Zero-cycles on $K3$ surfaces over number fields

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Plan

- Preliminaries
- Motivation
- Sketch of proof
- Relation with other results

Notation

- X is a smooth, projective, geometrically connected variety over a number field k
- $Z_0(X) =$ group of zero-cycles on X
- $CH_0(X) = Z_0(X)/$rational equivalence
- $A_0(X) = \ker(\deg : CH_0(X) \to \mathbb{Z})$
- F/k a field extension $X_F = X \otimes_k F$
- $k \leftrightarrow \mathrm{Spec}\,(k)$
- $\Omega = \{$places of k$\}$
- k_v the completion of k at the place $v \in \Omega$
- $\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$

Pairings for rational points

$$\mathrm{Br}(X) \times X(k) \to \mathrm{Br}(k)$$

$$\mathrm{Br}(X) \times \prod_{v\in\Omega} X(k_v) \to \oplus_v \mathrm{Br}(k_v) \xrightarrow{\oplus inv} \mathbb{Q}/\mathbb{Z} \quad (*)$$

$$X(k) \overset{diag}{\hookrightarrow} \prod_{v\in\Omega} X(k_v)$$

$$X(k) \perp \mathrm{Br}(X) \quad wrt \quad (*)$$

"The Brauer-Manin obstruction to the existence of a rational point on X is the only one"
is the following statement

$$\exists (P_v) \in \prod_{v \in \Omega} X(k_v), \quad (P_v) \perp \mathrm{Br}(X)$$

$$\Downarrow$$

$$\exists P \in X(k)$$

Pairings for zero-cycles

$$\mathrm{Br}(X) \times Z_0(X) \to \mathrm{Br}(k)$$

$$\mathrm{Br}(X) \times \prod_{v\in\Omega} Z_0(X_{k_v}) \to \mathbb{Q}/\mathbb{Z} \quad (**)$$

$$\mathrm{Br}(X) \times \prod_{v\in\Omega} CH_0(X_{k_v}) \to \mathbb{Q}/\mathbb{Z}$$

$$Z_0(X) \to \prod_{v\in\Omega} Z_0(X_{k_v})$$

$$\mathrm{Im}\left(Z_0(X) \to \prod_{v\in\Omega} Z_0(X_{k_v})\right) \perp \mathrm{Br}(X) \quad wrt \quad (**)$$

"The Brauer-Manin obstruction to the existence of a zero-cycle of degree one on X is the only one"
is the following statement

$$\exists (z_v) \in \prod_{v\in\Omega} Z_0(X_{k_v}), \quad \deg(z_v) = 1 \;\; \forall \;\; v \in \Omega, \quad (z_v) \perp \mathrm{Br}(X)$$

$$\Downarrow$$

$$\exists z \in Z_0(X), \quad \deg(z) = 1$$

Motivation

Conjecture 1 (Colliot-Thélène, Sansuc ; Kato and Saito)

Let X is a smooth, projective, geometrically connected variety over a number field k.
The Brauer-Manin obstruction to the existence of a zero-cycle of degree one on X is the only one

This is known as CONJECTURE (E_1). These is also a statement about zero-cycles of degree zero known as CONJECTURE (E_0). There is an even stronger statement (from the combination) known as CONJECTURE (E). These other conjectures will only concern us at the very end.

Known cases of Conjecture 1

- X a curve such that the Tate-Shafarevich group of its Jacobian is finite (Saito)
- Conic bundle surfaces over $\mathbb{P}^1$ (Salberger)
- Smooth compactifications of homogeneous spaces of connected linear algebraic groups with connected geometric stabilizer. (Liang, using results of Borovoi on the Hasse principle)
- X, such that there exists $f : X \to C$, where C is a curve with the following properties: the geometric generic fibre is rationally connected and the fibres above "sufficiently many" points satisfy that the Brauer Manin obstruction to the Hasse principle and weak approximation is the only one. (Harpaz and Wittenberg)
  There is also theoretical motivation for the conjecture (s) (Colliot-Thélène survey 1995)

Conjecture 2 (Skorobogatov)

Let X/k be a $K3$ surface.
The Brauer-Manin obstruction to the existence of a rational point on X is the only one

Growing evidence:

- Results for Kummer surfaces (Harpaz and Skorobogatov)
- Results for elliptic fibrations. (Colliot-Thélène, Swinnerton-Dyer, Skorobogatov)

Motivation

Conjecture 1 (Colliot-Thélène, Sansuc ; Kato and Saito)

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Conjecture 2 (Skorobogatov)

Let X/k be a $K3$ surface.
The Brauer-Manin obstruction to the existence of a rational point on X is the only one

Theorem (I. 2018)

Let X/k be a $K3$ surface.
Suppose that for all finite extensions F/k the Brauer-Manin obstruction to the existence of a rational point on X_F is the only one.
Then the Brauer-Manin obstruction to the existence of a zero-cycle of degree 1 on X is the only one.

Remarks

- Result holds for arbitrary degree
- If Conjecture 2 is false in general but survives for a subclass of $K3$ surfaces, the Theorem still says something for that subclass.
- Transfer "evidence" for Conjecture 2 to "evidence" for Conjecture 1

Sketch of proof

Idea of Liang (2013):

• (The conclusion holds for X) if and only if (The conclusion holds for $X \times \mathbb{P}^1$)

• Use the trivial fibration:

$$\begin{array}{c} X \times \mathbb{P}^1 \\ \big\downarrow f \\ \mathbb{P}^1 \end{array}$$

We will now describe the simplest case of the fibration method for zero cycles. This is all that we will need. For the much more general version see Harpaz and Wittenberg 2016. In any case all the arguments that we will use are already in Liang 2013 (although in a slightly different language)

We have the trivial fibration

$$\begin{array}{c} Y = X \times \mathbb{P}^1 \\ \big\downarrow f \\ \mathbb{P}^1 \end{array}$$

Fix the following

- a finite $B \subset \mathrm{Br}(X)$,
- a finite $S \subset \Omega$, which is "large enough".
- $H \subset \mathbb{P}^1$, where H is a Hilbert subset.

The Hilbert subset is important to us and will come into play in the end. However the actual definition of a Hilbert subset is actually irrelevant for this talk.

Let V/k an irreducible variety. Let U be a dense open subset, and W an irreducible variety with a finite, étale map $W \to U$. Consider the set of points of U above which the fiber is irreducible. Finite intersection of such subsets of V are called Hilbert subsets.

NB. As defined a Hilbert subset is not a subset of $V(k)$. Liang uses the terminology "generalised Hilbertian subset" and reserves the terminology Hilbert subset only for subsets of $V(k)$.

Step 1. Reduction to reduced, effective cycles

INPUT:

- $(z_v) \perp B$ with
- $\deg(z_v) = d \ \forall \ v \in \Omega$

OUTPUT:

- $(z'_v) \perp B$ with
- $\deg(z'_v) = d' \ \forall \ v \in \Omega$ and
- z'_v reduced and effective

Reduced means: $f_*(z'_v) \in Z_0(\mathbb{P}^1_{k_v})$ is reduced zero-cycle.
In the above we also have some control over the value of the new degree d'.
The argument goes back at least to Colliot-Thélène and Swinnerton-Dyer 1994.

Step 2. Everything above a point
INPUT:

- $(z_v) \perp B$ with
- $\deg(z_v) = d \ \forall \ v \in \Omega$
- z_v reduced and effective
- condition on d which we can always guarantee by previous step

OUTPUT:

- $(z'_v) \perp B$ with
- $\deg(z'_v) = d \;\; \forall \;\; v \in \Omega$ and
- (I) z'_v reduced and effective
- (II) $P \in Z_0(\mathbb{P}^1_k)$ such that $f_*(z'_v) = P \in Z_0(\mathbb{P}^1_{k_v}) \;\; \forall \;\; v \in S$

(III) We can take P to be a closed point of H

Denote $F = k(P)$. We have the fibre Y_P/F.
A small argument shows that Properties (I)+(II)+(III) together with the "large enough" property of S imply the

Crucial observation

We have a family of local points

$$(T_w) \in \prod_{w \in \Omega_F} Y_{P,F_w}$$

which is orthogonal to $\mathrm{Im}\,(B \to \mathrm{Br}(Y_P))$

The end result of step 1 and step 2:
We have a family of local points

$$(T_w) \in \prod_{w \in \Omega_F} Y_{P,F_w}$$

of Y_P/F which is orthogonal to $\mathrm{Im}\,(B \to \mathrm{Br}(Y_P))$.
Recall that Y_P/F is simply X_F/F.
Now suppose that

$$B \twoheadrightarrow \mathrm{Br}(Y_P)/\mathrm{Br}_0(Y_P)$$

We can apply the assumption that the Brauer-Manin obstruction to the existence of a rational point on X_F is the only one, and conclude that there exists an F-rational point on X_F. This gives us a zero cycle of degree $[F : k]$.
(We can get the degree of our original zero cycle, since any changes in degrees that happened occurred by adding/subtracting global zero-cycles)

For the method to work we need to ensure that

$$B \twoheadrightarrow \mathrm{Br}(Y_P)/\mathrm{Br}_0(Y_P)$$

More precisely all we need is the following.

Property A

There exists a Hilbert subset $H \subset \mathbb{P}^1_k$ with the following property: For any closed point $P \in H$ the map $B \to \mathrm{Br}(X_{k(P)})/\mathrm{Br}_0(X_{k(P)})$ is surjective

(Actually need something weaker as we can assume something about the degree of the field extension $k(P)/k$)

For general X:

- To start we need $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite. We then take $B \subset \mathrm{Br}(X)$ a finite generating subgroup.
- Then we need to prove **Property (A)** for X.

When X/k is a $K3$ surface:

- We can start since $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite for a $K3$ surface over a number field (Skorobogatov and Zarhin).
- In order to prove **Property (A)** we use a boundedness result of Orr and Skorobogatov, namely: There exists $C_{d,X}$ such that

$$|\mathrm{Br}(\overline{X})^{\mathrm{Gal}(\bar{k}/N)}| \leq C_{d,X}$$

  for any field extension N/k with $[N : k] \leq d$

Relations with other results

Work of Liang 2013

Assumptions on X:

- $NS(X_{\overline{k}})$ is torsion free,
- $H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) = H^2(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}}) = 0$,

These imply that both $\mathrm{Br}(X_{\overline{k}})$ and $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ are finite. To prove **Property (A)** use a result of Harari (at least for the "weak approximation" part that we did not mention- for existence part can argue directly...)

Work of Balestrieri and Newton 2018
Assumptions on X: X/k is Kummer

- We can start since $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite (Skorobogatov and Zarhin).

Using **Property (A)** seems out of reach (?)
But Creutz and Viray proved the following

$$X(\mathbb{A}_k)^{\mathrm{Br}(X)} = \emptyset \Rightarrow X(\mathbb{A}_k)^{\mathrm{Br}(X)[2^\infty]} = \emptyset$$

for Kummer varieties

The result of Creutz and Viray has the implication that we want something weaker in **Property (A)**: Instead of

$$B \twoheadrightarrow \mathrm{Br}(Y_P)/\mathrm{Br}_0(Y_P)$$

we now only need

$$B \twoheadrightarrow \mathrm{Br}(Y_P)/\mathrm{Br}_0(Y_P)[2^\infty]$$

(The appearance of 2 causes some problems with zero-cycles of even degree)

The real work is in showing this modified property. This is what Balestrieri and Newton do.

Caveat: This is only my impression of what they did. I believe that they prove enough for the above to go through. But maybe they prove more! They follow Liang's language closely so the translation to what we said is not that straightforward.

To move to general fibrations we need a substitute for Harari's result. The results of Cadoret and Charles on uniform boundedness of show that there might be hope for families of $K3$ surfaces!

Problem with our result

We would like to make a precise connection from our conclusions with CONJECTURE (E). We are unable to do so, but maybe that is unavoidable given our lack of understanding of $A_0(X_{k_v})$.
Let F be a finite extension of $\mathbb{Q}_p$. Let V/F is a $K3$ surface which admits a smooth proper model over $\mathcal{O}_F$.
By results of Kato, Saito and Saito, Sato $A_0(X)$ is the direct sum of a finite p-group and a group that is divisible by all primes distinct from p.
What can be said about the p-primary part of $A_0(V)$?

CONJECTURE (E_0)
The sequence

$$\varprojlim_n A_0(X)/n \to \prod_{v\in\Omega} \varprojlim_n A_0(X_{k_v})/n \to \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z})$$

is exact.
CONJECTURE (E_1)
If there is a family of local zero-cycles (z_v) of degree one orthogonal to $\mathrm{Br}(X)$ then there exists a zero-cycle of degree one on X.

CONJECTURE (E)
The sequence

$$\widehat{CH_0(X)} \to \widehat{CH_{0,A}(X)} \to \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z})$$

is exact