Algebraic number theory

Test 1

18 February, 2011

You can use all statements from lectures or problem sheets without proof.

1. Decide which of the following elements of $\mathbb{Z}[\sqrt{2}]$ are units, irreducibles, or neither (a very brief explanation will suffice):
$1+\sqrt{2}$, $2+\sqrt{2}$, $3+\sqrt{2}$, $4+\sqrt{2}$.

In Questions 2 and 3, p and q are prime numbers,

$$\alpha = \sqrt[3]{p^2 q}, \quad \beta = \frac{1}{p}\alpha^2, \quad K = \mathbb{Q}(\alpha), \quad \mathbb{Z}[\alpha] = \{l + m\alpha + n\alpha^2 | l, m, n \in \mathbb{Z}\}.$$

2. (a) Prove that $\beta \in \mathcal{O}_K$ and $\beta \notin \mathbb{Z}[\alpha]$.
(b) Using the result of part (a) decide if $\mathbb{Z}[\alpha]$ is an integrally closed ring.

3. Find an integral dependence relation for $\alpha + \beta$.