Specialization of Néron-Severi group in positive characteristic

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K3 surfaces and Galois representations - Shepperton, England
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- $f : Y \to X$ smooth proper morphism;
- For $x \in X$, Y_x and $Y_{\overline{x}}$ corresponding fibres.

Specialization of representations

- Smooth and proper base change:

$$\begin{tikzcd}
\pi_1(k(\eta)) \arrow[r, "\rho_{\ell,\eta}"] \arrow[d, two heads] & GL(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1))) \arrow[dd, "\simeq"] \\
\pi_1(X) \arrow[ur, "\rho_\ell"] \arrow[dr] & \\
\pi_1(k(x)) \arrow[r, "\rho_{\ell,x}"] \arrow[u] & GL(H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)))
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- Write:

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- Consider the inclusion

$$\Pi_{\ell,x} \subseteq \Pi_\ell$$

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- For $x \in |X|$, injective map:

$$sp_{\eta,x} : NS(Y_{\overline{\eta}}) \otimes \mathbb{Q} \hookrightarrow NS(Y_{\overline{x}}) \otimes \mathbb{Q}$$

Main result

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Corollary

If $f : Y \to X$ smooth and proper there exists a $x \in |X|$ such that $sp_{\eta,x}$ is an isomorphism.

Applications

- If X curve, f projective then $\exists C := C(\ell, Y \to X)$ such that
$$|Br(Y_{\overline{x}})[\ell^{\infty}]^{\pi_1(x)}| \leq C$$
for all $x \in X(k)$ such that Y_x satisfies the Tate conjecture for divisors.

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- (E.A) Z smooth projective variety of dimension ≥ 3. There are infinitely many k-rational hyperplane sections W with $NS(W) \otimes \mathbb{Q} = NS(Z) \otimes \mathbb{Q}$.

Main ideas in the proof when $p = 0$: Cadoret's talk

- Variational Hodge conjecture (i.e. Lefschetz theorem on (1,1)-classes + Hodge II (P.Deligne)) $\Rightarrow$ specialization of $NS(Y_{\overline{x}})$ in Betti cohomology controlled via the action of topological fundamental group of $X_{\mathbb{C}}$.

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- Comparison between singular and étale cohomology $\Rightarrow$ action studied via the relationship between Π_ℓ and $\Pi_{\ell,x}$

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- 1 is replaced with the variational Tate conjecture in crystalline cohomology;
- 2 is replaced with:
  - Relation between F-crystals and F-overconvergent isocrystals;
  - Comparison between ℓ-adic and overconvergent monodromy groups via Tannakian formalism and independence.

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    - Objects: $(\mathcal{U} \to \mathcal{T}, \gamma)$, $\mathcal{U} \subseteq \mathcal{X}$ Zariski open, $\mathcal{U} \to \mathcal{T}$ nilpotent immersion of W schemes, γ P.D. structure on $Ker(\mathcal{O}_\mathcal{T} \to \mathcal{O}_\mathcal{U})$;

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  - Leray spectral sequence: $E_2^{i,j} := H^i(\mathcal{X}, R^j\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W}) \otimes \mathbb{Q} \Rightarrow H^i_{crys}(\mathcal{Y})$.

Variational Tate conjecture in crystalline cohomology

Commutative diagram

For $\mathfrak{t} \in |\mathcal{X}|$:

$$\begin{tikzcd}
 & H^2_{crys}(\mathcal{Y}) \arrow[d, "\downarrow Leray"'] \arrow[dr, "i_{\mathfrak{t}}^*"] & Pic(\mathcal{Y}) \otimes \mathbb{Q} \arrow[l, "ch_{\mathcal{Y}}"'] \arrow[dr, "i_{\mathfrak{t}}^*"] & \\
 & H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W}) \otimes \mathbb{Q} \arrow[r, hook] & H^2_{crys}(\mathcal{Y}_{\mathfrak{t}}) & Pic(\mathcal{Y}_{\mathfrak{t}}) \otimes \mathbb{Q} \arrow[l, "ch_{\mathcal{Y}_{\mathfrak{t}}}"]
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Fact (M.Morrow '14)

If $\mathfrak{f}$ is projective, for every $z \in Pic(\mathcal{Y}_{\mathfrak{t}}) \otimes \mathbb{Q}$ the following are equivalent:

1. *There exists $\tilde{z} \in Pic(\mathcal{Y}) \otimes \mathbb{Q}$ such that $ch_{\mathcal{Y}_{\mathfrak{t}}}(z) = i_{\mathfrak{t}}^*(ch_{\mathcal{Y}}(\tilde{z}))$;*

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2. *$ch_{\mathcal{Y}_{\mathfrak{t}}}(z)$ lies in $H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W})^{F=q} \otimes \mathbb{Q}$.*

Models

$$\begin{tikzcd}
Y_x \arrow[r] \arrow[d] & Y \arrow[d] & Y_\eta \arrow[l] \arrow[d] \\
x \arrow[r] & X \arrow[d] & \eta \arrow[l] \\
 & k &
\end{tikzcd}$$

Models

Model over $\mathbb{F}_q$:

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\end{tikzcd}$$

Remark:

$\mathfrak{t}$ specialization of x, x specialization of η.

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- It is enough $Im(sp_{\eta,\mathfrak{t}}) = Im(sp_{x,\mathfrak{t}})$
- *VTCC*+diagram chasing $\Rightarrow$ enough to show

$$H^0(\mathcal{K}, R^2\mathfrak{f}_{\mathcal{K},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{K}}/K})^{F=q} = H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K})^{F=q}$$

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  - $\mathcal{O}_{\mathcal{X}/W} \otimes \mathbb{Q} := \mathcal{O}_{\mathcal{X}/K}$;
  - $R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K}$ (Coherence + Base change + ...).

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- **F-Isoc**$(\mathcal{X}) := \{(\mathcal{E}, \Phi) \mid \mathcal{E} \in$ **Isoc**$(\mathcal{X})$, $\Phi : F^*_{\mathcal{X}}\mathcal{E} \simeq \mathcal{E}\}$
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Pathologies

1. Different behaviour from ℓ-adic representations;
2. Infinite dimensional cohomology if $\mathcal{X}$ not proper.

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  - global monodromy theorem (Crew, Kedlaya).

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- *There is a functor Forg* : $\textbf{F-Isoc}^{\dagger}(\mathcal{X}) \to \textbf{F-Isoc}(\mathcal{X})$ *(Berthelot-Ogus);*

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We want to show:

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- VTCC: relation between algebraic cycles and isocrystals;
- Berthelot, Ogus, Kedlaya, Shiho: relation between isocrystals and overconvergent isocrystals.

From representations to overconvergent isocrystals

To do:

From ℓ-adic representations to overconvergent isocrystals.

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Solution:

Tannakian formalism.

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Independence

- $\mathcal{F} := R^2 f_* \mathbb{Q}_\ell(1)$
- $< \mathcal{F} >$ Tannakian category with Tannakian group $G(\mathcal{F})$
- $G(\mathcal{F}) = \overline{\Pi}_\ell$, $G(\mathcal{F}_\mathcal{K}) = \overline{\Pi}_{\ell,x}$
- By assumption $G(\mathcal{F}) = G(\mathcal{F}_\mathcal{K})$.

Proposition

$$G(\mathcal{F}_\mathcal{K}) = G(\mathcal{F}) \text{ if and only if } G(\mathcal{M}_\mathcal{K}) = G(\mathcal{M})$$

Idea of proof

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  - Theory of weights: invariants determined by L functions;
  - L functions do not depend on ℓ or p.