Finiteness theorems for K3 surfaces over arbitrary fields

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Shepperton, May 2018

Outline

Rational points on K3 surfaces

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One invariant very relevant to these problems is the group $\operatorname{Aut} X$ of k-automorphisms of X.

Cones

Let X be a projective K3 surface over an algebraically closed field k.

- Inside $(\mathrm{Pic}\, X)_{\mathbb{R}}$ we have the cone

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- Within $\mathcal{C}_X$, the ample cone is cut out by the (-2)-curves C.

Automorphisms and cones

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- (Where $\operatorname{Nef}(X)$ meets the boundary of $\mathcal{C}_X$, it need not be locally polyhedral.)

Finiteness theorems

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- Nikulin has classified the lattices arising as Picard lattices of K3 surfaces such that $\mathrm{O}(\operatorname{Pic} X)/W(\operatorname{Pic} X)$ is finite. For each rank $\rho \geq 3$, there are only finitely many (up to isomorphism).

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How many of the preceding statements remain true over an arbitrary base field k?

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- The positive, ample and nef cones of X are just the intersections with $(\operatorname{Pic} X)_{\mathbb{R}}$ of the positive, ample and nef cones of $\bar{X}$.

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- For example, suppose that X contains a pair of disjoint, conjugate (-2)-curves C_1, C_2. The class $[C_1 + C_2] \in \operatorname{Pic} X$ defines a wall of the ample cone that does not correspond to a (-2)-class defined over k.

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- Define $R_X = W(\operatorname{Pic} \bar{X})^{\operatorname{Aut}(\bar{k}/k)}$. This is a Coxeter group.

The main theorem

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Let X be a K3 surface over any field of characteristic $\neq 2$.

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Separably closed fields

The first step is to reduce to separably closed fields – after that, we can use Galois theory.

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- This also shows that all (-2)-curves on $\bar{X}$ are defined over k.
- Similarly, $\mathrm{H}^0(X, T_X) = 0$ shows that the automorphism scheme $\mathbf{Aut}_{X/k}$ is étale over k, and so $\operatorname{Aut} X \to \operatorname{Aut} \bar{X}$ is an isomorphism.

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- Γ_k acts on $W(\mathrm{Pic}\, X^s)$ by conjugation in $\mathrm{O}(\mathrm{Pic}\, X^s)$; we have $\sigma s_\delta \sigma^{-1} = s_{\sigma\delta}$. Define $R_X = W(\mathrm{Pic}\, X^s)^{\Gamma_k}$.

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- It is clear that the action of R_X on $\operatorname{Pic} X^s$ preserves $(\operatorname{Pic} X^s)^{\Gamma_k}$, but not immediately obvious that it preserves $\operatorname{Pic} X$.
- Fortunately, we can see this from an explicit description of R_X.

Description of R_X

Theorem (Hée; Lusztig; Geck, Iancu)

Let (W, T) be a Coxeter system. Let G be a group of permutations of T that induce automorphisms of W. Let F be the set of G-orbits $I \subset T$ for which W_I is finite, and for $I \in F$ let ℓ_I be the longest element of (W_I, I). Then $(W^G, \{\ell_I : I \in F\})$ is a Coxeter system.

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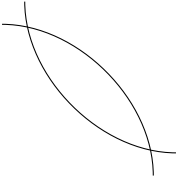

If two (-2)-curves have intersection number ≥ 2, then the corresponding reflections generate an infinite dihedral group in W. So a Galois orbit containing two such curves will not lie in F, and will not contribute a generator to R_X.

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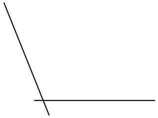

If an orbit consists of two (-2)-curves C, C' intersecting with multiplicity 1, then $s_C, s_{C'}$ generate a subgroup of W isomorphic to $S_3 = W(A_2)$; the longest element is the (-2)-class $[C] + [C']$, giving a Galois-invariant reflection $s_{[C]+[C']} = s_C s_{C'} s_C = s_{C'} s_C s_{C'}$.

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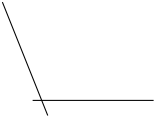

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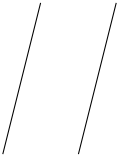

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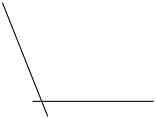

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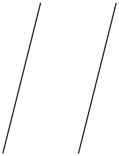

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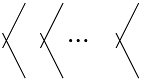
In general, the only orbits contributing to R_X are disjoint unions of these.



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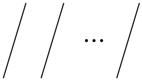

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It is now easy to show that $\operatorname{Nef} X \cap \mathcal{C}_X$ is a fundamental domain for the action of R_X on $\mathcal{C}_X$, as follows.

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5. To show that two translates of $(\operatorname{Nef} X \cap \mathcal{C}_X)$ intersect only in their boundaries, use $\partial(\operatorname{Nef} X) = \partial(\operatorname{Nef} X^s) \cap (\operatorname{Pic} X)_{\mathbb{R}}$.

Descending finiteness

To prove that $\operatorname{Aut} X \to \mathrm{O}(\operatorname{Pic} X)/R_X$ has finite kernel and cokernel:

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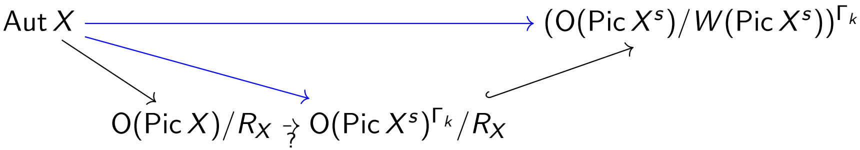

Theorem

Let Λ be a lattice and $H \subset \mathrm{O}(\Lambda)$ a subgroup such that $M = \Lambda^H$ is non-degenerate. Then:

1. *the natural map $\mathrm{O}(\Lambda, M) \to \mathrm{O}(M)$ has finite cokernel;*
2. *if $M^\perp$ is definite, then $\mathrm{O}(\Lambda, M) \to \mathrm{O}(M)$ has finite kernel, and the centraliser $Z_{\mathrm{O}(\Lambda)} H$ has finite index in $\mathrm{O}(\Lambda, M)$.*

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- That the action of $\operatorname{Aut} X$ on $\operatorname{Nef}^e X$ admits a rational polyhedral fundamental domain follows as in the complex case (using the general fact that $\mathrm{O}(\operatorname{Pic} X)$ has such a fundamental domain).
- There are finitely many orbits under $\operatorname{Aut}(X)$ of classes of irreducible curves of given self-intersection: this also follows as in the complex case.

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- Over $\mathbb{C}$, a K3 surface having intersection matrix M would have infinite automorphism group, whereas a K3 surface having intersection matrix N would have finite automorphism group.
- We construct a K3 surface X over $\mathbb{Q}$ such that $\operatorname{Pic} X$ has intersection matrix M, but $\operatorname{Pic} \bar{X}$ has intersection matrix N. So $\operatorname{Aut} \bar{X}$, and *a fortiori* $\operatorname{Aut} X$, is finite.

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Lemma

Let X be a K3 surface over $\mathbb{Q}$ with an elliptic fibration $\pi\colon X \to \mathbb{P}^1$ that has a section. Suppose that π has four conjugate fibres of type I_2 or III and that the rank of $\operatorname{Pic} \bar{X}$ is at most 6. Then $\operatorname{Pic} X$ and $\operatorname{Pic} \bar{X}$ have intersection matrices M and N, respectively.

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It turns out that such a surface cannot be embedded with small codimension in projective space, so we do the next best thing: find a smooth quartic surface with an elliptic fibration but no section, whose relative Jacobian is the desired X.

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The smooth quartic surface $Y \subset \mathbb{P}^3_{\mathbb{Q}}$ given by the equation

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contains the line $w = z = 0$. Projection away from this line defines a fibration with no section, having four conjugate singular fibres of type I_2.

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- To prove that the rank of Pic $\bar{Y}$ is 6, we use good reduction at 3 and point-counting over the fields $\mathbb{F}_{3^n}$ for $n \leq 8$.
- It follows that the Jacobian of the fibration on Y is an example of the type we are looking for.

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- We take X to be the intersection of a quadric and a cubic in $\mathbb{P}^4$, containing a pair of disjoint Galois-conjugate conics and having geometric Picard number 3.

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- A K3 surface over $\mathbb{C}$ having this Picard lattice would contain no (-2)-curves, so would have infinite automorphism group.
- However, X does contain a Galois-conjugate disjoint pair of (-2)-curves, and in fact contains many.

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- The two reflections in the (-4)-classes $[C] + [C']$ and $[D] + [D']$ generate an infinite dihedral subgroup of R_X, showing that $\mathrm{O}(\operatorname{Pic} X)/R_X$ is finite, and hence so is $\operatorname{Aut} X$.

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- Actually writing down equations for an example is more involved than in Example I, since we must use reduction at two primes to show that $\operatorname{Pic} \bar{X}$ has rank 3.

Example III

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Theorem

Let k be a field of characteristic zero, let $c \in k^\times$ be such that $[k(\zeta_8, \sqrt[4]{c}) : k] = 16$, and let $X \subset \mathbb{P}^3_k$ be the surface

$$x^4 - y^4 = c(z^4 - w^4).$$

Then $\operatorname{Aut} X$ *is finite.*

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Our proof is computational.

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  from Example I. The group $O(\Lambda)$ is not hard to describe.
- Listing all the lines and conics on $\bar{X}$ gives enough elements of R_X to prove that $O(\operatorname{Pic} X)/R_X$ is finite.