IMAGES OF GALOIS REPRESENTATIONS AND THE MUMFORD–TATE CONJECTURE

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JOINT WORK WITH

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THE MUMFORD–TATE CONJECTURE

General setting:

- $F \subset \mathbb{C}$ is a finitely generated field of characteristic 0
- Y/F is smooth projective (more generally: a pure motive)

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We want to study $H^n(Y)(m)$ for some n, m.

- $H_{\mathrm{B}} := H^n(Y_{\mathbb{C}}, \mathbb{Q}(m))$: polarizable Hodge structure
- $H_\ell := H^n(Y_{\bar{F}}, \mathbb{Q}_\ell(m))$: Galois representation

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- $\mathbb{Q}\mathrm{HS} \supset \langle H_{\mathrm{B}} \rangle :=$ tensor subcategory generated by H_{B}

This means: $\langle H_{\mathrm{B}} \rangle$ is the smallest subcategory that contains H_{B} and is stable under $\oplus$, $\otimes$, $(\quad)^{\vee}$ and taking subquotients.

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Mumford–Tate group: algebraic group

$$G_{\mathrm{B}} \subset \mathrm{GL}(H_{\mathrm{B}})$$

(over $\mathbb{Q}$) with the property that

$$\langle H_{\mathrm{B}} \rangle \simeq \mathrm{Rep}(G_{\mathrm{B}}; \mathbb{Q})$$

For $T \in \langle H_{\mathrm{B}} \rangle$ and $t \in T$:

$$t \text{ is a Hodge class} \iff t \text{ is invariant under the action of } G_{\mathrm{B}}$$

(Hodge class $=$ rational class of Hodge type $(0, 0)$)

Galois side

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Let

$$\langle H_\ell \rangle \subset \left(\ell\text{-adic representations of } \mathrm{Gal}(\bar{F}/F)\right)$$

be the tensor subcategory generated by H_ℓ. Then

$$\langle H_\ell \rangle \simeq \mathrm{Rep}(G_\ell; \mathbb{Q}_\ell)\,.$$

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- *From now on we assume F is such that G_ℓ is connected.*
- Conjecturally G_ℓ is reductive; this is not known in general (OK for abelian motives)

If $T_\ell \in \langle H_\ell \rangle$ and $t \in T_\ell$ then

$$t \text{ is a Tate class} := t \text{ is invariant under } G_\ell$$

MUMFORD–TATE CONJECTURE:

Under the comparison isomorphism $H_{\mathrm{B}} \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H_\ell$ we have

$$G_{\mathrm{B}} \otimes \mathbb{Q}_\ell \overset{?}{=} G_\ell$$

as algebraic subgroups of $\mathrm{GL}(H_\ell)$.

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Remark: if we take $H = H^2(Y)(1)$ then the Hodge conjecture is known (Lefschetz theorem on divisor classes); in this case

$$\text{MTC} \implies \text{TC for divisor classes}$$

Status of the MTC

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- MTC is "true on centers" (Vasiu, Ullmo–Yafaev)

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MAIN RESULTS

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- $$\rho_{Y,\ell} \colon \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}(H)\big(\mathbb{Z}_\ell\big)$$
  the ℓ-component of ρ_Y

Suppose the MTC is true: $G_{\mathrm{B}} \otimes \mathbb{Q}_\ell = G_\ell$. This means:

the image of $\rho_{Y,\ell}$ is Zariski-dense in $G_{\mathrm{B}}(\mathbb{Q}_\ell)$

Bogomolov + Faltings (p-adic Hodge theory) in fact gives:

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Question: Can we make this more precise, also varying ℓ ?

Example (Serre, Inventiones 1972):

E/F elliptic curve with $\mathrm{End}(E_{\bar{F}}) = \mathbb{Z}$ then the image of

$$\rho_E \colon \mathsf{Gal}(\bar{F}/F) \to \mathsf{GL}\Big(\varprojlim E[n](\bar{F})\Big) \cong \mathsf{GL}_2(\hat{\mathbb{Z}})$$

is *open* in $\mathsf{GL}_2(\hat{\mathbb{Z}})$.

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(3) If $H_{\mathbb{Q}}$ is Hodge-maximal, $\mathrm{Im}(\rho)$ is open in $G_{\mathrm{B}}(\mathbb{A}_{\mathrm{f}})$.

This confirms a conjecture of Serre (1976). Parts (1), (2) have independently been obtained by Hindry and Ratazzi.

Hodge-maximality

Definition. — Let V be a $\mathbb{Q}$-Hodge structure, given by

$$h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}\,,$$

and

$$M \subset \mathrm{GL}(V)$$

the Mumford–Tate group. Then V is *Hodge-maximal* if there does **not** exist a non-trivial isogeny $M' \to M$ of connected $\mathbb{Q}$-groups such that $h\colon \mathbb{S} \to M_{\mathbb{R}}$ lifts to $h'\colon \mathbb{S} \to M'_{\mathbb{R}}$.

Remark. Hodge-maximality is a necessary condition for $\mathrm{Im}(\rho) \subset G_{\mathrm{B}}(\mathbb{A}_{\mathrm{f}})$ to be open.

Sketch: Suppose we do have an isogeny $M' \to M$ with h lifting to h'.

- Wintenberger: the ℓ-adic Galois representations ρ_ℓ lift to $\rho'_\ell \colon \mathrm{Gal}(\bar{F}/F) \to M'(\mathbb{Q}_\ell)$

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For abelian varieties: H_1 is not always Hodge-maximal

COROLLARY OF THE MAIN THEOREM FOR ABELIAN VARIETIES

For $n > 0$ let $F \subset F[n]$ be the field extension generated by the coordinates of the points in $Y[n](\bar{F})$. Assume the MTC for Y is true. Then:

Corollary of the Main Theorem for Abelian varieties

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(1) Given ℓ there is a constant $C(\ell) = C(Y, \ell)$ such that

$$\left[F[\ell^i] : F\right] = C(\ell) \cdot \ell^{i \cdot \dim(G_{\mathrm{B}})}$$

for all i big enough.

(2) If H_1 is Hodge-maximal then there is a constant $C = C(Y)$ such that

$$\left[F[n] : F\right] = C \cdot n^{\dim(G_{\mathrm{B}})}$$

for all n divisible enough.

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- $H^2(Y(\mathbb{C}), \mathbb{Q})$ is *not* Hodge-maximal.

3

The Galois representation associated with a Shimura variety

OUTLINE OF THE PROOF OF THE MAIN THEOREMS

- To a component of a Shimura variety $S_0 \subset \mathrm{Sh}_K(G, X)$ we are going to associate a representation

  $$\phi\colon \pi_1(S_0) \to K \subset G(\mathbb{A}_\mathrm{f})$$

  of the étale fundamental group.
- Main technical result: the image of ϕ is "big".
- We deduce the main theorems about AV and K3's by using that their moduli spaces (essentially) are Shimura varieties, and by using a result of Cadoret–Kret about Galois generic points.

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For $K \subset G(\mathbb{A}_{\mathsf{f}})$ a compact open subgroup we have the associated scheme

$$\mathrm{Sh}_K(G, X) \qquad \text{over } E$$

with

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If $K_1 \subset K_2$ then we have an associated morphism

$$\mathrm{Sh}_{K_1,K_2}\colon \mathrm{Sh}_{K_1}(G, X) \to \mathrm{Sh}_{K_2}(G, X)$$

and if K_1 is normal in K_2 this is a Galois cover with group K_2/K_1. (Assume K_2 is neat.)

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By construction, for $K \subset K_0$ compact open we then have an étale cover

$$S_K \to S_0$$

and if $K \triangleleft K_0$ then this is Galois with group K_0/K.

For $K \lhd K_0$, let

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Taking the limit over all K we obtain

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What does this actually mean?

(Y, λ) principally polarized abelian variety, $\dim(Y) = g$,

Weil pairing:

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We want to compare this with the standard symplectic pairing

$$\psi_n \colon (\mathbb{Z}/n\mathbb{Z})^{2g} \times (\mathbb{Z}/n\mathbb{Z})^{2g} \to (\mathbb{Z}/n\mathbb{Z})$$

Definition. — A *Jacobi level n structure* on (Y, λ) is a pair (α, ζ) consisting of isomorphisms of group schemes

$$\alpha\colon (\mathbb{Z}/n\mathbb{Z})^{2g} \xrightarrow{\sim} Y[n]\,, \qquad \zeta\colon (\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} \mu_n$$

such that the diagram

$$\begin{tikzcd}
(\mathbb{Z}/n\mathbb{Z})^{2g} \times (\mathbb{Z}/n\mathbb{Z})^{2g} \arrow[r, "\psi_n"] \arrow[d, "\alpha\times\alpha", "\wr"'] & (\mathbb{Z}/n\mathbb{Z}) \arrow[d, "\zeta", "\wr"'] \\
Y[n] \times Y[n] \arrow[r, "e_n^\lambda"] & \mu_n
\end{tikzcd}$$

is commutative.

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We have a diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(A_{g,1} \otimes \bar{\mathbb{Q}}) \arrow[r] \arrow[d, two heads, "\phi_{\text{geom}}"] & \pi_1(A_{g,1}) \arrow[r] \arrow[d, "\phi"] & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \arrow[r] \arrow[d, two heads, "\chi"] & 1 \\
1 \arrow[r] & \mathrm{Sp}_{2g}(\hat{\mathbb{Z}}) \arrow[r] & \mathrm{CSp}_{2g}(\hat{\mathbb{Z}}) \arrow[r] & \hat{\mathbb{Z}}^{\times} \arrow[r] & 1
\end{tikzcd}$$

The homomorphism

$$\chi\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \hat{\mathbb{Z}}^{\times}$$

(in this case the cyclotomic character) describes the action of Galois on the set of irreducible components of $\varprojlim_n A_{g,n} \otimes \bar{\mathbb{Q}}$.

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If we choose roots of unity ζ_n for all n in a compatible manner, we have a tower of irreducible moduli schemes $A_{g,(n)} \otimes \bar{\mathbb{Q}}$ parametrizing ppav with *symplectic level n structure*, and $A_{g,(n),\bar{\mathbb{Q}}} \to A_{g,(1),\bar{\mathbb{Q}}}$ is Galois with group $\mathsf{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z})$.

This tower corresponds with the homomorphism

$$\phi_{\mathsf{geom}}\colon \pi_1(A_{g,1} \otimes \bar{\mathbb{Q}}) \twoheadrightarrow \mathsf{Sp}_{2g}(\hat{\mathbb{Z}}),$$

which is surjective because the $A_{g,(n),\bar{\mathbb{Q}}}$ are all irreducible.

Back to the general case: to the Shimura datum (G, X) and the geometrically irreducible component

$$S_0 \subset \mathrm{Sh}_{K_0}(G, X)_F$$

over the number field F we have associated the homomorphism

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Using Deligne's description of the action of Galois on the set of geometric irreducible components of the tower of Shimura varieties, we prove:

Main Theorem about the homomorphism ϕ

Let $\mathscr{G}$ be an integral model of G such that $K_0 \subset \mathscr{G}(\hat{\mathbb{Z}})$.

(1) The index $\left[\mathscr{G}(\mathbb{Z}_\ell) : \mathrm{Im}(\phi_\ell)\right]$ is bounded when ℓ varies. ($\phi_\ell = \ell$-adic component of ϕ)

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(3) If (G, X) is maximal, $\mathrm{Im}(\phi) \subset G(\mathbb{A}_\mathrm{f})$ is open.

Some technical details on the proof.

Set

$$\mathrm{Sh}(G,X) = \varprojlim_{K} \mathrm{Sh}_K(G,X)\,.$$

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The set of geometric irreducible components together with the action of $\mathrm{Gal}(\bar{E}/E)$ on it allows a purely group-theoretic description:

Let $\mathrm{ad}\colon G \to G^{\mathrm{ad}}$ be the adjoint map, let $G^{\mathrm{ad}}(\mathbb{R})^+ \subset G^{\mathrm{ad}}(\mathbb{R})$ be the topological identity component, and let

$$G(\mathbb{Q})_+ := \left\{ g \in G(\mathbb{Q}) \;\middle|\; \mathrm{ad}(g) \in G^{\mathrm{ad}}(\mathbb{R})^+ \right\}.$$

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Then $\pi_0\big(\mathrm{Sh}(G,X)_{\bar{\mathbb{Q}}}\big)$ is a torsor under

$$G(\mathbb{A}_{\mathsf{f}})/G(\mathbb{Q})_+^-\,.$$

This is an abelian profinite group.

The Galois group $\mathrm{Gal}(\bar{E}/E)$ acts on the set of geometric irreducible components through its maximal abelian quotient, and the action is given by a reciprocity homomorphism

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We reduce our main theorem about the homomorphism ϕ to the following result about the reciprocity homomorphism:

Theorem

The cokernel of the reciprocity map has finite exponent, and if (G, X) is maximal then it is a finite discrete group.

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(3) The general case.

DEDUCING THE MAIN THEOREMS ABOUT AV AND K3'S

We focus on the result for abelian varieties; the case of K3 surfaces is analogous.

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Let (Y, λ) be a ppav over $F \subset \mathbb{C}$, let $G = G_{\mathrm{B}}$ be the Mumford–Tate group. We obtain a Shimura datum (G, X) and, as before,

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We may arrange everything in such a way that (Y, λ) corresponds to a point $y \in S_0(F)$. This gives

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(S_0 \otimes \bar{F}) \arrow[r] & \pi_1(S_0) \arrow[r] & \mathrm{Gal}(\bar{F}/F) \arrow[r] \arrow[l, bend right, "y_*"'] & 1 \end{tikzcd}$$

We show: $\phi \circ y_*$ is the usual Galois representation of $\mathrm{Gal}(\bar{F}/F)$ on $H_1(Y)$.

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In our main result we assume that the MTC for Y is true. By the result of Bogomolov mentioned earlier, it follows that the image of $\phi_\ell \circ y_*$ is open in the image of ϕ_ℓ.

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THEOREM (CADORET–KRET)

If, for some ℓ, the image of $\phi_\ell \circ y_*$ is open in the image of ϕ_ℓ then in fact the image of $\phi \circ y_*$ is open in the image of ϕ.

Together with our results about the image of ϕ, the main theorem follows:

- Assumption that MTC is true + Bogomolov $\Rightarrow$

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Combining these we obtain that the image of ρ_Y is “big”.

THANK YOU FOR

YOUR ATTENTION