mathrm{Hom}(A[\ell], A^\vee[\ell])_{\text{sym}}$.

Proof. This crucially uses the subtle fact that the Weil pairings for A and $A^\vee$ differ by sign [Lan83, Ch. VII, §2, Thm. 5(iii), p. 193], that is,

$$e_{\ell,A^\vee}(y, x) = -e_{\ell,A}(x, y)$$

for all $x \in A[\ell]$, $y \in A^\vee[\ell]$. Thus $\phi \in \mathrm{Hom}(A[\ell], A^\vee[\ell])_{\text{sym}}$ if and only if

$$e_{\ell,A}(x, \phi y) = -e_{\ell,A^\vee}(\phi y, x) = -e_{\ell,A}(y, \phi x).$$

Equivalently, ϕ is symmetric if and only if the bilinear form $e_{\ell,A}(x, \phi y)$ is skew-symmetric:

$$e_{\ell,A}(x, \phi y) = -e_{\ell,A}(y, \phi x), \quad x, y \in A[\ell].$$

On the other hand, $\wedge^2_{\mathbb{Z}/\ell} A[\ell]$ is by definition the quotient of $A[\ell] \otimes_{\mathbb{Z}/\ell} A[\ell]$ by the $\mathbb{Z}/\ell$-submodule generated by $x \otimes x$ for $x \in A[\ell]$. When $\ell \neq 2$, this submodule is generated by the elements of the form $x \otimes y + y \otimes x$ for $x, y \in A[\ell]$. Using this and the way our identifications have been set, it is clear that a homomorphism $\phi : A[\ell] \to A^\vee[\ell]$ comes from an element of $\mathrm{Hom}(\wedge^2_{\mathbb{Z}/\ell} A[\ell], \mu_\ell)$ if and only if $e_{\ell,A}(x, \phi x) = 0$ for all $x \in A[\ell]$. When $\ell \neq 2$, $\phi \in \mathrm{Hom}(A[\ell], A^\vee[\ell])$ comes from an element of $\mathrm{Hom}(\wedge^2_{\mathbb{Z}/\ell} A[\ell], \mu_\ell)$ if and only if $e_{\ell,A}(x, \phi y) + e_{\ell,A}(y, \phi x) = 0$ for all $x, y \in A[\ell]$. □

Let us now recall some basic properties of abelian varieties over an arbitrary field k. For abelian varieties A and B we write $\mathrm{Hom}(A, B)$ for the group of homomorphisms $A \to B$ (defined over k). A divisor D on A^{s} defines the homomorphism $A^{\text{s}} \to (A^\vee)^{\text{s}}$ sending $a \in A(\bar{k})$ to the linear equivalence class of $T_a^*(D) - D$

in $\mathrm{Pic}^0(A^{\mathrm{s}})$, where T_a is the translation by a in A^{s}. If L is the class of D in $\mathrm{NS}(A^{\mathrm{s}})$, then this map depends only on L, and is denoted by $\varphi_L : A^{\mathrm{s}} \to (A^\vee)^{\mathrm{s}}$ [Mum74, §8]. For $\alpha \in \mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})$ we denote by $\alpha^\vee \in \mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})$ the transpose of α. Then $\varphi_L^\vee = \varphi_L$. Moreover, if we define

$$\mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})_{\mathrm{sym}} = \{u \in \mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}}) \mid u = u^\vee\},$$

then the group homomorphism

$$\mathrm{NS}(A^{\mathrm{s}}) \longrightarrow \mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})_{\mathrm{sym}}, \quad L \mapsto \phi_L,$$

is an isomorphism [Lan83], [Mum74, §20, formula (I) and Thm. 1 on p. 186, Thm. 2 on p. 188 and Remark on p. 189]. For any $\alpha \in \mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})$ we have $(\alpha^\vee)^\vee = \alpha$, and thus

$$\alpha + \alpha^\vee \in \mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})_{\mathrm{sym}}. \tag{15.4}$$

We have $\mathrm{Hom}(A, B) = \mathrm{Hom}_\Gamma(A^{\mathrm{s}}, B^{\mathrm{s}}) = \mathrm{Hom}(A^{\mathrm{s}}, B^{\mathrm{s}})^\Gamma$. Since $\mathrm{Hom}(A^{\mathrm{s}}, B^{\mathrm{s}})$ has no torsion, the group $\mathrm{Hom}(A, B)/\ell$ is a subgroup of $\mathrm{Hom}(A^{\mathrm{s}}, B^{\mathrm{s}})/\ell$.

Let us now assume that $\ell \neq \mathrm{char}(k)$. The action of homomorphisms on points of order ℓ defines a natural map of Γ-modules

$$\mathrm{Hom}(A^{\mathrm{s}}, B^{\mathrm{s}}) \longrightarrow \mathrm{Hom}(A[\ell], B[\ell]).$$

A homomorphism that contains $A[\ell]$ in its kernel factors through the multiplication by ℓ map, hence the image of $\mathrm{Hom}(A^{\mathrm{s}}, B^{\mathrm{s}})$ in $\mathrm{Hom}(A[\ell], B[\ell])$ is $\mathrm{Hom}(A^{\mathrm{s}}, B^{\mathrm{s}})/\ell$. We thus obtain an embedding

$$\mathrm{Hom}(A, B)/\ell \subset \mathrm{Hom}_\Gamma(A[\ell], B[\ell]).$$

Now let $B = A^\vee$. Then for any $\alpha \in \mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})$ and any $x, y \in A[\ell]$ we have

$$e_{\ell, A^\vee}(\alpha x, y) = e_{\ell, A}(x, \alpha^\vee y),$$

see [Lan83, Ch. VII, §2, Thm. 4], [Mum74, p. 186]. Thus $\mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})_{\mathrm{sym}}/\ell$ is a subgroup of $\mathrm{Hom}(A[\ell], A^\vee[\ell])_{\mathrm{sym}}$. Note that if $\ell \neq 2$, then using (15.4) we see that this subgroup consists precisely of the elements of $\mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})/\ell$ that define symmetric homomorphisms on ℓ-torsion subgroups:

$$\mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})_{\mathrm{sym}}/\ell = \mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})/\ell \ \cap \ \mathrm{Hom}(A[\ell], A^\vee[\ell])_{\mathrm{sym}}. \tag{15.5}$$

(To see that the natural inclusion of the left hand side into the right hand side is an isomorphism, note that any α in the right hand side lifts to $\frac{\ell+1}{2}(\alpha + \alpha^\vee)$.)

Now we are ready to prove the main result of this section.

Theorem 15.3.3 *Let A be an abelian variety over a field k that is finitely generated over its prime subfield. Then $\mathrm{Br}(A^{\mathrm{s}})[\ell]^\Gamma = 0$ for almost all primes ℓ. Hence the subgroup of $\mathrm{Br}(A^{\mathrm{s}})^\Gamma$, which consists of the elements of order prime to $\mathrm{char}(k)$, is finite.*

Proof. The second statement follows from the first statement and Corollary 15.2.2. The first statement is a consequence of the following variant of the Tate conjecture on homomorphisms first stated by Zarhin in [Zar77]: for abelian varieties A and B over k the natural injection

$$\mathrm{Hom}(A,B)/\ell \hookrightarrow \mathrm{Hom}_\Gamma(A[\ell],B[\ell]) \tag{15.6}$$

is an isomorphism for almost all ℓ. In the finite characteristic case this is due to Zarhin [Zar77, Thm. 1.1]. When k is a number field, [Zar85, Cor. 5.4.5] based on the results of Faltings [Fal83] says that for almost all ℓ we have

$$\mathrm{End}(A)/\ell = \mathrm{End}_\Gamma(A_\ell). \tag{15.7}$$

The same proof works over arbitrary fields that are finitely generated over $\mathbb{Q}$, if one replaces the reference to [Zar85, Prop. 3.1] by the reference to the corollary on p. 211 of [Fal86]. Applying (15.7) to the abelian variety $A\times B$, one deduces that (15.6) is a bijection.

By (15.5) this gives an isomorphism

$$\mathrm{Hom}(A,A^\vee)_{\mathrm{sym}}/\ell \tilde{\longrightarrow} \mathrm{Hom}_\Gamma(A[\ell],A^\vee[\ell])_{\mathrm{sym}}.$$

By the discussion before the theorem the left hand side is canonically isomorphic to $\mathrm{Hom}_\Gamma(A^{\mathrm{s}},(A^\vee)^{\mathrm{s}})_{\mathrm{sym}}/\ell = \mathrm{NS}(A^{\mathrm{s}})^\Gamma/\ell$, whereas the right hand side is canonically isomorphic to $\mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}},\mu_\ell)^\Gamma$. Since the Néron–Severi group of an abelian variety is torsion-free and $\mathrm{NS}(A^{\mathrm{s}})^\Gamma$ is a saturated subgroup of $\mathrm{NS}(A^{\mathrm{s}})$, the natural map $\mathrm{NS}(A^{\mathrm{s}})^\Gamma/\ell \to (\mathrm{NS}(A^{\mathrm{s}})/\ell)^\Gamma$ is injective. We conclude that the embedding $(\mathrm{NS}(A^{\mathrm{s}})/\ell)^\Gamma \subset \mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}},\mu_\ell)^\Gamma$ is actually an equality, because the cardinalities of these finite groups are the same. As was recalled in the beginning of the section, this implies that for almost all ℓ we have $\mathrm{Br}(A^{\mathrm{s}})[\ell]^\Gamma=0$. □

15.4 Varieties dominated by products

Products of varieties

Theorem 15.4.1 *Let k be a field finitely generated over $\mathbb{Q}$. Let X and Y be smooth, projective and geometrically integral varieties over k. Then*

$$\big(\mathrm{Br}(\overline{X}\times\overline{Y})/(\mathrm{Br}(\overline{X})\oplus\mathrm{Br}(\overline{Y}))\big)^\Gamma$$

is a finite group.

Proof. By Corollary 4.6.7 the Γ-module $\mathrm{Br}(\overline{X})\oplus\mathrm{Br}(\overline{Y})$ is a direct summand of $\mathrm{Br}(\overline{X}\times\overline{Y})$. Since $\mathrm{Br}(\overline{X}\times\overline{Y})$ is a torsion group such that $\mathrm{Br}(\overline{X}\times\overline{Y})[n]$ is finite for every positive integer n, the same is true for the group in the statement of the theorem. Thus it is enough to prove the following statements.

(a) *For every prime ℓ we have $V_\ell\big((\mathrm{Br}(\overline{X}\times\overline{Y})/(\mathrm{Br}(\overline{X})\oplus\mathrm{Br}(\overline{Y})))^\Gamma\big)=0$.*

(b) *For almost all primes ℓ we have $\big(\mathrm{Br}(\overline{X}\times\overline{Y})[\ell]/(\mathrm{Br}(\overline{X})[\ell]\oplus\mathrm{Br}(\overline{Y})[\ell])\big)^\Gamma=0$.*

Let us prove (a). We can pass to the limit in the isomorphism of Corollary 4.6.8, taking into account what was said in Remark 4.6.9. This produces an isomorphism of Γ-modules between

$$V_\ell(\mathrm{Br}(\overline{X}\times\overline{Y}))/(V_\ell(\mathrm{Br}(\overline{X}))\oplus V_\ell(\mathrm{Br}(\overline{Y}))) = V_\ell\big(\mathrm{Br}(\overline{X}\times\overline{Y})/(\mathrm{Br}(\overline{X})\oplus\mathrm{Br}(\overline{Y}))\big)$$

and the quotient of $\mathrm{Hom}_{\mathbb{Q}_\ell}\big(V_\ell(B^\vee),V_\ell(A)\big)$ by $\mathrm{Hom}(\overline{B^\vee},\overline{A})\otimes\mathbb{Q}_\ell$ (embedded via a natural map given by the action on torsion points). Hence we obtain

$$\begin{aligned} V_\ell\big((\mathrm{Br}(\overline{X}\times\overline{Y})/(\mathrm{Br}(\overline{X})\oplus\mathrm{Br}(\overline{Y})))^\Gamma\big) &= \\ V_\ell\big(\mathrm{Br}(\overline{X}\times\overline{Y})/(\mathrm{Br}(\overline{X})\oplus\mathrm{Br}(\overline{Y}))\big)^\Gamma &= \\ \big(\mathrm{Hom}_{\mathbb{Q}_\ell}\big(V_\ell(B^\vee),V_\ell(A)\big)/\mathrm{Hom}(\overline{B^\vee},\overline{A})\otimes\mathbb{Q}_\ell\big)^\Gamma. \end{aligned}$$

By the fundamental results of Faltings [Fal83, Fal86] Γ-modules $V_\ell(B^\vee)$ and $V_\ell(A)$ are semisimple and

$$\mathrm{Hom}_\Gamma(V_\ell(B^\vee),V_\ell(A)) = \mathrm{Hom}(B^\vee,A)\otimes\mathbb{Q}_\ell.$$

By a theorem of Chevalley [Che54, p. 88] the semisimplicity of Γ-modules $V_\ell(B^\vee)$ and $V_\ell(A)$ implies the semisimplicity of $\mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(B^\vee),V_\ell(A))$. From this we deduce (a).

Let us prove (b). By Corollary 4.6.8 and Remark 4.6.9 it is enough to show

$$\big(\mathrm{Hom}(B^\vee[\ell],A[\ell])/(\mathrm{Hom}(\overline{B^\vee},\overline{A})/\ell)\big)^\Gamma = 0$$

for almost all primes ℓ. Since $\mathrm{Hom}(\overline{B^\vee},\overline{A})^\Gamma=\mathrm{Hom}(B^\vee,A)$, the exact sequence

$$0\longrightarrow \mathrm{Hom}(\overline{B^\vee},\overline{A})^\Gamma/\ell \longrightarrow \big(\mathrm{Hom}(\overline{B^\vee},\overline{A})/\ell\big)^\Gamma \longrightarrow \mathrm{H}^1(k,\mathrm{Hom}(\overline{B^\vee},\overline{A}))$$

implies, in view of the finiteness of $\mathrm{H}^1(k,\mathrm{Hom}(\overline{B^\vee},\overline{A}))$, that for all but finitely many primes ℓ we have

$$\big(\mathrm{Hom}(\overline{B^\vee},\overline{A})/\ell\big)^\Gamma = \mathrm{Hom}(B^\vee,A)/\ell.$$

If we further assume that $\ell > 2\dim(A)+2\dim(B)-2$, then, by a theorem of Serre [Ser94], the semisimplicity of the Γ-modules $B^\vee[\ell]$ and $A[\ell]$ implies the semisimplicity of $\mathrm{Hom}(B^\vee[\ell],A[\ell])$. Hence we obtain

$$\begin{aligned} \big(\mathrm{Hom}(B^\vee[\ell],A[\ell])/(\mathrm{Hom}(\overline{B^\vee},\overline{A})/\ell)\big)^\Gamma &= \\ \mathrm{Hom}(B^\vee[\ell],A[\ell])^\Gamma/(\mathrm{Hom}(\overline{B^\vee},\overline{A})/\ell)^\Gamma &= \\ \mathrm{Hom}_\Gamma(B^\vee[\ell],A[\ell])/(\mathrm{Hom}(B^\vee,A)/\ell) &= 0. \end{aligned}$$

This is zero for almost all ℓ, since (15.6) is a bijection for almost all ℓ. □

Corollary 15.4.2 *Let k be a field finitely generated over $\mathbb{Q}$ such that $\mathrm{H}^3(k,\bar{k}^*)=0$, for example, a number field. Let X and Y be smooth, projective and geometrically integral varieties over k. Then the cokernel of the natural map*

$$\mathrm{Br}(X)\oplus\mathrm{Br}(Y)\longrightarrow\mathrm{Br}(X\times Y)$$

is finite.

Proof. From the functoriality of the spectral sequence (4.7) and the assumption $\mathrm{H}^3(k,\bar{k}^*)=0$, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{Br}(X\times Y) & \to & \mathrm{Br}(\overline{X}\times\overline{Y})^\Gamma & \to & \mathrm{H}^2(k,\mathrm{Pic}(\overline{X}\times\overline{Y})) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Br}(X)\oplus\mathrm{Br}(Y) & \to & \mathrm{Br}(\overline{X})^\Gamma\oplus\mathrm{Br}(\overline{Y})^\Gamma & \to & \mathrm{H}^2(k,\mathrm{Pic}(\overline{X}))\oplus\mathrm{H}^2(k,\mathrm{Pic}(\overline{Y}))
\end{array}$$

The middle vertical map is clearly injective. Next, the kernel of the right hand vertical map is finite. Indeed, in view of the exact sequence (4.25) it is enough to remark that the abelian group $\mathrm{Hom}(\overline{B^\vee},\overline{A})$ is free and finitely generated, hence $\mathrm{H}^1(k,\mathrm{Hom}(\overline{B^\vee},\overline{A}))$ is finite.

By Theorem 15.4.1 this diagram shows that the subgroup of $\mathrm{Br}(X\times Y)$ generated by $\mathrm{Br}_1(X\times Y)$ and the images of $\mathrm{Br}(X)$ and $\mathrm{Br}(Y)$, has finite index. By Proposition 4.6.4, $\mathrm{Br}_1(X\times Y)$ is finite modulo $\mathrm{Br}_1(X)\oplus\mathrm{Br}_1(Y)$, so we are done. □.

Varieties dominated by products of curves

A smooth, projective and geometrically integral variety X over a field k is called a variety *dominated by a product of curves* if there is a dominant rational map from a product of geometrically integral curves to X.

Theorem 15.4.3 *Let k be a field finitely generated over $\mathbb{Q}$. Let X be a variety dominated by a product of curves. Then $\mathrm{Br}(\overline{X})^\Gamma$ is finite.*

Proof. If V and W are smooth, projective and geometrically integral varieties over a field k which is finitely generated over $\mathbb{Q}$, then the cokernel of the natural map $\mathrm{Br}(\overline{V})^\Gamma\oplus\mathrm{Br}(\overline{W})^\Gamma\to\mathrm{Br}(\overline{V}\times\overline{W})^\Gamma$ is finite by Theorem 15.4.1. The Brauer group of a smooth, projective, integral curve over an algebraically closed field is zero (Theorem 4.5.1). Thus if Z is a product of smooth, projective and geometrically integral curves over k, then $\mathrm{Br}(\overline{Z})^\Gamma$ is finite.

Since $\mathrm{char}(k)=0$, we can assume that there is a smooth, projective and geometrically integral variety Y over k, a birational morphism $Y\to Z$, where Z is a product of smooth, projective and geometrically integral curves, and a dominant, generically finite morphism $f: Y\to X$. By the birational invariance of the Brauer group (Corollary 5.2.6) we have $\mathrm{Br}(\overline{Y})^\Gamma=\mathrm{Br}(\overline{Z})^\Gamma$. By Theorem 3.5.4 the natural map $\mathrm{Br}(\overline{X})\hookrightarrow\mathrm{Br}(\bar{k}(X))$ is injective. The standard restriction-corestriction argument then gives that the kernel of $f^*:\mathrm{Br}(\overline{X})\to\mathrm{Br}(\overline{Y})$ is killed by the degree $[\bar{k}(Y):\bar{k}(X)]$. Since $\mathrm{Br}(\overline{X})$ is a torsion group of cofinite type, this kernel is finite. Hence $\mathrm{Br}(\overline{X})^\Gamma$ is finite. □

Corollary 15.4.4 *Let k be a field finitely generated over $\mathbb{Q}$. Let $f(t)$ and $g(t)$ be separable polynomials of degree $d\geq 2$. Let $F(x,y)$ and $G(x,y)$ be homogeneous forms of degree d such that $f(t)=F(t,1)$ and $g(t)=G(t,1)$. Let $X\subset\mathbb{P}^3_k$ be the surface with equation $F(x,y)=G(z,w)$, for example, a diagonal surface. Then the Brauer group $\mathrm{Br}(X)$ is finite modulo $\mathrm{Br}(k)$.*

Proof. The surface X is dominated by the product of smooth plane curves of degree d, namely, the curves $z^d = F(x,y)$ and $z^d = G(x,y)$. Since $\mathrm{Pic}(\overline{X})$ is torsion-free, the group $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ is finite. The finiteness of $\mathrm{Br}(\overline{X})^\Gamma$ follows from Theorem 15.4.3. □

15.5 K3 surfaces

Preliminaries on K3 surfaces

For a detailed introduction to the geometry of K3 surfaces we refer the reader to Huybrechts' book [Huy16], see also [Voi02, §7.2]. Here we briefly recall the definition and the basic geometric properties of K3 surfaces.

A smooth, projective and geometrically integral surface X over a field k is called a *K3 surface* if $\Omega^2_X \cong \mathcal{O}_X$ and $\mathrm{H}^1(X,\mathcal{O}_X)=0$. Standard examples of K3 surfaces are quartic surfaces in $\mathbb{P}^3_k$ and double covers of $\mathbb{P}^2_k$ ramified in a smooth sextic curve.

Let $k=\mathbb{C}$. Using Serre duality and the Riemann–Roch theorem one finds that the classical (Betti) cohomology group $\mathrm{H}^2(X,\mathbb{Z})$ is a free abelian group of rank 22. We have the cup-product

$$\cup : \mathrm{H}^2(X,\mathbb{Z}) \times \mathrm{H}^2(X,\mathbb{Z}) \longrightarrow \mathrm{H}^4(X,\mathbb{Z}) \cong \mathbb{Z},$$

where the last isomorphism is due to the fact that $\dim(X)=2$. This is a symmetric bilinear pairing. The Poincaré duality implies that this pairing is a perfect duality, that is, it induces an isomorphism

$$\mathrm{H}^2(X,\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{H}^2(X,\mathbb{Z}),\mathbb{Z}).$$

Thus the determinant of the matrix of this bilinear form with respect to a $\mathbb{Z}$-basis of $\mathrm{H}^2(X,\mathbb{Z})$ lies in $\mathbb{Z}^*=\{\pm 1\}$. Topological arguments (Wu's formula, Thom–Hirzebruch index theorem) give that the associated integral quadratic form is even, i.e. $x\cup x \in 2\mathbb{Z}$ for any $x\in \mathrm{H}^2(X,\mathbb{Z})$, and of signature $(3,19)$. By the classification of even integral quadratic forms, this implies that $\mathrm{H}^2(X,\mathbb{Z})$ can be written as the orthogonal direct sum $L=\mathrm{E}_8(-1)^{\oplus 2}\oplus U^{\oplus 3}$. Here E_8 is the (positive definite) root lattice of the root system E_8; the lattice $\mathrm{E}_8(-1)$ is obtained by multiplication of the form on E_8 by -1, and U is the hyperbolic lattice of rank 2 (whose associated quadratic form is $q(x_1,x_2)=x_1x_2$).

Hodge structures of complex tori

Let M be a finitely generated free abelian group. Following Deligne, an *integral Hodge structure* on M is a representation of the 2-dimensional real torus $\mathbb{S}=\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ in $\mathrm{GL}(M_\mathbb{R})$. Then we have a Hodge decomposition $M_\mathbb{C}=\oplus_{p,q}M^{p,q}$ such that $z\in\mathbb{S}(\mathbb{R})=\mathbb{C}^*$ acts on $M^{p,q}$ by $z^p\bar{z}^q$. The space $M^{q,p}$ is the complex conjugate of $M^{p,q}$. If $p+q=n$ for each summand $M^{p,q}$, then the Hodge structure is called pure of weight n.

A complex torus is $\mathbb{C}^g/\Lambda$, where $\Lambda \cong \mathbb{Z}^{2g}$ is a full lattice, i.e. $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}^g$. To give a complex torus is the same as to give an integral Hodge structure of type $\{(1,0),(0,1)\}$ on Λ. (The complex structure on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is defined by the action of $i \in \mathbb{C}^* = \mathbb{S}(\mathbb{R})$.) An abelian variety is a complex torus with a polarisation, which is an integral skew-symmetric form on Λ satisfying some conditions. (This can be also rephrased by saying that the integral Hodge structure is polarisable.)

An example of a complex torus is the Jacobian of a smooth projective curve. For a curve C of genus g the spaces $\mathrm{H}^{1,0} \cong \mathrm{H}^0(C, \Omega^1_C)$ and $\mathrm{H}^{0,1} \cong \mathrm{H}^1(C, \mathcal{O}_C)$ have dimension g, so the Hodge decomposition

$$\mathrm{H}^1(C,\mathbb{Z})_{\mathbb{C}} = \mathrm{H}^1(C,\mathbb{C}) = \mathrm{H}^{1,0} \oplus \mathrm{H}^{0,1}$$

gives rise to a complex torus. Explicitly, integrating g linearly independent holomorphic 1-forms over $2g$ elements of a $\mathbb{Z}$-basis of $\mathrm{H}_1(C,\mathbb{Z})$ produces a full lattice $\Lambda \subset \mathbb{C}^g$. Then one shows that the complex torus $\mathbb{C}^g/\Lambda$ has a polarisation, so is an abelian variety. This is the Jacobian of C.

Hodge structures of K3 type

In a very rough analogy to the Jacobian of a curve, one would like to associate an abelian variety to a polarised K3 surface. The Hodge decomposition on the second integral cohomology group of a complex K3 surface X is

$$\mathrm{H}^2(X,\mathbb{Z})_{\mathbb{C}} = \mathrm{H}^2(X,\mathbb{C}) = \mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2},$$

where $\mathrm{H}^{2,0} \cong \mathrm{H}^0(X, \Omega^2_X)$ and $\mathrm{H}^{0,2} \cong \mathrm{H}^2(X, \mathcal{O}_X)$ are both 1-dimensional vector spaces over $\mathbb{C}$. This is a Hodge structure of pure weight 2.

Choose a non-zero $\omega \in \mathrm{H}^{2,0}$. Since $\mathrm{H}^{4,0} = 0$ we have $\omega \cup \omega = 0$. The complex conjugate $\overline{\omega}$ is a non-zero element of $\mathrm{H}^{0,2}$. Since the pairing

$$\mathrm{H}^{2,0} \times \mathrm{H}^{0,2} \longrightarrow \mathrm{H}^{2,2} = \mathrm{H}^4(X,\mathbb{C}) \cong \mathbb{C}$$

is non-degenerate and the cup-product is symmetric, $\omega \cup \overline{\omega}$ is a non-zero real number. Actually, $\omega \cup \overline{\omega} > 0$. Since $\mathrm{H}^{3,1} = \mathrm{H}^{1,3} = 0$, we have $\mathrm{H}^{2,0} \perp \mathrm{H}^{1,1}$ and $\mathrm{H}^{0,2} \perp \mathrm{H}^{1,1}$. It is convenient to twist this Hodge structure by 1 in order to obtain a Hodge structure of weight 0:

$$\mathrm{H}^2(X,\mathbb{Z}(1))_{\mathbb{C}} = \mathrm{H}^{1,-1} \oplus \mathrm{H}^{0,0} \oplus \mathrm{H}^{-1,1}.$$

The advantage of this is that now the image of $\mathbb{S}$ lies in $\mathrm{SO}(\mathrm{H}^2(X,\mathbb{Z}))_{\mathbb{R}}$. (Twisting by 1 also means rescaling the image of the integral cohomology inside the complex cohomology by $2\pi i$.)

The Picard group of a complex K3 surface is a free abelian group. Its rank ρ is called the *Picard number*. The cycle class map gives an embedding

$$\mathrm{Pic}(X) \hookrightarrow \mathrm{H}^2(X,\mathbb{Z}(1)).$$

By the Lefschetz theorem $\mathrm{Pic}(X) = \mathrm{H}^2(X,\mathbb{Z}(1)) \cap \mathrm{H}^{(0,0)}$. From this we obtain $1 \leq \rho \leq 20$. The orthogonal complement to $\mathrm{Pic}(X)$ in $\mathrm{H}^2(X,\mathbb{Z}(1))$ is called the *transcendental lattice* and is denoted by $T(X)$.

Definition 15.5.1 *Let M be a finitely generated free abelian group with a non-degenerate integral symmetric bilinear form (x, y). An integral Hodge structure on M is called a* **Hodge structure of K3 type***, if the Hodge decomposition is*

$$M_{\mathbb{C}} = M^{1,-1} \oplus M^{0,0} \oplus M^{-1,1},$$

where $M^{1,-1} \perp M^{0,0}$, $\dim(M^{1,-1}) = 1$ and for a non-zero $\omega \in M^{1,-1}$ we have

$$(\omega^2) = 0, \quad (\omega, \overline{\omega}) > 0.$$

Take a primitive element $\lambda \in L$ such that $(\lambda^2) > 0$. Let $d = \frac{1}{2}(\lambda^2) \in \mathbb{Z}$. It can be proved that the primitive elements $x \in L$ with $(x^2) = 2d$ form an orbit of $\mathrm{Aut}(L)$. Hence the isomorphism class of the orthogonal complement $\lambda^\perp \subset L$ depends only on d. It follows that the lattice $\lambda^\perp$ is isomorphic to the orthogonal direct sum

$$L_d = \mathrm{E}_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus (-2d). \tag{15.8}$$

The signature of L_d is $(2, 19)$. This associates to a K3 surface with a primitive polarisation of degree $2d$ an integral Hodge structure of K3 type on L_d.

Associating to an integral Hodge structure of K3 type on L_d the 1-dimensional complex space $\mathrm{H}^{1,-1}$ defines a point in the *period domain*

$$\Omega_d = \{x \in \mathbb{P}(L_{d,\mathbb{C}}) \mid (x^2) = 0,\ (x, \bar{x}) > 0\} = \mathrm{SO}(2,19)(\mathbb{R})/\mathrm{SO}(2)(\mathbb{R})\times\mathrm{SO}(19)(\mathbb{R}),$$

see [Voi02, Thm. 7.18]. One identifies Ω_d with the Grassmannian of positive definite oriented 2-dimensional real subspaces of $L_d \otimes \mathbb{R} \simeq \mathbb{R}^{21}$, by attaching to x the plane spanned by $\mathrm{Re}(x), \mathrm{Im}(x)$ in this order. Thus

$$\Omega_d = \mathrm{SO}(2,19)(\mathbb{R})/\mathrm{SO}(2)(\mathbb{R}) \times \mathrm{SO}(19)(\mathbb{R}).$$

Ω_d has two isomorphic connected components that are interchanged by the complex conjugation (or reversing the orientation).

15.6 The Kuga–Satake variety

Hodge structures of curves and K3 surfaces are quite different, so we cannot construct an analogue of Jacobian for K3 surfaces without more work. Nevertheless, we have the following very important result. The classical Torelli theorem can be stated as follows: the isometry class of the integral Hodge structure on $\mathrm{H}^1(C, \mathbb{Z})$, where C is a smooth and connected complex curve, uniquely determines C. The Torelli theorem for K3 surfaces of Piatetskii-Shapiro and Shafarevich [PSS71] leads to the following result: the isometry class of the integral Hodge structure on $\mathrm{H}^2(X, \mathbb{Z})$, where X is a complex K3 surface, uniquely determines X, see [Huy16, Thm. 7.5.3].

Another obstacle is that the cup-product pairing on $\mathrm{H}^2(X, \mathbb{Z})$ is symmetric, whereas for an abelian variety one would need a skew-symmetric pairing, such as the one given by the cup-product on $\mathrm{H}^1(C, \mathbb{Z})$. To overcome this issue one employs the Clifford algebra of the quadratic form on $\mathrm{H}^2(X, \mathbb{Z})$.

Clifford algebra and spinor group

Let us recall the general construction of the Clifford algebra and the spinor group, see [BouIX, §9].

Let M be a finitely generated free abelian group with a non-degenerate quadratic form $q : M \to \mathbb{Z}$. Define the Clifford algebra $\mathrm{Cl}(M)$ as the quotient of the full tensor algebra $\oplus_{n\geq 0} M^{\otimes n}$ by the two-sided ideal I generated by the elements $x \otimes x - q(x)$ for $x \in M$. There is an isomorphism of abelian groups $\mathrm{Cl}(M) \simeq \oplus_{n=0}^{\mathrm{rk}(M)} \wedge^n M$, hence $\mathrm{rk}(\mathrm{Cl}(M)) = 2^{\mathrm{rk}(M)}$. The multiplication by -1 on M acts on $\oplus_{n\geq 0} M^{\otimes n}$. Since $x \otimes x - q(x)$ is invariant, we have $I = I^+ \oplus I^-$, where I^+ is the subgroup of invariant elements and I^- is the subgroup of anti-invariant elements. Thus we can define

$$\mathrm{Cl}^+(M) = (\oplus_{n\geq 0} M^{\otimes 2n})/I^+, \quad \mathrm{Cl}^-(M) = (\oplus_{n\geq 0} M^{\otimes 2n+1})/I^-,$$

where the first equality is the quotient of a ring by an ideal, whereas the second one is the quotient of a (left or right) $\oplus_{n\geq 0} M^{\otimes 2n}$-module $\oplus_{n\geq 0} M^{\otimes 2n+1}$ by the submodule I^-. The natural embedding of M into $\oplus_{n\geq 0} M^{\otimes n}$ gives rise to an injective map $M \to \mathrm{Cl}^-(M)$. Define the *Clifford group*

$$\mathrm{GSpin}(M) = \{g \in \mathrm{Cl}^+(M)^* | gMg^{-1} = M\}.$$

The group $\mathrm{GSpin}(M)$ acts by conjugation on M preserving the quadratic form. This gives an exact sequence of algebraic groups over $\mathbb{Q}$:

$$1 \longrightarrow \mathbb{G}_{m,\mathbb{Q}} \longrightarrow \mathrm{GSpin}(M)_{\mathbb{Q}} \longrightarrow \mathrm{SO}(M)_{\mathbb{Q}} \longrightarrow 1.$$

The *adjoint* action of $\mathrm{GSpin}(M)$ on $\mathrm{Cl}^+(M)$, i.e. the action by conjugations, gives rise to a representation of $\mathrm{GSpin}(M)_{\mathbb{Q}}$, which is isomorphic to the direct sum of $\wedge^{2n} M_{\mathbb{Q}}$ for $n \geq 0$.

The *spinor group* $\mathrm{Spin}(M)_{\mathbb{Q}}$ is the algebraic group over $\mathbb{Q}$ defined by the exact sequence

$$1 \longrightarrow \mathrm{Spin}(M)_{\mathbb{Q}} \longrightarrow \mathrm{GSpin}(M)_{\mathbb{Q}} \longrightarrow \mathbb{G}_{m,\mathbb{Q}} \longrightarrow 1,$$

where the third arrow is the spinor norm.

It is instructive to consider the case of an orthogonal direct sum of n hyperbolic planes $U^{\oplus n}$, i.e. rank 2 lattices with $\mathbb{Z}$-basis e_i, f_i such that $q(e_i) = q(f_i) = 0$, $(e_i, f_i) = 1$, for $i = 1, \ldots, n$, where $(a, b) = q(a+b) - q(a) - q(b)$ is the associated bilinear form. Let Λ be the full exterior algebra of $\mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n$. Then $\mathrm{Cl}(U^{\oplus n})$ is isomorphic to $\mathrm{End}(\Lambda)$, see the proof of [BouIX, §9, no. 4, Thm. 2]. Next, $\mathrm{Cl}^+(U^{\oplus n})$ is isomorphic to $\mathrm{End}(\Lambda^+) \oplus \mathrm{End}(\Lambda^-)$, where Λ^+ and Λ^- are the even and odd parts of Λ, respectively.

From this it follows that if $\mathrm{rk}(M)$ is *even*, then $\mathrm{Cl}(M_{\mathbb{C}})$ is isomorphic to a matrix algebra. The unique simple module of this simple algebra is called the *spinor representation*. We have $\mathrm{Cl}(M_{\mathbb{C}}) = \mathrm{End}_{\mathbb{C}}(W)$. The restriction of W to $\mathrm{Cl}^+(M_{\mathbb{C}})$ splits into the direct sum of two non-isomorphic *semi-spinor representations*, so that $\mathrm{Cl}^+(M_{\mathbb{C}}) = \mathrm{End}_{\mathbb{C}}(W_1) \oplus \mathrm{End}_{\mathbb{C}}(W_2)$, where $W = W_1 \oplus$

W_2 and $\dim(W_1) = \dim(W_2)$. The spaces W_1 and W_2 are non-isomorphic representations of $\mathrm{Spin}(M)_{\mathbb{C}}$.

If $\mathrm{rk}(M)$ is *odd*, then $\mathrm{Cl}^+(M_{\mathbb{C}})$ is isomorphic to a matrix algebra $\mathrm{End}_{\mathbb{C}}(W)$, where W is called the spinor representation. In this case the full Clifford algebra $\mathrm{Cl}(M_{\mathbb{C}})$ is the direct sum of two isomorphic matrix algebras, see [BouIX, §9, no. 4, Thm. 3]. More precisely, if $e_0, \ldots, e_{2n}$ is an orthonormal basis of q over $\mathbb{C}$, then one can choose a sign so that $\tau = \pm e_0 \ldots e_{2n}$ is in the centre of $\mathrm{Cl}(M_{\mathbb{C}})$ and $\tau^2 = 1$. Then $\mathrm{Cl}(M_{\mathbb{C}})$ is the direct sum of its two-sided ideals $\mathrm{Cl}^+(M_{\mathbb{C}})(1+\tau)$ and $\mathrm{Cl}^+(M_{\mathbb{C}})(1-\tau)$. Thus $\mathrm{Cl}(M_{\mathbb{C}})$ is isomorphic to $\mathrm{End}_{\mathbb{C}}(W)^{\oplus 2}$, i.e. the two resulting representations of $\mathrm{Spin}(M)_{\mathbb{C}}$ are isomorphic to W.

Kuga–Satake construction I

Let us apply this to the second cohomology of a polarised complex K3 surface X. Fix a primitive ample class $\lambda \in \mathrm{H}^2(X, \mathbb{Z}(1))$ and define P as the orthogonal complement to λ in $\mathrm{H}^2(X, \mathbb{Z}(1))$, so that $\mathrm{rk}(P) = 21$. We have

$$P_{\mathbb{C}} = P^{1,-1} \oplus P^{0,0} \oplus P^{-1,1}.$$

Kuga and Satake [KS67] showed how to define a canonical complex structure on the real vector space $\mathrm{Cl}^+(P_{\mathbb{R}})$. We can normalise $\omega \in P^{1,-1}$ so that $(\omega, \overline{\omega}) = 2$. Write $\omega = \omega_1 + i\omega_2$, where $\omega_1, \omega_2 \in \mathrm{H}^2(X, \mathbb{R})$. Then $(\omega_1^2) = (\omega_2^2) = 1$ and $(\omega_1, \omega_2) = 0$. By the definition of the Clifford algebra, the following holds in $\mathrm{Cl}(P_{\mathbb{R}})$:

$$\omega_1^2 = \omega_2^2 = 1, \quad \omega_1\omega_2 = -\omega_2\omega_1.$$

Let $I = \omega_1\omega_2 \in \mathrm{Cl}^+(P_{\mathbb{R}})$. (It is immediate to check that I does not depend on ω.) Then $I^2 = -1$, so the left multiplication by I defines a complex structure on the real vector space $\mathrm{Cl}^+(P_{\mathbb{R}})$, thus making $\mathrm{Cl}^+(P_{\mathbb{R}})/\mathrm{Cl}^+(P)$ a complex torus. It has a polarisation [Huy16, Ch. 4, 2.2], so is an abelian variety.

Kuga–Satake construction II, d'après Deligne

In Deligne's version [Del72] one equips $\mathrm{Cl}^+(P)$ with an integral Hodge structure of type $\{(1,0),(0,1)\}$ as follows. Since $\mathbb{S}$ preserves the quadratic form on $P_{\mathbb{R}}$, we have a homomorphism $h : \mathbb{S} \to \mathrm{SO}(P)_{\mathbb{R}}$ whose kernel is $\{\pm 1\}$. For any $a, b \in \mathbb{R}$, not both equal to 0, we have $a + bI \in \mathrm{GSpin}(P)(\mathbb{R})$. Deligne points out that this is a canonical lifting of $h : \mathbb{S} \to \mathrm{SO}(P)_{\mathbb{R}}$ to $\tilde{h} : \mathbb{S} \hookrightarrow \mathrm{GSpin}(P)_{\mathbb{R}}$. (Indeed, if we write $z = a + bi$, then $a + bI \in \mathrm{Cl}^+(P_{\mathbb{R}})$ and $x \mapsto (a+bI)x(a+bI)^{-1}$ acts on ω as multiplication by $z\bar{z}^{-1}$, on $\overline{\omega}$ as multiplication by $\bar{z}z^{-1}$, and on $P^{0,0} \cap P_{\mathbb{R}}$ as the identity.) This means that the adjoint action of $\mathrm{GSpin}(P_{\mathbb{Q}})$ on P induces our original Hodge structure of K3 type on P.

Lemma 15.6.1 *The left action of* $\mathrm{GSpin}(P)_{\mathbb{Q}}$ *on* $\mathrm{Cl}^+(P_{\mathbb{Q}})$ *induces an integral Hodge structure of type* $\{(1,0),(0,1)\}$ *on* $\mathrm{Cl}^+(P_{\mathbb{Q}})$*. The same is true for* $\mathrm{Cl}(P_{\mathbb{Q}})$*.*

Proof. The adjoint representation of $\mathrm{GSpin}(P)_{\mathbb{Q}}$ on $\mathrm{Cl}^+(P_{\mathbb{Q}})$ is isomorphic to the direct sum of $\wedge^{2n} P_{\mathbb{Q}}$ for $n \geq 0$. The Hodge structure on P is of

K3 type, hence the induced Hodge structure on each $\wedge^{2n}P$ is of Hodge type $\{(1,-1),(0,0),(-1,1)\}$. Thus the Hodge structure on $\mathrm{Cl}^+(P)$, is also of Hodge type $\{(1,-1),(0,0),(-1,1)\}$.

The action of $\mathbb{S} \subset \mathrm{GSpin}(P)_\mathbb{R}$ by left multiplication induces an integral Hodge structure on $\mathrm{Cl}^+(P)$. We would like to determine its type. Note that the $\mathbb{C}$-algebra $\mathrm{Cl}^+(P_\mathbb{C})$ can be identified with a matrix algebra $\mathrm{End}_\mathbb{C}(W)$, where the complex vector space W is the unique simple module of $\mathrm{Cl}^+(P_\mathbb{C})$. Hence the action of $\mathrm{GSpin}(P)_\mathbb{C}$ on $\mathrm{Cl}^+(P_\mathbb{C})$ by left multiplication is isomorphic to $W^{\dim(W)}$. The adjoint representation $\mathrm{GSpin}(P)_\mathbb{C}$ on $\mathrm{Cl}^+(P_\mathbb{C})$ is isomorphic to $\mathrm{End}_\mathbb{C}(W) = W \otimes_\mathbb{C} W^*$, where $W^* = \mathrm{Hom}_\mathbb{C}(W, \mathbb{C})$. Thus the type of the Hodge structure on $\mathrm{Cl}^+(P)$ defined by left multiplication of $\mathbb{S} \subset \mathrm{GSpin}(P)_\mathbb{R}$ must be $\{(a,b),(b,a)\}$ with $a - b = \pm 1$, otherwise the Hodge structure on $W \otimes_\mathbb{C} W^*$ cannot be of type $\{(1,-1),(0,0),(-1,1)\}$. But $\mathbb{R}^* \subset \mathbb{C}^*$ acts on $\mathrm{Cl}^+(P_\mathbb{C})$ tautologically, so the weight of W is $a + b = 1$. Thus the type is $\{(1,0),(0,1)\}$.

The right multiplication by $x \in M$, $q(x) \neq 0$, is an isomorphism of $\mathbb{Q}$-vector spaces $\mathrm{Cl}^+(P_\mathbb{Q}) \to \mathrm{Cl}^-(P_\mathbb{Q})$ which preserves the left action of $\mathrm{GSpin}(P)_\mathbb{Q}$. This shows that integral Hodge structure on $\mathrm{Cl}(P_\mathbb{Q})$ is of type $\{(1,0),(0,1)\}$. □

It can be shown that the integral Hodge structures on $\mathrm{Cl}^+(P)$ and $\mathrm{Cl}(P)$ are polarisable, so we actually obtain abelian varieties and not just complex tori. The complex abelian variety $\mathrm{Cl}^+(P_\mathbb{R})/\mathrm{Cl}^+(P)$ is sometimes called the *even Kuga–Satake variety* of (X, λ). The complex abelian variety $\mathrm{Cl}(P_\mathbb{R})/\mathrm{Cl}(P)$ is usually called the *Kuga–Satake variety* of (X, λ).

15.7 Moduli spaces of K3 surfaces and Shimura varieties

Moduli spaces of polarised K3 surfaces

Let $O(L_d)$ be the orthogonal group of the lattice L_d defined in (15.8) with associated period domain Ω_d. Write $L_d^* = \mathrm{Hom}(L_d, \mathbb{Z})$. We have a natural injective map $L_d \to L_d^*$. Its cokernel is the discriminant group of L_d. Define

$$\widetilde{O}(L_d) = \{g \in O(L_d) \mid g \text{ acts trivially on } L_d^*/L_d \simeq \mathbb{Z}/2d\}.$$

Equivalently, $\widetilde{O}(L_d)$ is the stabiliser of λ in $O(L)$. The key (difficult) facts are:

(1) $\widetilde{O}(L_d)\backslash\Omega_d$ is a quasi-projective irreducible variety over $\mathbb{C}$ (Baily–Borel);

(2) there is a coarse moduli space M_d of K3 surfaces with a primitive polarisation of degree $2d$;

(3) M_d is a Zariski open subscheme of $\widetilde{O}(L_d)\backslash\Omega_d$.

Note that M_d is not smooth, though it is smooth as an orbifold (or as a Deligne–Mumford stack). It is constructed as a categorical quotient of the open subscheme of the relevant Hilbert scheme parameterising K3 surfaces in a given projective space by the action of the projective linear group. Fact (3) uses local and global Torelli theorems, and surjectivity of the period map, see [Huy16, Cor. 6.4.3]. This description is a K3 analogue of the coarse moduli space of elliptic curves $\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}$ or the moduli space of dimension g principally polarised abelian varieties $\mathcal{A}_g = \mathrm{Sp}(2g,\mathbb{Z})\backslash\mathcal{H}_g$, where $\mathcal{H}$ is the usual upper half-plane and $\mathcal{H}_g$ is the Siegel upper half-plane. Here Ω_d, $\mathcal{H}$, $\mathcal{H}_g$ are Hermitian symmetric domains, so property (1) follows from the Baily–Borel theorem about quotients of Hermitian symmetric domains by torsion-free arithmetic sugroups of their automorphism groups. (One needs to first apply the Baily–Borel theorem to a torsion-free finite index subgroup of $\widetilde{O}(L_d)$, and then take a quotient of a variety by a finite group action.)

Replacing $O(L_d)$ by the index 2 subgroup $\mathrm{SO}(L_d)$ gives rise to an unramified cover $\widetilde{M}_d \to M_d$. Here $\widetilde{M}_d$ is a Zariski open subset of $\widetilde{\mathrm{SO}(L_d)}\backslash\Omega_d$, where $\widetilde{\mathrm{SO}(L_d)} = \mathrm{SO}(L_d) \cap \widetilde{O}(L_d)$. This replaces the non-connected orthogonal group by the connected special orthogonal group. The degree of $\widetilde{M}_d \to M_d$ is 2 unless $d = 1$. In the exceptional case $d = 1$ the group $O(L_1) = \widetilde{O}(L_1)$ contains -1 which acts trivially on Ω_1, hence $O(L_1) = \{\pm 1\} \times \mathrm{SO}(L_1)$ and thus $\widetilde{M}_1 \to M_1$ is an isomorphism.

We have seen that a point in Ω_d is a homomorphism $\mathbb{S} \to \mathrm{SO}(L_d)_{\mathbb{R}}$. The action of $\mathrm{SO}(L_d)(\mathbb{R})$ on Ω_d is transitive, so Ω_d can be identified with the conjugacy class of h in $\mathrm{Hom}(\mathbb{S}, \mathrm{SO}(L_d)(\mathbb{R}))$. This is similar to the classical identification of $\mathcal{H} = \mathrm{SL}(2)(\mathbb{R})/\mathrm{SO}(2)(\mathbb{R})$ with the conjugacy class of $\mathbb{S} \subset \mathrm{GL}(2)^+_{\mathbb{R}}$. (Here $\mathrm{GL}(2)^+_{\mathbb{R}}$ is given by the condition $\det(x) > 0$; note that $\mathrm{GL}(2)(\mathbb{R})/\mathbb{S} = \mathcal{H}^{\pm}$.)

In modern language $\mathcal{A}_g$ and $\widetilde{\mathrm{SO}(L_d)}\backslash\Omega_d$ are the sets of complex points of Shimura varieties. To exploit the connection between moduli spaces of primitively polarised K3 surfaces and Shimura varieties, we now give a very brief introduction to Shimura varieties, referring the reader to Deligne's foundational paper [Del79] and Milne's lecture notes [Mil05] for a systematic treatment.

Orthogonal Shimura variety

A *Shimura datum* is a pair (G, X), where G is a connected reductive algebraic group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class in $\mathrm{Hom}(\mathbb{S}, G(\mathbb{R}))$ satisfying certain axioms ensuring that each connected component of X is a Hermitian symmetric domain. Morphisms of Shimura data are defined in the obvious way. In the K3 case, let $\mathrm{SO}(L_d)$ be the group scheme over $\mathbb{Z}$ whose functor of points associates to a ring R the group $\mathrm{SO}(L_d \otimes_{\mathbb{Z}} R)$. Then $(\mathrm{SO}(L_d)_{\mathbb{Q}}, \Omega_d)$ is a Shimura datum. In the case of principally polarised abelian varieties the Shimura datum is $(\mathrm{GSp}_{2g,\mathbb{Q}}, \mathcal{H}_g^{\pm})$.

A *congruence subgroup* is a subgroup of $G(\mathbb{Q})$ cut out by a compact open subgroup $K \subset G(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$, where $\mathbf{A}_{\mathbb{Q},\mathrm{f}} = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the ring of finite adèles. (Equivalently, a congruence subgroup is a subgroup that has a subgroup of finite index which preserves a lattice modulo some integer N in a rational representation.) Deligne's definition of the Shimura variety defined by the Shimura datum (G, X)

and a compact open subgroup $K \subset G(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$ is

$$\mathrm{Sh}_K(G,X)_{\mathbb{C}} = G(\mathbb{Q})\backslash X \times G(\mathbf{A}_{\mathbb{Q},\mathrm{f}})/K,$$

where $G(\mathbb{Q})$ acts diagonally on both factors on the left, whereas K acts on $G(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$ on the right. The crucial fact is that any Shimura variety $\mathrm{Sh}_K(G,X)$ is defined over a number field; it is a so called *canonical model*. The set $\mathrm{Sh}_K(G,X)(\mathbb{C})$ is a disjoint union of the quotients $\Gamma\backslash X_+$, where X_+ is a connected component of X and Γ is a congruence subgroup of the stabiliser of X_+ in $G(\mathbb{Q})$.

We now go back to the K3 surfaces Shimura datum $(\mathrm{SO}(L_d)_{\mathbb{Q}}, \Omega_d)$. Let $\mathbb{K} \subset \mathrm{SO}(L_d)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$ be a compact open subgroup. The canonical model of the associated Shimura variety $\mathrm{Sh}_K(L_d) := \mathrm{Sh}_{\mathbb{K}}(\mathrm{SO}(L_d)_{\mathbb{Q}}, \Omega_L)$ is a quasi-projective variety over $\mathbb{Q}$. By construction, the $\mathbb{C}$-points of $\mathrm{Sh}_K(L_d)$ parameterise $\mathbb{Z}$-Hodge structures on L_d of K3 type, see Definition 15.5.1.

Suppose that $\mathbb{K}$ is neat. (See R. Pink's thesis [Pin, pp. 4-5] for the definition of neatness and the fact that every compact open subgroup of $\mathrm{SO}(L_d)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$ contains a neat subgroup of finite index.) Then for each prime ℓ there is a lisse $\mathbb{Z}_\ell$-sheaf $L_{d,\ell}$ on $\mathrm{Sh}_K(L_d)$ defined by the inverse system of finite étale covers $\mathrm{Sh}_{\mathbb{K}(\ell^m)}(L_d) \to \mathrm{Sh}_K(L_d)$, where $\mathbb{K}(\ell^m)$ is the largest subgroup of $\mathbb{K}$ that acts trivially on L/ℓ^m. Thus, to a k-point x of $\mathrm{Sh}_K(L_d)$ there corresponds a representation $\mathrm{Gal}(\bar{k}/k) \to \mathrm{SO}(L_d \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$. Putting together these representations for all ℓ gives a representation

$$\phi_x : \mathrm{Gal}(\bar{k}/k) \longrightarrow \mathrm{SO}(L_d \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}). \tag{15.9}$$

We refer to this as the *monodromy representation*.

Spin Shimura variety

From a lattice with signature $(2,n)$, $n \geq 1$, one can also construct a spin Shimura variety. Recall that $\mathrm{Cl}(L_d)$ is the Clifford algebra of L_d, and $\mathrm{Cl}^+(L_d) \subset \mathrm{Cl}(L_d)$ is the even Clifford algebra. Let $\mathrm{GSpin}(L_d)$ be the group $\mathbb{Z}$-scheme whose functor of points associates to a ring R the group of invertible elements g of $\mathrm{Cl}^+(L_d \otimes_{\mathbb{Z}} R)$ such that $g(L_d \otimes_{\mathbb{Z}} R)g^{-1} = L_d \otimes_{\mathbb{Z}} R$.

Recall that $h : \mathbb{S} \to \mathrm{SO}(L_d)_{\mathbb{R}}$ canonically lifts to $\tilde{h} : \mathbb{S} \to \mathrm{GSpin}(L_d)_{\mathbb{R}}$. It follows that the $\mathrm{GSpin}(L_d)(\mathbb{R})$-conjugacy class of $\tilde{h} : \mathbb{S} \to \mathrm{GSpin}(L_d)_{\mathbb{R}}$ maps bijectively to Ω_d, which is the $\mathrm{SO}(L_d)(\mathbb{R})$-conjugacy class of h. This shows that the homomorphism $\mathrm{GSpin}(L_d) \to \mathrm{SO}(L_d)$ naturally extends to a morphism of Shimura data

$$(\mathrm{GSpin}(L_d)_{\mathbb{Q}}, \Omega_d) \longrightarrow (\mathrm{SO}(L_d)_{\mathbb{Q}}, \Omega_d).$$

If $\widetilde{\mathbb{K}} \subset \mathrm{GSpin}(L_d)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$ is a compact open subgroup, we write $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d)$ for the Shimura variety $\mathrm{Sh}_{\widetilde{\mathbb{K}}}(\mathrm{GSpin}(L_d)_{\mathbb{Q}}, \Omega_d)$. We can take $\mathbb{K}$ to be the image of $\widetilde{\mathbb{K}}$ in $\mathrm{SO}(L)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$; indeed, by [And96, 4.4], $\mathbb{K}$ is compact and open in $\mathrm{SO}(L_d)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$. The natural group homomorphism $\mathrm{GSpin}(L_d)_{\mathbb{Q}} \to \mathrm{SO}(L_d)_{\mathbb{Q}}$ induces a morphism

$\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d) \to \mathrm{Sh}_{\mathbb{K}}(L_d)$. This morphism is finite and surjective, and is defined over $\mathbb{Q}$, see [And96, App. 1].

Let $\widetilde{\mathbb{K}}_N \subset \mathrm{GSpin}(L_d)(\hat{\mathbb{Z}})$ be the set of elements congruent to 1 modulo N in $\mathrm{Cl}^+(L_d \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})$. If $\widetilde{\mathbb{K}} \subset \widetilde{\mathbb{K}}_N$ for $N \geq 3$, then $\widetilde{\mathbb{K}}$ and $\mathbb{K}$ are neat and the morphism $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d) \to \mathrm{Sh}_{\mathbb{K}}(L_d)$ is étale. This morphism restricts to an isomorphism on each geometric connected component [Riz10, §5.5, (32)]. Thus $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d) \to \mathrm{Sh}_{\mathbb{K}}(L_d)$ has a section defined over a number field E which only depends on $\widetilde{\mathbb{K}}$.

Kuga–Satake construction III: the Kuga–Satake abelian scheme

The choice of a polarisation of the integral Hodge structure on $\mathrm{Cl}(L_d)$ defines a morphism of Shimura data

$$(\mathrm{GSpin}(L_d)_{\mathbb{Q}}, \Omega_d) \longrightarrow (\mathrm{GSp}_{2g,\mathbb{Q}}, \mathcal{H}_g^{\pm}),$$

where $g = 2^{20}$. Moreover, there is a finite morphism of Shimura varieties from $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d)$ to a moduli space of abelian varieties, defined over $\mathbb{Q}$. In order to construct this, we find a skew-symmetric form on $\mathrm{Cl}(L_d)$ following [Huy16, Ch. 4, 2.2]. For this we choose orthogonal elements $f_1, f_2 \in L_d$ satisfying $(f_1^2), (f_2^2) > 0$ and define a skew-symmetric form on $\mathrm{Cl}(L_d)$ by $\pm\mathrm{Tr}(f_1 f_2 v^* w)$, where $\mathrm{Tr}(x)$ is the trace of the left multiplication by $x \in \mathrm{Cl}(L_d)$ on $\mathrm{Cl}(L_d)$. The action of $\mathrm{GSpin}(L_d)$ on this form is multiplication by the spinor norm (see [Huy16, Ch. 4, Prop. 2.5] for proofs of these facts, as well as the correct choice of sign). The group $\mathrm{GSpin}(L_d)$ injects into the group of symplectic similitudes $\mathrm{GSp}(\mathrm{Cl}(L_d))$ of this form.

If $\widetilde{\mathbb{K}} \subset \widetilde{\mathbb{K}}_N$, then we have a morphism from $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d)$ to the Shimura variety $\mathrm{Sh}_{\Gamma_N}(\mathrm{GSp}(\mathrm{Cl}(L_d))_{\mathbb{Q}}, \mathcal{H}^{\pm})$, where Γ_N is the subgroup of $\mathrm{GSp}(\mathrm{Cl}(L_d))(\hat{\mathbb{Z}})$ consisting of the elements that are congruent to 1 modulo N. The latter Shimura variety is identified with the moduli variety $\mathcal{A}_{g,\delta,N}$ parameterising abelian varieties of dimension $g = 2^{n+1}$, polarisation type δ (explicitly computable in terms of L and f_1, f_2) and level structure of level N. If $N \geq 3$, then $\mathcal{A}_{g,\delta,N}$ is a fine moduli space carrying a universal family of abelian varieties.

Recall that E is a number field over which there exists a section of the morphism of Shimura varieties $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d)_E \to \mathrm{Sh}_{\mathbb{K}}(L_d)_E$. The definition of the Kuga–Satake abelian scheme depends on the choice of E.

Definition 15.7.1 *The* **Kuga–Satake abelian scheme** $f : A \to \mathrm{Sh}_{\mathbb{K}}(L_d)_E$ *is defined as the pullback of the universal family of abelian varieties on* $\mathcal{A}_{g,\delta,N}$ *to* $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d)$*, and then, after extending the ground field from* $\mathbb{Q}$ *to* E*, to* $\mathrm{Sh}_{\mathbb{K}}(L_d)_E$*.*

The left multiplication by the elements of $L_d \subset \mathrm{Cl}(L_d)$ on $\mathrm{Cl}(L_d)$ gives a homomorphism $L_d \hookrightarrow \mathrm{End}_{\mathbb{Z}}(\mathrm{Cl}(L_d))$ whose cokernel is torsion-free. Since $\mathrm{Cl}(L_d) = R^1 f_{\mathrm{an},*}\mathbb{Z}$ as sheaves on $\mathrm{Sh}_{\mathbb{K}}(L_d)_{\mathbb{C}}$, this gives rise to a morphism of variations of $\mathbb{Z}$-Hodge structures

$$L_d \hookrightarrow \mathrm{Cl}(L_d) \hookrightarrow \mathrm{End}_{\mathbb{Z}}(R^1 f_{\mathrm{an},*}\mathbb{Z}). \tag{15.10}$$

Via the comparison theorems we get a morphism of $\mathbb{Z}_\ell$-sheaves

$$L_{d,\ell} \hookrightarrow \mathrm{End}_{\mathbb{Z}_\ell}(R^1 f_* \mathbb{Z}_\ell). \tag{15.11}$$

Back to moduli spaces of K3 surfaces

Recall that M_d introduced in the beginning of this section is the coarse moduli space over $\mathbb{Q}$ of primitively polarised K3 surfaces of degree $2d$; this is a quasi-projective variety defined over $\mathbb{Q}$. Let $\widetilde{M}_d$ be the coarse moduli space over $\mathbb{Q}$ of triples (X, λ, u) such that X is a K3 surface over a field of characteristic 0, λ is a primitive polarisation of X of degree $2d$, and u is an isometry

$$\det(P^2(\overline{X}, \mathbb{Z}_2(1))) \longrightarrow \det(L_d \otimes_{\mathbb{Z}} \mathbb{Z}_2),$$

where $P^2(\overline{X}, \mathbb{Z}_2(1))$ is the orthogonal complement of the image of λ in the 2-adic étale cohomology $\mathrm{H}^2(\overline{X}, \mathbb{Z}_2(1))$. We have an unramified cover $\widetilde{M}_d \to M_d$ (of degree 2 unless $d = 1$, when this is an isomorphism). By the work of Rizov and Madapusi Pera based on the Torelli theorem [PSS71], there is an open immersion $\widetilde{M}_d \hookrightarrow \mathrm{Sh}_{\mathbb{K}_d}(L_d)$ defined over $\mathbb{Q}$, where

$$\mathbb{K}_d = \{g \in \mathrm{SO}(L_d \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}) : g \text{ acts trivially on } L_d^*/L_d\}. \tag{15.12}$$

For a proof that this immersion is defined over $\mathbb{Q}$, see [MP15, Cor. 5.4] (see also [Riz10, Thm. 3.9.1]).

To a polarised K3 surface (X, λ) defined over a field k of characteristic 0 one can attach two Galois representations: the representation in étale cohomology and the monodromy representation. The first of them comes from the natural action of the Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$ on $\mathrm{H}^2_{\text{ét}}(\overline{X}, \hat{\mathbb{Z}}(1))$. For a prime ℓ define $P^2(\overline{X}, \mathbb{Z}_\ell(1))$ as the orthogonal complement to λ in $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$. Choose an isometry $u : \det(P^2(\overline{X}, \mathbb{Z}_2(1))) \tilde{\longrightarrow} \det(L_d \otimes_{\mathbb{Z}} \mathbb{Z}_2)$. After replacing k by a quadratic extension we can assume that Γ acts trivially on $\det(P^2(\overline{X}, \mathbb{Z}_2(1)))$. By [Sai12, Cor. 3.3] the quadratic character through which Γ acts on the 1-dimensional vector space $\det(\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)))$ does not depend on ℓ. Thus Γ acts trivially on $\det(P^2(\overline{X}, \mathbb{Z}_\ell(1)))$ for all primes ℓ, hence the representation $\rho_X : \Gamma \to \mathrm{O}(P^2(\overline{X}, \hat{\mathbb{Z}}(1)))$ attached to X takes values in $\mathrm{SO}(P^2(\overline{X}, \hat{\mathbb{Z}}(1)))$.

The triple (X, λ, u) defines a k-point x in $\widetilde{M}_{2d} \subset \mathrm{Sh}_{\mathbb{K}_d}(L_d)$. Choose a neat compact open subgroup $\mathbb{K}'_d \subset \mathbb{K}_d$ and let x' be a lift of x to $\mathrm{Sh}_{\mathbb{K}'_d}(L_d)$, so that x' is defined over a finite extension k' of k. Let $\Gamma' = \mathrm{Gal}(\bar{k}/k')$ and let $\phi_{x'} : \Gamma' \to \mathrm{SO}(L_d \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})$ denote the monodromy representation associated with the point x', as defined at (15.9).

Lemma 15.7.2 *Let (X, λ) be a primitively polarised K3 surface over a field k of characteristic 0. There exists a finite extension k'/k of explicitly bounded degree such that the adelic Galois representations $\rho_{X|\Gamma'} : \Gamma' \to \mathrm{SO}(P^2(\overline{X}, \hat{\mathbb{Z}}(1)))$ and $\phi_{x'} : \Gamma' \to \mathrm{SO}(L_d \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})$ are isometric, where $\Gamma' = \mathrm{Gal}(\bar{k}/k')$.*

Proof. This is an immediate consequence of [MP16, Prop. 5.6 (1)]. □

The conclusion of the work done in this section is the following proposition.

Proposition 15.7.3 *Let k be a field of characteristic 0, and let (X,λ) be a primitively polarised K3 surface over k. Let $P^2(\overline{X},\mathbb{Z}_\ell(1))$ be the orthogonal complement to λ in $\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))$, where ℓ is a prime. There exists a finite extension k'/k and an abelian variety A over k' with the following properties.*

(i) *For any prime ℓ there is an embedding of Γ'-modules, where $\Gamma'=\mathrm{Gal}(\bar{k}/k')$:*

$$P^2(\overline{X},\mathbb{Z}_\ell(1))\hookrightarrow \mathrm{End}_{\mathbb{Z}_\ell}(\mathrm{H}^1_{\text{ét}}(\overline{A},\mathbb{Z}_\ell)). \tag{15.13}$$

(ii) *Let $k\subset\mathbb{C}$, and let $P^2(X_\mathbb{C},\mathbb{Z}(1))$ be the orthogonal complement to λ in $\mathrm{H}^2(X_\mathbb{C},\mathbb{Z}(1))$. There is an embedding of integral Hodge structures of weigth 0:*

$$P^2(X_\mathbb{C},\mathbb{Z}(1))\hookrightarrow \mathrm{End}_{\mathbb{Z}}(\mathrm{H}^1(A_\mathbb{C},\mathbb{Z})). \tag{15.14}$$

The two embeddings are compatible via comparison isomorphisms between classical and ℓ-adic étale cohomology.

Proof. Let x be the k-point in M_d defined by (X,λ). After replacing k by a quadratic extension k'/k we can assume that x lifts to a k'-point on $\widetilde{M}_d\hookrightarrow \mathrm{Sh}_{\mathbb{K}_d}(L_d)$, where $\mathbb{K}_d$ is defined in (15.12). Pick a neat compact open subgroup $\tilde{\mathbb{K}}_d$ in $\mathrm{GSpin}(L_d)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$, for example, the set of elements congruent to 1 modulo N in $\mathrm{Cl}^+(L_d\otimes\hat{\mathbb{Z}})$, where $N\geq 3$. Let $\mathbb{K}'_d$ be the intersection of $\mathbb{K}_d$ with the image of $\tilde{\mathbb{K}}_d$ in $\mathrm{SO}(L_d)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$. Then $\mathbb{K}'_d$ is a neat compact open subgroup of $\mathbb{K}_d$. We enlarge k' so that x comes from a k'-point s on the cover $\mathrm{Sh}_{\mathbb{K}'_d}(L_d)$ of $\mathrm{Sh}_{\mathbb{K}_d}(L_d)$. We extend k' further to include the number field E over which there is a section of the morphism of Shimura varieties $\mathrm{Sh}^{\mathrm{spin}}_{\tilde{\mathbb{K}}_d}(L_d)_E\to\mathrm{Sh}_{\mathbb{K}'_d}(L_d)_E$. Now we have the Kuga–Satake abelian scheme $f: A\to\mathrm{Sh}_{\mathbb{K}'_d}(L_d)$, so $A=f^{-1}(s)$ is an abelian variety over k'. Now (15.13) is just the specialisation of (15.11) at the k'-point s. Similarly, (15.14) is the specialisation of (15.10). □

Given a polarised K3 surface X over k we can call an abelian variety A from Proposition 15.7.3 a *Kuga–Satake variety* of X. Indeed, for $k\subset\mathbb{C}$, by construction $A_\mathbb{C}$ is isomorphic to the complex Kuga–Satake variety of the complex K3 surface $X_\mathbb{C}$ as defined at the end of the previous section. What we gain now is that A is defined over a finite extension of k.

It is worth noting that Lemma 15.7.2 replaces Proposition 6.4 and Lemma 6.5.1 in Deligne's pioneering work [Del72] (written before the machinery of Shimura varieties was fully developed) in establishing that (15.13) is an isomorphism of Galois modules, cf. [Del72, Prop. 6.5].

15.8 Tate conjecture and the Brauer group of K3 surfaces

From Proposition 15.7.3 we obtain an embedding of $\mathrm{Gal}(\bar{k}/k')$-modules

$$P^2(\overline{X},\mathbb{Q}_\ell(1))\hookrightarrow \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{H}^1(A\times_{k'}\bar{k},\mathbb{Q}_\ell)). \tag{15.15}$$

Similarly, avoiding the finitely many primes dividing $2d$, one obtains an embedding of $\mathrm{Gal}(\bar{k}/k')$-modules

$$P^2(\overline{X}, \mu_\ell) \hookrightarrow \mathrm{End}_{\mathbb{F}_\ell}(A[\ell]). \tag{15.16}$$

Deligne used this to prove the Weil conjectures for K3 surfaces over finite fields (before he proved them for arbitrary varieties), but this theory has many other applications. For example, if k is finitely generated over $\mathbb{Q}$, then the semisimplicity of the Galois module $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$ for a K3 surface X follows from (15.15) and the semisimplicity for abelian varieties, as proved by Faltings.

Theorem 15.8.1 *Let X be a K3 surface over a field k finitely generated over $\mathbb{Q}$. Then the Tate conjecture holds for X, that is, we have*

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))^\Gamma = \mathrm{Pic}(\overline{X})^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}_\ell.$$

Proof. Let A be a Kuga–Satake abelian variety of X defined over a finite extension of k, as constructed in Proposition 15.7.3.

The profinite, hence compact group Γ acts continuously on the discrete group $\mathrm{Pic}(\overline{X})$, so this action factors through a finite quotient $\mathrm{Gal}(k'/k)$ of Γ, for some finite Galois extension k' of k. Thus it is enough to prove the theorem under the additional assumption that Γ acts trivially on $\mathrm{Pic}(\overline{X})$ and on $\mathrm{End}(\overline{A})$. We need to show that the Γ-invariant subspace of $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$ is $\mathrm{Pic}(\overline{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$. By Faltings, the Tate conjecture holds for A. Thus the Γ-invariant subspace of $\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{H}^1(\overline{A}, \mathbb{Q}_\ell))$ is $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$. Hence the image of $P^2(\overline{X}, \mathbb{Q}_\ell(1))^\Gamma$ in $\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{H}^1(\overline{A}, \mathbb{Q}_\ell))$ belongs to the $\mathbb{Q}_\ell$-span of the intersection of the image of $P^2(X_{\mathbb{C}}, \mathbb{Q}(1))$ in $\mathrm{End}_{\mathbb{Q}}(\mathrm{H}^1(A_{\mathbb{C}}, \mathbb{Q}))$ with $\mathrm{End}(\overline{A}) \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathrm{End}_{\mathbb{Q}}(\mathrm{H}^1(A_{\mathbb{C}}, \mathbb{Q}))$. But such elements of $\mathrm{End}_{\mathbb{Q}}(\mathrm{H}^1(A_{\mathbb{C}}, \mathbb{Q}))$ have Hodge type $(0,0)$. Hence every element of $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))^\Gamma$ is a $\mathbb{Q}_\ell$-linear combination of classes of type $(0,0)$ in $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Q}(1))$. By the Lefschetz theorem, each such class is algebraic. □

The following result was obtained in [SZ08], using Deligne's version of the Kuga–Satake construction [Del72].

Theorem 15.8.2 (Skorobogatov–Zarhin) *Let X be a K3 surface over a field k finitely generated over $\mathbb{Q}$. Then $\mathrm{Br}(\overline{X})^\Gamma$ is finite.*

Proof. The ℓ-primary torsion subgroup $\mathrm{Br}(\overline{X})^\Gamma\{\ell\}$ is finite for all primes ℓ. This follows from Theorems 15.2.1 and 15.8.1.

To prove that $\mathrm{Br}(\overline{X})^\Gamma[\ell] = 0$ for almost all ℓ, by the Kummer exact sequence it is enough to prove that for almost all ℓ we have $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_\ell)^\Gamma = (\mathrm{Pic}(\overline{X})/\ell)^\Gamma$. For this it is enough to show that $(T(X)/\ell)^\Gamma = 0$ for almost all ℓ, where $T(X)$ is the transcendental lattice of X. By the Lefschetz theorem and the non-degeneracy of the intersection pairing on $\mathrm{Pic}(\overline{X})$, the transcendental lattice $T(X)$ does not contain non-zero elements of Hodge type $(0,0)$. Hence the image of $T(X)$ in $\mathrm{End}_{\mathbb{Z}}(\mathrm{H}^1(A_{\mathbb{C}}, \mathbb{Z}))$ has trivial intersection with $\mathrm{End}(\overline{A})$. It follows that the image of $T(X)/\ell$ in $\mathrm{End}_{\mathbb{F}_\ell}(A[\ell])$ intersects trivially with $\mathrm{End}(\overline{A})/\ell = \mathrm{End}(A)/\ell$ for almost all ℓ. By Faltings and Zarhin, for almost all ℓ we have

$$\mathrm{End}_{\mathbb{F}_\ell}(A[\ell])^\Gamma = \mathrm{End}(A)/\ell.$$

Thus $(T(X)/\ell)^\Gamma = 0$ for almost all ℓ. □

Corollary 15.8.3 *Let X be a K3 surface over a field k finitely generated over $\mathbb{Q}$. The group* $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ *is finite.*

Remark 15.8.4 Let k be a field of characteristic $p > 0$ which is finitely generated over $\mathbb{F}_p$. Then the subgroups of $\mathrm{Br}(\overline{X})^\Gamma$ and $\mathrm{Br}(X)/\mathrm{Br}_0(X)$, which consist of the elements of order prime to p, are both finite. For $p > 2$ this is proved in [SZ15] using work of Rizov and Madapusi Pera [MP15], and Zarhin [Zar76, Zar77, Zar85]. For $p = 2$ this is proved by K. Ito in [Ito18] using instead of [MP15] a more recent work of Wansu Kim and Madapusi Pera proving the Tate conjecture and essentially establishing the Kuga–Satake construction in characteristic 2.

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