Singular K3 surfaces of class number two

(joint with Frithjof Schulze)

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K3 surfaces and Galois representations,
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CM elliptic curves

Elliptic curve $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, $\text{im}(\tau) > 0$

E has complex multiplication (CM) $\Leftrightarrow$ End$(E) \supsetneq \mathbb{Z}$
$\Leftrightarrow \tau$ quadratic over $\mathbb{Q}$

Consequence: infinitely many CM elliptic curves, dense in moduli

Examples:
Elliptic curves with extra automorphisms (j-invariant $j = 0, 12^3$), and without, e.g. $j = -3315, 2^3 3^3 11^3$

More precisely: End$(E) = \mathcal{O}$ is an order in $K = \mathbb{Q}(\tau)$

Examples: first three have End $= \mathcal{O}_K$, last has End $= \mathbb{Z}[2i]$.

Notation: d_K discriminant of (ring of integers $\mathcal{O}_K$ of) K

Class group theory

Write $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$ ($f \in \mathbb{N}$), then there are class groups

$$
\begin{array}{ccl}
Cl(\mathcal{O}) & = & \{\text{fractional } \mathcal{O}\text{-ideals of } K\}/(\text{principal } \mathcal{O}\text{-ideals}) \\
\updownarrow & 1:1 & \qquad d = f^2 d_K \\
Cl(d) & = & \{\text{primitive positive-definite even binary} \\
 & & \quad \text{quadratic forms of discriminant } d\}/SL_2(\mathbb{Z})
\end{array}
$$

with elements

$$Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \; a, c \in \mathbb{N}, b \in \mathbb{Z}, \;\; b^2 - 4ac = f^2 d_K.$$

Unique reduced form: $-a \leq b \leq a \leq c$,
with $b \geq 0$ if $a = c$ or $|b| = a$.

Follows: $Cl(d)$ finite, **class number** $h(d) = \#Cl(d)$.

CM theory

1. Deuring: CM elliptic curves are modular, i.e. related to Hecke characters

2. ring class field $H(d) = K(j(E))$ for any E with End$= \mathcal{O}$,

Corollary: E is defined over $H(d)$, and at best over degree two subfield $\mathbb{Q}(j)$

3. Shimura: Gal$(H(d)/K) \cong Cl(d)$ acts faithfully and transitively on all such E (so there are $h(d)$ in number)

Corollary: Exactly 13 CM elliptic curves over $\mathbb{Q}$

4. $\forall L$ number field: $\#\{\text{CM } E/L\} < \infty$, or even

$$\forall N \in \mathbb{N}: \quad \#\{\text{CM } E/L; [L:\mathbb{Q}] \leq N\} < \infty.$$

Similar problem in higher dimension?
$\longrightarrow$ singular K3 surfaces (more fruitful than abelian surfaces)

Singular K3 surfaces

K3 surface X: smooth, projective surface with

$$h^1(X, \mathcal{O}_X) = 0, \ \ \omega_X = \mathcal{O}_X.$$

Examples: double sextics, smooth quartics in $\mathbb{P}^3$, ...

Here: work over $\mathbb{C}$, so Picard number

$$\rho(X) = \text{rk}\,\mathrm{NS}(X) \leq h^{1,1}(X) = 20 \quad \text{(Lefschetz)}$$

Much of arithmetic concentrated in isolated case $\rho = 20$: **singular K3 surfaces** (in the sense of exceptional)

Example: Fermat quartic

$$X = \{x_0^4 + x_1^4 + x_2^4 + x_4^4 = 0\} \subset \mathbb{P}^3.$$

48 lines have intersection matrix of rank 20 and determinant -64; hence they generate $\mathrm{NS}(X)$ up to finite index. [Non-trivial: showing that the lines generate $\mathrm{NS}(X)$...]

Transcendental lattice

Transcendental lattice $T(X) = \mathrm{NS}(X)^\perp \subset H^2(X, \mathbb{Z})$

identified with positive-definite, even, binary quadratic form

$$Q(X) = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}.$$

(unique up to conjugation in $\mathrm{SL}(2, \mathbb{Z})$)
as before – except that $Q(X)$ need not be primitive!

Example: Given that the Fermat has NS of discriminant -64 generated by lines, one can compute the discriminant group

$$\mathrm{NS}^\vee / \mathrm{NS} \cong (\mathbb{Z}/8\mathbb{Z})^2 \cong T^\vee / T \quad (\textit{Nikulin})$$

By inspection of $Cl(-4), Cl(-16)$ and $Cl(-64)$, this implies

$$Q = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

Torelli for singular K3 surfaces

Torelli: $X \cong Y \iff T(X) \cong T(Y)$

Follows: Fermat quartic, up to isomorphism, uniquely determined by $T = \ldots$.

Funny side-remark: there is another model as smooth quartic, this time with 56 lines! (Degtyarev–Itenberg–Sertöz, Shimada–Shioda)

History: Same result first proved and used for singular abelian surfaces, i.e. with $\rho = 4$ (Shioda–Mitani)

Discriminant $d = \det \mathrm{NS}(X) = b^2 - 4ac < 0$.

Application: if any, then there is a unique K3 surface of each discriminant

$$d = -3, -4, -7, -8, -11, -19, -43, -67, -163.$$

[since $h(d) = 1$, and $Q(X)$ is automatically primitive as an even quadratic form.]

Singular K3 surfaces of class number one

Class number of singular K3 X : $h(d)$

Have seen: at most 9 singular K3 surfaces with class number one and fundamental discriminant (i.e. $d = d_K$ for some imaginary quadratic K)

Cheap examples: Vinberg's most algebraic K3 surfaces X_3, X_4, for instance as (isotrivial) elliptic surfaces

$$X_3 : \qquad y^2 + t^2(t-1)^2 y = x^3$$
$$X_4 : \qquad y^2 = x^3 - t^3(t-1)^2 x$$

Compute ρ, d: trivial lattice spanned by zero section and fiber components in NS: Here $U + E_6^3$ resp. $U + D_4 + E_7^2$. Follows $\rho = 20$, and obtain $d = -3, -4$ from finite index overlattice generated by torsion section $(0,0)$.

[Fun features: unirational in char $p \equiv -1 \mod |d|$, explicit dynamics, ...]

Non-fundamental discriminants

Recall non-fundamental discriminants of class number one:

$$d = -12, -16, -27, -28$$

For each d, there are thus two possible quadratic forms $Q(X)$ on a singular K3 surface of discriminant d. E.g.

$$d = 12 \Longrightarrow Q(X) = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$$

(exactly one of which is divisible).

In practice: distinguish forms by divisibility/degree of primitivity/discriminant groups/discriminant forms... (works and applies in general)

Problem: General construction?! (over $\mathbb{C}$/over $\mathbb{Q}$/...)

Kummer surfaces

Classical construction (accounting for one 'K' in K3):
A abelian surface $\Longrightarrow$

$$A/\langle -1 \rangle \text{ has } 16\ A_1 \text{ sing} \Longrightarrow \mathrm{Km}(A) = \widetilde{A/\langle -1 \rangle} \text{ K3}$$

(converse also (Nikulin): 16 nodal curves $\Longrightarrow$ Kummer)

Properties: $\rho(\mathrm{Km}(A)) = \rho(A) + 16$
$T(\mathrm{Km}(A)) = T(A)[2]$ (scaled inters. form)

Follows: Fermat quartic, singular K3 with $Q(X) = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

could be Kummer, but other K3's like X_3, X_4 *not* (compare attempt at proving subjectivity of period map for K3's...)

Singular abelian surfaces

Shioda–Mitani: Any positive-definite even binary quadratic form Q is attained by some singular abelian surface A

Proof: Write $Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ as before. Set

$$\tau = \frac{-b + \sqrt{d}}{2a}, \qquad \tau' = \frac{b + \sqrt{d}}{2},$$

and consider

$$A = E_\tau \times E_{\tau'}$$

for complex tori $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. □

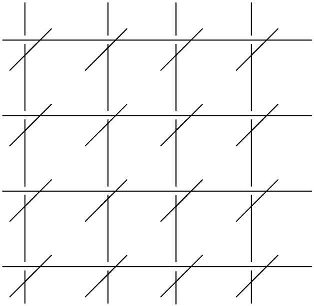
Application: Fermat $= \mathrm{Km}(E_i \times E_{2i})$,

$\omega \in \mu_3$ primitive $\Longrightarrow \mathrm{Km}(E_\omega^2)$ has $Q = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

Surjectivity of period map

Shioda–Inose: Any positive-definite even binary quadratic form Q is attained by some singular K3 surface X

Proof: Consider associated singular abelian surface A
$\Longrightarrow \mathrm{Km}(A)$ has wrong quadratic form $2Q$, but classical configuration of 24 nodal curves:



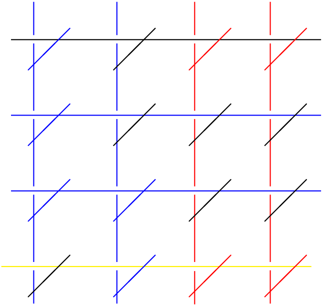
$$\begin{array}{ccc} \mathrm{Km}(A) & \to & \mathbb{P}^1 \\ \downarrow & & \\ \mathbb{P}^1 & & \end{array}$$

(Fibre components (I_0^*) and torsion sections of isotrivial elliptic fibrations induced from projections onto factors of A)

Auxilliary elliptic fibration

Key feature of K3 surfaces: may admit several different elliptic (or genus one) fibrations (like the two before)

Here: blue divisor (with multiplicities) has Kodaira type $\mathrm{II}^* \Rightarrow$ induces elliptic fibration



$$\mathrm{Km}(A) \searrow \mathbb{P}^1$$

yellow curve = section; red curves contained in two different reducible fibres F_1, F_2 (Kodaira types I_0^* or I_1^*)

Shioda–Inose structure

Consider quadratic base change ramified at F_1, F_2
$\Longrightarrow$ gives another K3 surface X

Check: $T(A) = T(X)$ □

Terminology: **Shioda–Inose structure**

$$\begin{array}{ccc} A & & X = \mathrm{SI}(A) \\ & \searrow \quad \swarrow & \\ & \mathrm{Km}(A) & \end{array}$$

(Extended to certain K3 surfaces of Picard number $\rho \geq 17$ by Morrison.)

Corollary: Every singular K3 surface is defined over some number field, and it is modular ($\Rightarrow$ Hecke character)

Livne: singular K3 over $\mathbb{Q}$, discriminant $d \Rightarrow$
$\exists$ associated wt 3 modular form with CM in $K = \mathbb{Q}(\sqrt{d})$
(converse by Elkies-S.)

Fields of definition

Inose $+\varepsilon$: Singular K3 X admits model over $\mathbb{Q}(j+j', jj') \subset H(d)$ (Inose's pencil: elliptic fibration with two fibres of type II^*)

Corollary: $h(d) = 1 \Longrightarrow X$ over $\mathbb{Q}$

[all elliptic curves involved have CM with class number one]

Problem: can we do better in general?

Example: Fermat quartic: $Q(X) = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$.

0. original quartic in $\mathbb{P}^3$;
1. $X = \mathrm{Km}(E_i \times E_{2i})$;
2. $X = \mathrm{SI}(E_i \times E_{4i})$.
3. smooth quartic in $\mathbb{P}^3$ with 56 lines (Shimada–Shioda)

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2. $X = \mathrm{SI}(E_i \times E_{4i})$ – over $\mathbb{Q}(\sqrt{2})$
3. smooth quartic in $\mathbb{P}^3$ with 56 lines (Shimada–Shioda) – over $\mathbb{Q}(\sqrt{-2})$

Long-term goal – classification

Goal: Classify all singular K3 surfaces over $\mathbb{Q}$

Comment: $\# >> 13$ (but finite, see below)

Today: Any singular K3 surface of class number two is defined over $\mathbb{Q}$ (Schulze–S.)

Example: Fermat!

Bigger framework: **arithmetic Torelli Theorem** (conjectural)

Input needed: obstructions against being defined over $\mathbb{Q}$

Will see: two old obstructions, one new

Intertwined: proof of prototypical cases of today's theorem

First obstruction: genus

Theorem 1 (Shimada, S.).
X singular K3. Then

$$\{T(X^\sigma); \sigma \in \mathrm{Aut}(\mathbb{C})\} = \text{genus of } T(X).$$

Corollary:
$X/\mathbb{Q} \implies$ the genus of $T(X)$ consists of a single class

Equivalently: let m denote the degree of primitivity of $T(X)$. Then $Cl(d/m^2)$ is only 2-torsion.

Consequences: ok for class number two, but *not* if $T(X)$ is primitive of class number three

Second obstruction: Galois action

Theorem 2 (Elkies, S.).
X singular K3 of discriminant d with NS defined over L. Then

$$H(d) \subseteq L(\sqrt{d})$$

Meaning: $X/\mathbb{Q} \Rightarrow$ Galois action of 'size' $h(d)$ on $\mathrm{NS}(X)$

Proof combines modularity, Artin–Tate conjecture (details to follow), class group theory

Consequence: $\mathrm{NS}/\mathbb{Q} \Rightarrow h(d) = 1$.

Example: Vinberg's X_3, X_4

Indeed: X admits model over $\mathbb{Q}$ with $\mathrm{NS}/\mathbb{Q} \Leftrightarrow$
$Q(X)$ primitive of class number one ($\# = 13$)

Easy to see: Q as above $\longrightarrow \tau = \tau(Q) \longrightarrow E = E_\tau$ (CM, $h = 1$) $\longrightarrow X = \mathrm{SI}(E^2)/\mathbb{Q}$

Use Inose's pencil on X from Shioda–Inose structure:

essential data presently: 2 fibres of type II^*, 1 fibre of type I_2, 1 section P of ht $|d|/2$ ($d < -4$)

fibres automatically over $\mathbb{Q}$ (by construction), no Galois action, so if Galois acts non-trivially on NS, then on $\mathrm{MW} = \mathbb{Z}P$. Only possibility

$$P^\sigma = -P.$$

Hence P defined over quadratic extension, and corresponding quadratic twist has *all* of NS defined over $\mathbb{Q}$.

If Q is not primitive, say 2-divisible, then $X = \mathrm{Km}(A) \Rightarrow \mathrm{NS}(A)$ not over $\mathbb{Q}$ (because $H^2 = \wedge^2 H^1$ as Galois module) $\Rightarrow$ same for $\mathrm{NS}(X)$

Consequence: for singular K3 of class number two to be defined over $\mathbb{Q}$, need order 2 Galois action on NS which cannot be twisted away!

Finiteness

Just like for CM elliptic curves, we derive:

Corollary (Shafarevich):

$$\forall N \in \mathbb{N}: \quad \#\{\text{singular K3}/L; [L:\mathbb{Q}] \leq N\} < \infty.$$

Proof: X/L, H very ample $\Rightarrow$ Galois acts on $H^\perp \subset \mathrm{NS}(X)$; this is negative-definite, hence has finite isometry group; in fact, size can be bounded a priori. □

Problem: Could it suffice for a singular K3 to be defined over $\mathbb{Q}$ to ensure that the two obstructions are met?

Class number two – recap

Recall: want to show that all singular K3 of class number 2 are defined over $\mathbb{Q}$

What's available?

1. Inose's pencil over $\mathbb{Q}(j+j', jj')$

for $h=2, Q$ primitive, get:

- Q principal form (identity in $Cl(d)$) $\Leftrightarrow a=1 \Leftrightarrow \tau=\tau'$, $\mathbb{Q}(j+j', jj') \neq \mathbb{Q}$, no Galois action up to twist as before
- Q non-principal $\Rightarrow j'=j^\sigma \Rightarrow \mathbb{Q}(j+j', jj') = \mathbb{Q}$.

2. imprimitive Q, say $Q=mQ', 1<m<7$

Kuwata: cyclic degree m base changes of Inose's pencil lead to elliptic K3's X' with all Mordell–Weil ranks from 1 to 18 except for 15 (gap closed by Kloosterman)

Shioda: $T(X') = T(X)[m] \Rightarrow$ reduction to case 1. for several imprimitive Q (including Kummer case $m=2$)

3. Extremal elliptic K3 surfaces ($\rho = 20$, but MW finite)

Shimada-Zhang: lattice theoretical classification

Beukers–Montanus: equations (and designs d'enfant) for all semi-stable fibrations

S.: many non-semi-stable cases

4. Isolated examples: E.g.

Peters–Top–van der Vlugt: K3 quartic associated to Melas code

Degtyarev–Itenberg–Sertöz: smooth quartic/$\mathbb{Q}$ with 56 lines over $\mathbb{Q}(\sqrt{2})$ [not isomorphic to the Fermat]

Approach: elliptic fibrations

Idea (for theoretical and practical reasons): use elliptic fibration (with section) on X; implies

$$\mathrm{NS}(X) = U + M$$

$\longrightarrow$ have to impose Galois action on M.

Kneser–Nishiyama method: Determine all possible M by embedding 'partner lattice' $M^\perp$ into Niemeier lattices ($M^\perp$ negative definite of rank $26 - \rho(X)$ with same discriminant form as $T(X)$, exists by Nishiyama)

In practice: try out suitable M, ideally with small MW-rank [Note: 'fibre rank' read off from roots of M by theory of Mordell–Weil lattices (Shioda)]

First example

Take $Q = \begin{pmatrix} 2 & 0 \\ 0 & 56 \end{pmatrix}$.

Partner lattice: $M^\perp = \langle -8 \rangle + \langle A_4, v \rangle,\ v^2 = -4, v$ only meeting the second component of A_4 (looks like section of ht $14/5$)

Consider
$M^\perp \hookrightarrow N(E_7 + A_{17}) \Longrightarrow M = A_7 + \langle E_7, A_3, u \rangle,\ u^2 = -4, u$ meeting outer (simple) components of E_7, A_3

MWL: A_7, E_7, A_3 correspond to reducible fibres,
u corresponds to section P of ht $4 - 3/2 - 3/4 = 7/4$.

Galois: may act independently as inversion on first fiber (I_8), and on second set of divisors (I_4, P) $\Rightarrow$ cannot be twisted away a priori

Parametrization

1. Work out family of elliptic K3 surfaces with

$$\mathrm{NS} \supseteq U + A_7 + E_7 + A_3$$

Start with $U + A_2 + A_4 + E_7 =$ easy to write down by hand as 5-dimensional family

$$y^2 = x^3 + (t^2u + at + 1)(t-1)^2x^2 + t^4(t-1)^5(tuv^2 - r^2) \\ -(2(-t^2uv + bt + r))t^2(t-1)^3x;$$

then promote = easy enough, though a bit complicated to write down; e.g., with parameter s,

$$u = \frac{1}{(s-1)^5 s^2}, \quad a = -\frac{s^3 - s^2 + s + 2}{(s-1)^3}$$

2. Search for member in family with section P of ht $7/4$
$=$ small enough to solve directly. Find

$$s = 8, \quad x(P) = -\frac{3^3 19}{2^4 7^5}(t-1)^2(7t+31).$$

In more complicated cases:

- use structure of parameter space as modular curve or Shimura curve, or
- win a parameter by 'guessing' s from point counts over various $\mathbb{F}_p$ using modularity and/or
- search for solution to system of equations in some $\mathbb{F}_p$ and then apply p-adic Newton iteration.

Second example

Take $Q = 7\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Try, e.g.,

$$M = A_6 + \langle A_4 + D_7, P \rangle, \quad h(P) = 4 - \frac{6}{5} - \frac{7}{4} = 21/20.$$

Result: nice elliptic K3, but not over $\mathbb{Q}$.

Similar outcome for other M – why?

Artin–Tate conjecture

$X/\mathbb{F}_p$ K3 surface, $\ell \neq p \Rightarrow$ reciprocal characteristic polynomial of Frobenius

$$P(X,T) = \det(1 - \text{Frob}_p^* \, T; H^2_{ét}(\bar{X}, \mathbb{Q}_\ell)).$$

Artin–Tate conjecture: (equivalent to Tate conjecture (Milne))

$$p \frac{P(X,T)}{(1-pT)^{\rho(X)}}\Big|_{T=\frac{1}{p}} = |Br(X)| \cdot |\det(\mathrm{NS}(X))|$$

Note: $|Br(X)|$ always a square $\Rightarrow$ control over (square class of) $\det \mathrm{NS}(X)$

Situation: $X/\mathbb{Q}$ singular K3, p split in $K = \mathbb{Q}(\sqrt{d}) \Rightarrow$

$$P(X \otimes \mathbb{F}_p, T) = (1 - a_p T + p^2 T^2) \cdot (\text{cyclotomic factors})$$

where $a_p =$ coefficient of wt 3 eigenform with CM by K

In particular, $p \nmid a_p$, so $\rho(X \otimes \bar{\mathbb{F}}_p) = 20$ and Artin–Tate applies unconditionally

Presently X with Q given, $d = -147$; assume elliptic fibration with M defined over $\mathbb{Q} \Rightarrow$ Galois action by $L = \mathbb{Q}(\sqrt{-7})$ or $\mathbb{Q}(\sqrt{21})$ on I_7 fiber (after quadratic twist)

Take p split in K, but not in $L \Rightarrow I_7$ not over $\mathbb{F}_p \Rightarrow$

$$\rho(X \otimes \mathbb{F}_p) = 17, \quad \det \mathrm{NS}(X \otimes \mathbb{F}_p) = 2^5 21 \Rightarrow RHS = 42 \mod \mathbb{Q}^2$$

LHS: $P(X \otimes \mathbb{F}_p, T) = (1 - a_p T + p^2 T^2)(1 - T)^{17}(1 + T)^3$
where $a_p = \pm(\alpha^2 + \bar{\alpha}^2)$, $\alpha \in K = \mathbb{Q}(\sqrt{-3})$, $\alpha\bar{\alpha} = p$

LHS evaluates at $T = \frac{1}{p}$ as $\pm 8(\alpha \pm \bar{\alpha})^2 = 2$ or $6 \mod \mathbb{Q}^2$
— not compatible w/ RHS

Compatible elliptic fibration

Solution: 'synchronize' orthogonal summands in M with determinant divisible by 7; e.g.

$$M = A_2 + A_6 + \langle D_9, P \rangle, \quad h(P) = 4 - \frac{9}{4} = \frac{7}{4}.$$

Approach:

1. Family with $\mathrm{NS} \supseteq U + A_2 + A_6 + D_9$ obtained from previous work with Elkies: 2-dim'l family in λ, μ with

$$\mathrm{NS} \supseteq U + A_2 + A_4 + A_6 + D_4 \Rightarrow \text{ merge } A_4, D_4 \ (\lambda = 0).$$

2. Impose section P of ht $h(P) = 7/4$: easy enough:

$$\mu = \frac{63}{10}, \quad x(P) = -\frac{1008}{125}(7t - 5)t^3$$

Thank you!

Equations

Two-dimensional family with parameters $\lambda \in \mathbb{P}^1, \mu \neq 0$:

$$
\begin{aligned}
X_{\lambda,\mu}: \quad y^2 &= x^3 + (t-\lambda)A\,x^2 + t^2(t-1)(t-\lambda)^2 B\,x \\
&\quad + t^4(t-1)^2(t-\lambda)^3 C, \\
A &= \frac{1}{24}\Big(\frac{1}{9}(2\mu+9)^3 t^3 - (22\mu-9)(2\mu-27)t^2 \\
&\quad -27(14\mu-9)t - 81\Big), \\
B &= \mu\,\Big(\frac{1}{9}(2\mu+9)^3 t^2 - 2(10\mu-9)(2\mu-9)t \\
&\quad -27(2\mu-3)\Big), \\
C &= \frac{2}{3}\mu^2((2\mu+9)^3 t - 81(2\mu-3)^2).
\end{aligned}
$$

Singular fibers:

cusp	0	1	∞	λ	cubic with coefficients in μ
fiber	I_5	I_3	I_7	I_0^*	I_1, I_1, I_1