Superelliptic Jacobians, Brauer groups and Kummer varieties

Yuri G. Zarhin (Penn State/MPIM)

(based on a joint work with Alexei N. Skorobogatov)

Plan of the talk

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- $\mathrm{Br}(k)$ is the Brauer group of k (it is a torsion abelian group).

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The case $p = 2$ for $K3$ surfaces was settled by Kazuhiro Ito (2017)

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Here is a more elaborated version.

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This allows one to represent elements of $\mathrm{Br}(X)(\mathrm{non}-2)$ by explicit cup-products, and so evaluate them at local points.