Cubic fourfolds of discriminant 18 and odd-torsion Brauer–Manin obstructions to the Hasse Principle on general K3 surfaces

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We report on results of a collaboration

Everything today is joint work with Jennifer Berg.

Brauer–Manin recap

X/k a nice variety over a number field.

$H \subseteq \mathrm{Br}(X) := \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m)_{\mathrm{tors}}$ gives rise to an obstruction set

$$X(k) \subseteq X(\mathbb{A})^H \subseteq X(\mathbb{A}) := \prod_v X(k_v)$$

that contains $\overline{X(k)}$ (for product topology).

Brauer–Manin recap

Brauer–Manin obstruction to Hasse Principle:

$$X(\mathbb{A}) \neq \emptyset \quad \text{but} \quad X(\mathbb{A})^H = \emptyset \quad \text{for some } H \subseteq \mathrm{Br}(X).$$

Brauer–Manin obstruction to Weak Approximation:

$$X(\mathbb{A}) \ \setminus \ X(\mathbb{A})^H \neq \emptyset \quad \text{for some } H \subseteq \mathrm{Br}(X).$$

A conjecture of Skorobogatov

Conjecture (Skorobogatov, 2009)

The Brauer–Manin obstruction accounts for failures of the Hasse Principle and Weak Approximation on K3 surfaces.

More precisely, if X/k is a locally soluble K3 surface over a number field, then

$$\overline{X(k)} = X(\mathbb{A})^{\mathrm{Br}(X)}.$$

K3 surface examples

X/k a projective K3 surface over a number field.

$H = \mathrm{Br}(X)[2]$ can obstruct the Weak Approximation:

- Wittenberg, 2004
- Ieronymou, 2010.
- Hassett, V.-A., 2011.
- Elsenhans, Jahnel, 2013.
- Mckinnie, Sawon, Tanimoto, V.-A., 2017.

$H = \mathrm{Br}(X)[2]$ can obstruct the Hasse principle:

- Martin Bright, 2002.
- Hassett, V.-A., 2013.

$H = \mathrm{Br}(X)[3]$ or $\mathrm{Br}(X)[5]$ can obstruct the Weak Approximation:

- Preu, 2013.
- Ieronymou–Skorobogatov, 2015.

What about odd torsion and the Hasse Principle?

Question (Ieronymou–Skorobogatov, 2015)

Does there exist a locally soluble K3 surface X/k over a number field with $X(\mathbb{A})^{\mathrm{Br}(X)_{\mathrm{odd}}} = \emptyset$?

Theorem (Corn–Nakahara, 2017)

The degree 2 K3 surface $X/\mathbb{Q}$

$$w^2 = -3x^6 + 97y^6 + 97 \cdot 28 \cdot 7z^6$$

is locally soluble. The cyclic algebra

$$\mathscr{A} := \left(\mathbb{Q}(\sqrt[3]{28}, \zeta_3)/\mathbb{Q}(\zeta_3), \frac{w - \sqrt{-3}x^3}{w + \sqrt{-3}x^3} \right) \in \mathrm{Br}\,\mathbb{Q}(\zeta_3)(X)$$

extends to an element of $\mathrm{Br}(X_{\mathbb{Q}(\zeta_3)})$ that gives rise to a Brauer-Manin obstruction to the Hasse Principle on X.

What about odd torsion in the Brauer group?

The class $\mathscr{A}$ in Corn–Nakahara is algebraic.

Recall there is a filtration

$$\underbrace{\mathrm{Br}_0(X)}_{\substack{\mathrm{im}(\mathrm{Br}\,k \to \mathrm{Br}\,X) \\ \text{constant classes}}} \subseteq \underbrace{\mathrm{Br}_1(X)}_{\substack{\ker(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})) \\ \text{algebraic classes}}} \subseteq \mathrm{Br}(X).$$

CFT $\implies X(\mathbb{A})^{\mathrm{Br}_0(X)} = X(\mathbb{A})$, so we often consider the quotients

$$\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \quad \text{and} \quad \mathrm{Br}(X)/\mathrm{Br}_1(X)$$

when computing Brauer–Manin obstructions.

Ideas in Corn–Nakahara

Invert isomorphism

$$\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \xrightarrow{\sim} \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(\overline{X}))$$

coming from Hochschild–Serre spectral sequence.

Hardest step: writing down an explicit basis for $\mathrm{Pic}(\overline{X}) \simeq \mathbb{Z}^{20}$:

- Use readily available divisors to produce rank 20 sublattice.
- Add a special divisor from Dino Festi's 2016 PhD Thesis.
- Check the lattice obtained is saturated using intersection numbers from extra divisors at primes of supersingular reduction.

Can odd transcendental classes obstruct HP on a K3?

Theorem (Berg, V.-A., 2018)

There exists a K3 surface X *over* $\mathbb{Q}$ *of degree* 2*, together with an* $\mathscr{A} \in \mathrm{Br}(X)[3]$*, such that*

$$X(\mathbb{A}) \neq \emptyset \quad \text{and} \quad X(\mathbb{A})^{\mathscr{A}} = \emptyset.$$

Moreover, we have $\mathrm{Pic}(\overline{X}) \simeq \mathbb{Z}$*, and hence* $\mathrm{Br}_1(X)/\mathrm{Br}_0(X) = 0$*. In particular, there is no algebraic Brauer–Manin to the Hasse Principle on* X*.*

Can odd transcendental classes obstruct HP on a K3?

How can we find transcendental 3-torsion in $\mathrm{Br}(X)$?

$X/\mathbb{C}$: a complex projective K3 surface with $\mathrm{NS}(X) = \mathbb{Z}h$, $h^2 = 2d$.
Let

$$T_X := \mathrm{NS}(X)^{\perp} \subseteq \mathrm{H}^2(X, \mathbb{Z}) \cong U^3 \oplus E_8(-1)^2 =: \Lambda_{K3}$$

be the transcendental lattice of X.

$\mathrm{Br}(X) \simeq T_X^* \otimes \mathbb{Q}/\mathbb{Z}$, so there is a one-to-one correspondence

$$\{\langle\alpha\rangle \subset \mathrm{Br}\, X \text{ of order } n\} \quad \xleftrightarrow{1-1} \quad \{\text{surjections } T_X \to \mathbb{Z}/n\mathbb{Z}\}$$

Hence, to α as above, we may associate $T_\alpha \subseteq T_X$:

$$T_\alpha = \ker(\alpha\colon T_X \to \mathbb{Z}/n\mathbb{Z}).$$

Sublattices of index 3 in T_X

Theorem (Mckinnie, Sawon, Tanimoto, V.-A., 2017)

Let X be a complex projective K3 surface with $\operatorname{Pic} X \cong \mathbb{Z}h$, $h^2 = 2$, and let $\alpha \in (\operatorname{Br} X)[3]$. Then either

1. *there is a unique primitive embedding $T_\alpha \hookrightarrow \Lambda_{K3}$. This gives a degree 18 K3 surface Y associated to the pair $(X, \langle\alpha\rangle)$; OR*
2. *$T_\alpha(-1) \cong \langle h^2, T\rangle^\perp \subseteq \mathrm{H}^4(Y, \mathbb{Z})$, where Y is a cubic fourfold of discriminant 18 (h is the hyperplane class); OR,*
3. *$T_\alpha(-1)$ is a lattice with discriminant group $\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$.*

Cubic fourfolds of discriminant 18

Hence, lattice theory suggests:

Y cubic fourfold of discriminant 18 $\rightsquigarrow$ $(X, \langle\alpha\rangle)$,

where X is a K3 surface of degree 2 and $\langle\alpha\rangle \subset \mathrm{Br}(X)[3]$.

A cubic fourfold of discriminant 18 is a smooth cubic fourfold $Y \subseteq \mathbb{P}^5$, together with a rank two saturated lattice

$$\langle h^2, T\rangle \subset H^{2,2}(Y) \cap H^4(Y, \mathbb{Z})$$

of discriminant 18, where h is the hyperplane class,

	h^2	T
h^2	3	0
T	0	6

Hassett, 2000: such fourfolds exist.

Cubic fourfolds of discriminant 18

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of discriminant 18, where h is the hyperplane class,

	h^2	T
h^2	3	6
T	6	18

Hassett, 2000: such fourfolds exist.

What is the surface T?

Theorem (Addington, Hassett, Tschinkel, V.-A., 2016)

A general cubic fourfold Y of discriminant 18 contains an elliptic ruled surface T of degree 6, and the linear system of quadrics in $\mathbb{P}^5$ containing T is 3-dimensional.

Let $Q_0, Q_1, Q_2 \in \mathbb{C}[x_0, \ldots, x_5]_2$ be a basis for $\mathrm{H}^0(T, \mathscr{I}_T(2))$

$$\begin{array}{ccc} \mathrm{Bl}_T(Y) & & \\ \downarrow & \searrow^{\pi} & \\ Y & \dashrightarrow & \mathbb{P}(\mathrm{H}^0(T, \mathscr{I}_T(2))^*) = \mathbb{P}^2 \end{array}$$

$$\vec{x} = [x_0, \ldots, x_5] \longmapsto [Q_0(\vec{x}), Q_1(\vec{x}), Q_2(\vec{x})]$$

Fibers of π

Key insight:

General fiber of $\pi\colon \mathrm{Bl}_T(Y) \to \mathbb{P}^2$ is a del Pezzo surface of degree 6.

Geometrically, a dP6 is a blow-up of $\mathbb{P}^2$ at 3 non-colinear points.

The locus of fibers that are geometrically blow-ups of $\mathbb{P}^2$ at three distinct colinear points has image a sextic curve under π:

$$C : f(x,y,z) = 0.$$

The extension $K := \mathbb{C}(\mathbb{P}^2)(\sqrt{f})$ splits the two triples of pairwise skew exceptional curves in the generic fiber S of π.

K is the function field of our K3 surface X of degree 2!

Brauer elements of order 3

Let S_K be the (dP6) generic fiber of π base extended to K.

S_K contains two triples of pairwise skew exceptional curves:

$$\{E_1, E_2, E_3\} \quad \text{and} \quad \{L - E_1 - E_2, L - E_2 - E_3, L - E_1 - E_3\}.$$

These can be blown-down, respectively, with the morphisms associated to the linear systems

$$|L| \quad \text{and} \quad |2L - E_1 - E_2 - E_3|.$$

Brauer elements of order 3

We get

$$\phi_{|L|}\colon S_K \to X_1 \quad \text{and} \quad \phi_{|2L-E_1-E_2-E_3|}\colon S_K \to X_2$$

X_1 and X_2 are Severi-Brauer varieties of dimension 2 over K.

Hence X_1 and X_2 give rise to two classes $\mathscr{A}_1$ and $\mathscr{A}_2 \in (\operatorname{Br} K)[3]$.

Proposition (Corn, 2005)

$\mathscr{A}_1 \mathscr{A}_2 = \mathrm{Id}$ *in* $\operatorname{Br} K$*.*

Combining work of Corn, Kollár, and [AHTVA], can spread these classes to our K3 surface X.

We have recovered a pair $(X, \{\mathrm{Id}, \mathscr{A}_1, \mathscr{A}_2\})$!

Brauer elements of order 3: alternatively...

In each smooth fiber S of π, there are two families of twisted cubics, each one two dimensional, parametrized by

$$\mathbb{P}(\mathrm{H}^0(S,\mathscr{O}_S(L))) = \mathbb{P}^2 \quad \text{and} \quad \mathbb{P}(\mathrm{H}^0(S,\mathscr{O}_S(2L-E_1-E_2-E_3))) = \mathbb{P}^2.$$

These two $\mathbb{P}^2$'s come together over the locus $f(x,y,z)=0$ of the base $\mathbb{P}^2$ of π.

Let H be the relative Hilbert scheme that parametrizes twisted cubics in the fibers of π.

The Stein factorization of $H \to \mathbb{P}^2$ is

$$H \to X \to \mathbb{P}^2$$

where X is our K3 surface, and H is an étale $\mathbb{P}^2$ bundle over X.

Brauer elements of order 3: alternatively...

Hence H gives rise to an element $\mathscr{A} \in \mathrm{Br}(X)$.

Notes:

1. The covering involution $\iota \colon X \to X$ sending $w \mapsto -w$ induces

$$\iota^* \colon \mathrm{Br}(X) \to \mathrm{Br}(X)$$

   sending $\mathscr{A} \mapsto \mathscr{A}^{-1}$, so H actually encodes the *group* $\langle \mathscr{A} \rangle$.

2. If (Y, T) are defined over a field k (e.g., a number field), then so are H and X.

Computing the obstruction

Theorem (Wedderburn)

Every central simple algebra of degree 3 *over a field is cyclic.*

In principle, can write down cyclic representatives for $\mathscr{A}_1$ and $\mathscr{A}_2$.

Challenge

Do it.

Computing the obstruction

$$X(\mathbb{A})^{\mathscr{A}} = \left\{ (P_v) \in X(\mathbb{A}) : \sum_v \operatorname{inv}_v \mathscr{A}(P_v) = 0 \right\}.$$

To produce an example with $X(\mathbb{A})^{\mathscr{A}} = \emptyset$, we rig $(X, \mathscr{A})$ so that

$$\operatorname{inv}_v \mathscr{A}(P_v) = \begin{cases} 0 & \text{if } v \neq 3, \\ 1/3 \text{ or } 2/3 & \text{if } v = 3. \end{cases}$$

Key observation:

$\operatorname{inv}_v \mathscr{A}(P_v) = 0$ if and only if the fiber in H above $P_v \in X(k_v)$ is a *split* $\mathbb{P}^2$, i.e., is isomorphic to $\mathbb{P}^2$ over k_v.

Lang–Nishimura to the rescue

Lemma (Lang–Nishimura)

Let X and Y be birational smooth projective k-varieties. Then

$$X(k) \neq \emptyset \iff Y(k) \neq \emptyset.$$

The fiber above $P_v \in X(\mathbb{Q}_v)$ is birational to the dP6 fiber of π above the image of P_v in $\mathbb{P}^2$.

Apply Lang-Nishimura: to have $\mathrm{inv}_3 \mathscr{A}(P) \neq 0$ for all $P \in X(\mathbb{Q}_3)$, it suffices to have $\mathrm{Bl}_T(Y)(\mathbb{Q}_3) = \emptyset$.

Applying Lang-Nishimura again, it suffices to have $Y(\mathbb{Q}_3) = \emptyset$.

Construction of the cubic fourfold

Thus, the hardest part of our task is to produce a cubic fourfold $Y/\mathbb{Q}$ of discriminant 18, such that $Y(\mathbb{Q}_3) = \emptyset$.

Let $\mathbb{P}^5 := \operatorname{Proj} \mathbb{Q}[x_0, x_1, x_2, x_3, x_4, x_5]$, and define quadrics cut out by

$$Q_1 := -x_0x_3 + x_2x_3 - x_0x_4 + x_1x_4 + 3x_2x_4 + 5x_0x_5 - x_1x_5$$
$$Q_2 := -x_1x_3 + 5x_0x_4 - 2x_2x_4 - 2x_0x_5 + 5x_1x_5 + x_2x_5$$
$$Q_3 := -2x_2x_3 - x_0x_4 - 2x_1x_4 - 2x_2x_4 + x_1x_5$$

Each quadric contains the planes

$$\Pi_1 = \{x_0 = x_1 = x_2 = 0\} \quad \text{and} \quad \Pi_2 = \{x_3 = x_4 = x_5 = 0\}.$$

Construction of the cubic fourfold

The sextic elliptic surface T is obtained by saturating the ideal $\langle Q_1, Q_2, Q_3 \rangle$ with respect to $I(\Pi_1)I(\Pi_2)$. It is cut out by Q_1, Q_2, Q_3 and the two cubics

$$\begin{aligned}
&2x_3^3 + 5x_3^2x_4 + x_3x_4^2 + 14x_4^3 - 20x_3^2x_5 - 26x_3x_4x_5 \\
&\quad - 11x_4^2x_5 + 47x_3x_5^2 + 30x_4x_5^2 + 5x_5^3, \\
&2x_0^3 - x_0^2x_1 - 2x_0x_1^2 - x_1^3 + 47x_0^2x_2 + 10x_0x_1x_2 \\
&\quad + x_1^2x_2 - 11x_0x_2^2 - 18x_1x_2^2 - 4x_2^3
\end{aligned}$$

Lang-Nishimura to the rescue

The surface T is contained in the cubic fourfold Y cut out by

$$\begin{aligned}
&2x_0^3 - x_0^2x_1 - 2x_0x_1^2 - x_1^3 + 47x_0^2x_2 + 10x_0x_1x_2 \\
&+ x_1^2x_2 - 11x_0x_2^2 - 18x_1x_2^2 - 4x_2^3 + 18x_0^2x_3 + 18x_0x_1x_3 \\
&+ 9x_1^2x_3 + 18x_0x_2x_3 + 18x_1x_2x_3 + 18x_2^2x_3 + 9x_1x_3^2 + 6x_3^3 \\
&+ 36x_0^2x_4 + 9x_0x_1x_4 + 18x_1^2x_4 - 9x_0x_2x_4 + 18x_1x_2x_4 + 18x_2^2x_4 \\
&- 27x_0x_3x_4 + 18x_2x_3x_4 + 15x_3^2x_4 + 27x_0x_4^2 - 36x_2x_4^2 + 3x_3x_4^2 \\
&+ 42x_4^3 - 90x_0^2x_5 - 72x_0x_1x_5 - 45x_1^2x_5 - 18x_1x_2x_5 + 36x_0x_3x_5 \\
&- 45x_1x_3x_5 + 9x_2x_3x_5 - 60x_3^2x_5 - 54x_0x_4x_5 + 27x_1x_4x_5 - 18x_2x_4x_5 \\
&- 78x_3x_4x_5 - 33x_4^2x_5 - 90x_0x_5^2 + 141x_3x_5^2 + 90x_4x_5^2 + 15x_5^3 = 0.
\end{aligned}$$

We check that $Y(\mathbb{Z}/27\mathbb{Z}) = \emptyset$, so $Y(\mathbb{Q}_3) = \emptyset$.

The K3 surface

The K3 surface $X \subset \mathbb{P}(1,1,1,3)$ is given by

$$
\begin{aligned}
w^2 = {} & 17279788x^6 + 21966980x^5y + 5209685x^4y^2 - 10091766x^3y^3 \\
& - 9449085x^2y^4 - 3512294xy^5 - 510755y^6 + 81563000x^5z \\
& + 46799342x^4yz - 48304566x^3y^2z - 68669390x^2y^3z - 29936552xy^4z \\
& - 4960696y^5z + 132675265x^4z^2 - 24537700x^3yz^2 - 153420566x^2y^2z^2 \\
& - 94604246xy^3z^2 - 18001746y^4z^2 + 88262884x^3z^3 - 116707356x^2yz^3 \\
& - 139178230xy^2z^3 - 36604266y^3z^3 + 12231034x^2z^4 - 90599148xyz^4 \\
& - 40695955y^2z^4 - 11073000xz^5 - 22207274yz^5 - 3652475z^6.
\end{aligned}
$$

We show $\mathrm{Pic}(\overline{X}) \simeq \mathbb{Z}$. Hence $\mathrm{Br}_1(X)/\mathrm{Br}_0(X) = 0$.

Primes of bad reduction of Y

3, 5, 29, 2851, 1647622003,

8990396491695741359,

381640024919828593698301,

2329843929357212310902171133509290569012
6256356826741414312843163784586626801847,

7063057306288478297872948874470665724682
4151776978742375050861454515493652288934
3534041032125651313541554759455608434088
0768251657255814972524891.

Main Theorem

Theorem (Berg, V.-A., 2018)

There exists a K3 surface X over $\mathbb{Q}$ of degree 2, together with an $\mathscr{A} \in \mathrm{Br}(X)[3]$, such that

$$X(\mathbb{A}) \neq \emptyset \quad \textit{and} \quad X(\mathbb{A})^{\mathscr{A}} = \emptyset.$$

Moreover, we have $\mathrm{Pic}(\overline{X}) \simeq \mathbb{Z}$, and hence $\mathrm{Br}_1(X)/\mathrm{Br}_0(X) = 0$. In particular, there is no algebraic Brauer–Manin to the Hasse Principle on X.

A question

Definition (Creutz–Viray, 2017)

Let X be smooth projective geometrically integral variety over a number field k. We say degrees capture the Brauer-Manin obstruction on X if for any d that is the degree of a k-rational globally generated ample line bundle on X, we have

$$X(\mathbb{A})^{\mathrm{Br}(X)} = \emptyset \implies X(\mathbb{A})^{\mathrm{Br}(X)[d]} = \emptyset.$$

Question

The surface $X/\mathbb{Q}$ in Berg–V.-A. has a $\mathbb{Q}$-rational globally generated ample line bundle of degree 32. Is it true that

$$X(\mathbb{A})^{\mathrm{Br}(X)[2^\infty]} = \emptyset?$$