The theory of CM for K3 surfaces

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Introduction

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$X/\mathbb{C}$ has CM if $E(X) := \mathrm{End}_{\mathrm{Hdg}}(T(X)_{\mathbb{Q}})$ is a CM field (where complex conjugation is given by the adjunction with respect to the intersection pairing) and $\dim_{E(X)} T(X)_{\mathbb{Q}} = 1$.

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Fact:

One can show [Taelman] that for every CM number field E with the "right" dimension, i.e. $[E\colon \mathbb{Q}] \leq 20$, there exists a K3 surface X with CM by E.

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Remarks

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Question 2 was studied by Schütt when $\rho(X) = 20$, who gave an upper and lower bound for a field of definition.

Some basic definitions and facts

Facts:

- Every K3 surface $X/\mathbb{C}$ with CM comes equipped with a canonical embedding $\sigma_X : E(X) \hookrightarrow \mathbb{C}$. Its image $E \subset \mathbb{C}$ is the reflex field of $T(X)_{\mathbb{Q}}$.

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- If X is defined over a number field $K \subset \mathbb{C}$, we say that X has CM over K if all the cohomology classes of $E(X_{\mathbb{C}})$ are defined over K.
- X has CM over K if and only if $E \subset K$.

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Remark

This definiton is of transcendental nature: if $\tau \in \mathrm{Aut}(\mathbb{C})$, then usually $T(X)$ and $T(X^\tau)$ are not isomorphic as Hodge structures. However, we prove that the map $\tau^{ad} \colon E(X) \to E(X^\tau)$ sends $\mathcal{O}(X)$ isomorphically to $\mathcal{O}(X^\tau)$. This allows us to define principal K3 surfaces over every field that can be embedded into $\mathbb{C}$.

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The period map connect these two objects, over $\mathbb{C}$:

$$\mathcal{P}\colon \mathcal{F}_{2d,\mathbb{K}}(\mathbb{C}) \to Sh_{\mathbb{K}}(\mathrm{SO}(2,19),\Omega^{\pm})(\mathbb{C}).$$

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Since the canonical model of $Sh_{\mathbb{K}}(\mathrm{SO}(2,19),\Omega^{\pm})$ defined by Deligne is constructed by specifying the $\mathrm{Aut}(\mathbb{C})$-action on special points, the above Theorem is equivalent to the following (once we compute the reciprocity map..)

Corollary (Main theorem of CM)

Let $X/\mathbb{C}$ be a K3 surface with CM and let $E \subset \mathbb{C}$ be its reflex field. Let $\tau \in \mathrm{Aut}(\mathbb{C}/E)$ and $s \in \mathbb{A}_E^\times$ be an idèle such that $\mathrm{art}(s) = \tau_{|E^{ab}}$. There exists a unique Hodge isometry $\eta\colon T(X)_\mathbb{Q} \to T(X^\tau)_\mathbb{Q}$ such that the following triangle commutes

$$\begin{array}{ccc} T(X)_{\mathbb{A}_f} & \xrightarrow{\eta\otimes\mathbb{A}_f} & T(X^\tau)_{\mathbb{A}_f} \\ {\scriptstyle \frac{s_f}{\overline{s_f}}}\big\uparrow & \nearrow{\scriptstyle \tau^*} & \\ T(X)_{\mathbb{A}_f} & & \end{array}$$

Type

Let us fix a CM field E. Types are needed to parametrise tuples of the form $(T(X), B, \iota)$, where $T(X)$ is the transcendental lattice of a principal K3 with CM, $B \subset \mathrm{Br}(X)$ a finite subgroup invariant under the action of $\mathcal{O}(X)$ and $\iota\colon E \to E(X)$ an isomorphism.

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4. $\sigma\colon E \hookrightarrow \mathbb{C}$ an emdedding.

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$$I^* \otimes \mathbb{Q}/\mathbb{Z} \cong E/I^* \supseteq J/I^* \leftrightsquigarrow B \subset \mathrm{Br}(X) \cong T(X)^* \otimes \mathbb{Q}/\mathbb{Z};$$

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$$T(X) \otimes \mathbb{C} \supseteq T^{1,-1}(X) \leftrightsquigarrow \mathbb{C}_\sigma \subset E \otimes \mathbb{C} = \bigoplus_{\tau\colon E \hookrightarrow \mathbb{C}} \mathbb{C}_\tau.$$

K3 class fields and K3 class groups

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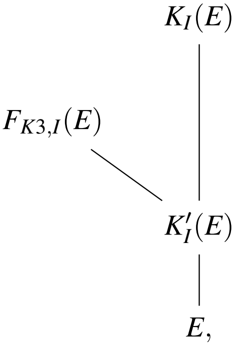

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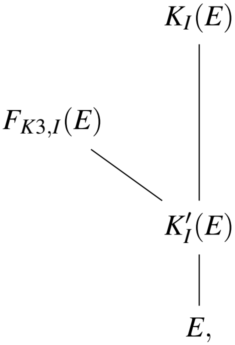

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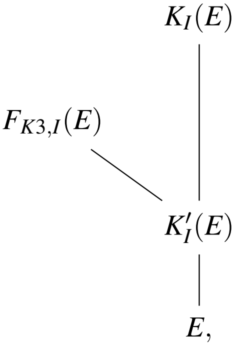

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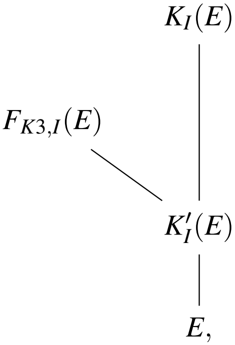

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- $\mathrm{Gal}(F_{K3,I}(E)/K'_I(E)) \cong \frac{\mathcal{O}_F^\times \cap N(E^{I,1})}{N(\mathcal{O}_E^I)}$.

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$$|G_{K3,I}(E)| = \frac{h_E \cdot \phi_E(I) \cdot [\mathcal{O}_F^\times : N(\mathcal{O}_E^I)] \cdot [E\colon F]}{h_F \cdot \phi_F(J) \cdot [\mathcal{O}_E^\times : \mathcal{O}_E^I] \cdot e(E/F, J) \cdot |H^1(E^{I,1})|}.$$

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Definition

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Main results

After having translated the main theorem of CM into the language of types, we prove:

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Theorem (A)

Let $(T(X), B, \iota)$ *be of type* (I, α, J, σ)*. Then the field of moduli of* $(T(X), B, \iota)$ *corresponds to* $F_{K3, I^* J^{-1}}(E)$*, the K3 class field of E modulo the ideal* $I^* J^{-1} \subset \mathcal{O}_E$*.*

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Theorem (B)

Let X/K be a principal K3 surface with CM over a number field K. There exists an ideal $I_B \subset \mathcal{O}_E$ such that

$$\mathcal{O}_E / I_B \cong \mathrm{Br}(\overline{X})^{\Gamma_K}$$

as $\mathcal{O}_E$-modules and

$$|G_{K3, I_B}(E)| \mid [K : E].$$

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2. Use the formula for $|G_{K3,I}(E)|$ to find all the ideals $I \subset \mathcal{O}_E$ such that $|G_{K3,I}(E)|$ divides $[K\colon E]$. Denote them $I_1, \cdots I_n$.
3. Now employ Theorem B, which says that $\mathrm{Br}(\overline{X_K})^{\Gamma_K} \cong \mathcal{O}_E/I_B$, with $I_B \subset \mathcal{O}_E$ a fractional ideal dividing one of the I_i's.

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$$0,\ \mathbb{Z}/3\times\mathbb{Z}/3,\ \mathbb{Z}/5,\ \mathbb{Z}/5\times\mathbb{Z}/5,\ \mathbb{Z}/2,\ \mathbb{Z}/2\times\mathbb{Z}/2,\ \mathbb{Z}/4\times\mathbb{Z}/2,\ \mathbb{Z}/4\times\mathbb{Z}/4$$

$$\mathbb{Z}/8\times\mathbb{Z}/4,\ \mathbb{Z}/8\times\mathbb{Z}/8,\ \mathbb{Z}/3\times\mathbb{Z}/3\times\mathbb{Z}/2,\ \mathbb{Z}/3\times\mathbb{Z}/3\times\mathbb{Z}/2\times\mathbb{Z}/2,$$

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Remark

Notice how the behaviour of the rational primes in E influences the groups $\mathrm{Br}(\overline{X})^{\Gamma_K}$.

Field of definition

Let $X/\mathbb{C}$ be a principal K3 surface with CM, and let (I, α, σ) be its type. Put

$$\mathcal{D}(X) := (\alpha) I \bar{I} \mathcal{D}_E,$$

where $\mathcal{D}_E$ is the different ideal of E. $\mathcal{D}(X) \subset \mathcal{O}_E$ is a well defined invariant of X, and the type map induces an isomorphism $\mathrm{disc}(T(X)) \cong \mathcal{O}_E/\mathcal{D}(X)$.

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Theorem

Let $X/\mathbb{C}$ be as above, and suppose that the natural map $\mu(E) \to \mathcal{O}_E/\mathcal{D}(X)$ is injective. Then X admits a model $\tilde{X}$ over $F_{K3,\mathcal{D}(X)}(E)$, which is uniquely determined by the two following properties:

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- *$\rho(\tilde{X}) = \rho(X)$;*
- *(Universal property) let Y be a K3 surface over a number field L with CM over L and suppose that $Y_{\mathbb{C}} \cong X$ and that $\rho(Y) = \rho(X)$. Then $F_{K3,\mathcal{D}(X)}(E) \subset L$ and $\tilde{X}_L \cong Y$.*

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- If the map $\mu(E) \to \mathcal{O}_E/\mathcal{D}(X)$ is not injective, we can still produce a similar result, but we have to impose some condition on the Hecke character associated to X...

THE END

Thank you for your attention!