Algebraic number theory

Test 1

18 February, 2011

1. (8 marks; 2 marks for each element)
$1+\sqrt{2}$ has norm -1, so this is unit.
$2+\sqrt{2}$ has norm 2 so this element is irreducible.
$3+\sqrt{2}$ has norm 7, so this is also irreducible.
$4+\sqrt{2} = \sqrt{2}(1+2\sqrt{2})$, where both factors are irreducible, so this element is neither a unit nor an irreducible.

2. (6 marks)
(a) (4 marks) $\beta = \sqrt[3]{pq^2}$ is a root of $t^3 - pq^2 = 0$, so β is an algebraic integer. Since $\beta \in K$ we have $\beta \in \mathcal{O}_K$.
1, α and α^2 form a basis of K as a vector space over $\mathbb{Q}$. So if $\beta \in \mathbb{Z}[\alpha]$, then β is an integer multiple of α^2, which not true.

(b) (2 marks) $\mathbb{Z}[\alpha]$ is not an integrally closed ring, because β is in the field of fractions K of $\mathbb{Z}[\alpha]$, is integral, and yet is not in $\mathbb{Z}[\alpha]$.

3. (6 marks)
Note that $\alpha\beta = pq$, $\alpha^2 = p\beta$, $\beta^2 = q\alpha$. It follows that $\mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\beta$ is a ring, and is stable under the multiplication by $\alpha + \beta$, which acts as the matrix

$$\begin{pmatrix} 0 & pq & pq \\ 1 & 0 & q \\ 1 & p & 0 \end{pmatrix}$$

Its characteristic polynomial is $t^3 - 3pqt - pq(p+q)$, so this gives the desired integral dependence relation. (You can also compute with the ring $\mathbb{Z}[\alpha]$ or with $\mathbb{Z}[\beta]$, the result is the same.)