J. London Math. Soc. (2) 89 (2014) 559–580
doi:10.1112/jlms/jdt080

Norms as products of linear polynomials

Damaris Schindler and Alexei Skorobogatov

Abstract

Let F be a number field, and let $F \subset K$ be a field extension of degree n. Suppose that we are given $2r$ sufficiently general linear polynomials in r variables over F. Let X be the variety over F such that the F-points of X bijectively correspond to the representations of the product of these polynomials by a norm from K to F. Combining the circle method with descent we prove that the Brauer–Manin obstruction is the only obstruction to the Hasse principle and weak approximation on any smooth and projective model of X.

1. *Introduction*

Let K/F be an extension of number fields of degree $n \geqslant 2$. We fix a basis $\xi_1, \ldots, \xi_n$ of K as an F-vector space, and write $N(\mathbf{z})$ for the norm form $N_{K/F}(z_1\xi_1 + \cdots + z_n\xi_n)$, where $\mathbf{z} = (z_1, \ldots, z_n)$. Let $L_1(\mathbf{t}), \ldots, L_{2r}(\mathbf{t})$, where $\mathbf{t} = (t_1, \ldots, t_r)$, be non-zero linear functions with coefficients in F, not necessarily homogeneous. Consider the Diophantine equation

$$\prod_{i=1}^{2r} L_i^{e_i}(\mathbf{t}) = cN(\mathbf{z}), \tag{1.1}$$

where $c \in F^*$, and $e_1, \ldots, e_{2r}$ are positive integers. It is known that already for $r = 1$ and $e_1 = e_2 = 1$, weak approximation for (1.1) can fail. Thus, one is naturally led to investigate whether the Brauer–Manin obstruction controls the Hasse principle and weak approximation on smooth and projective varieties birationally equivalent to the affine hypersurface (1.1). For $r = 1$, this was proved for $F = \mathbb{Q}$ in [**5**, **10**] (see also [**4**]), and recently generalized to an arbitrary number field F in [**16**]. In this paper, which is independent of [**16**], we combine the circle method of Hardy and Littlewood with the method of descent of Colliot-Thélène and Sansuc to extend these results to $r \geqslant 1$ and any number field F.

For the circle method part, we require the functions L_i to be sufficiently general. More precisely, we assume the following condition.

Condition I. *Let $\mathcal{L}$ be the set of linear functions $\{1, L_1, \ldots, L_{2r}\}$. For each $L \in \mathcal{L}$, there exist subsets $\mathcal{A} \subset \mathcal{L}$ and $\mathcal{B} \subset \mathcal{L}$ of linearly independent functions such that $|\mathcal{A}| = |\mathcal{B}| = r + 1$ and $\mathcal{A} \cap \mathcal{B} = \{L\}$.*

Our main result is the following theorem.

Theorem 1.1. *Let F be a number field. If $L_1, \ldots, L_{2r}$ satisfy Condition* I, *then the Brauer–Manin obstruction is the only obstruction to the Hasse principle and weak approximation on any smooth and proper model of the affine hypersurface X given by* (1.1). *When the set of F-points of X is not empty, it is Zariski dense in X.*

Received 22 October 2012; revised 30 October 2013; published online 16 January 2014.

2010 *Mathematics Subject Classification* 14G05 (primary) 11D57, 11G35, 11P55 (secondary).

The first author was partly supported by a DAAD scholarship. The second author was supported by the Centre Interfacultaire Bernoulli of the École Polytechnique Fédérale de Lausanne.

The calculation of the Brauer group of a smooth and proper model of X is a non-trivial open problem; see [**5**, **17**, **18**] for some results in this direction. The following corollary to Theorem 1.1 is based on the simplest case when the Brauer group is trivial, pointed out in [**5**, Corollary 2.7].

Corollary 1.2. *In the assumptions of Theorem* 1.1 *assume further that either of the following two conditions holds.*

(i) *One has* $(e_1, \dots, e_{2r}) = 1$ *and* K *does not contain a cyclic extension of* F *of degree* d *such that* $1 < d < n$.

(ii) n *is prime and* K *is not a Galois extension of* F.

Let X_{sm} *be the smooth locus of* X. *Then the image of the natural map*

$$X_{\mathrm{sm}}(F) \longrightarrow \prod_{\nu} X_{\mathrm{sm}}(F_\nu),$$

where F_ν *ranges over all completions of* F, *is dense in the product of local topologies.*

Our descent argument is summarized in Theorem 2.1 which closely follows [**5**]. We construct a smooth partial compactification X' of X_{sm} such that X' has no non-constant invertible regular functions and the geometric Picard group of X' is torsion-free. We define a convenient class of X'-torsors, called 'vertical' torsors. Such X'-torsors always exist and are birationally equivalent to the product of the variety Y given by

$$\sum_{j=1}^{2r} a_{ij} N(\mathbf{z}_j) + a_{i,2r+1} = 0, \quad 1 \leqslant i \leqslant r, \tag{1.2}$$

where $a_{ij} \in F$ and $\mathbf{z}_j = (z_{n(j-1)+1}, \dots, z_{nj})$, and the affine variety $N(\mathbf{z}) = a$ for some $a \in F^*$.

We always write $s = 2r + 1$ and $m = [F : \mathbb{Q}]$. It is easy to show (see the proof of Theorem 1.1 in Section 2) that Condition I implies that the coefficient $(r \times s)$-matrix $A = (a_{ij})$ satisfies the following rank condition.

Condition II. *If we remove any column of* A, *then the remaining columns can be partitioned into two* $(r \times r)$*-matrices of full rank.*

Using descent we deduce Theorem 1.1 from Theorem 1.3 and the well-known theorem of Sansuc that the Brauer–Manin obstruction is the only obstruction to the Hasse principle and weak approximation on smooth compactifications of principal homogeneous spaces of tori.

Theorem 1.3. *Let* Y *be the affine variety given by* (1.2), *where the matrix* A *satisfies Condition* II. *Then the image of the natural map*

$$Y_{\mathrm{sm}}(F) \longrightarrow \prod_{\nu} Y_{\mathrm{sm}}(F_\nu),$$

where F_ν *ranges over all completions of* F, *is dense in the product of local topologies.*

This theorem establishes the Hasse principle and weak approximation for Y_{sm}. To prove it we homogenize the system of equations (1.2) using an extra norm form, and then apply the Hardy–Littlewood circle method over F. Write $\mathcal{O}_F$ for the ring of integers of F. Let $B = (b_{ij})$ be an $(r \times s)$-matrix with entries in $\mathcal{O}_F$ that satisfies Condition II. Write $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_s)$, and set

$$f_i(\mathbf{x}) = \sum_{j=1}^{s} b_{ij} N(\mathbf{x}_j)$$

for $1 \leqslant i \leqslant r$. We look for integer solutions of the system of equations

$$f_i(\mathbf{x}) = 0, \quad 1 \leqslant i \leqslant r \tag{1.3}$$

in a certain box. Moreover, we want these solutions to satisfy congruence conditions. Let $\mathfrak{n} \subset \mathcal{O}_F$ be an integral ideal, and let $\omega_1, \ldots, \omega_m$ be a $\mathbb{Z}$-basis of $\mathfrak{n}$. This is also a basis of the real vector space $V = F \otimes_{\mathbb{Q}} \mathbb{R}$. Fix $\kappa > 0$ and $\mathbf{u} = (u_1, \ldots, u_{ns}) \in V^{ns}$, and define the box $\mathcal{B} = \mathcal{B}(\mathbf{u}, \kappa) \subset V^{ns}$ as

$$\mathcal{B}(\mathbf{u}, \kappa) = \{\mathbf{x} \in V^{ns} : |x_{ij} - u_{ij}| \leqslant \kappa \text{ for } 1 \leqslant i \leqslant ns \text{ and } 1 \leqslant j \leqslant m\},$$

where the real variables x_{ij} are defined by $x_i = \sum_{j=1}^{m} x_{ij}\omega_j$, and similarly for u_{ij}. Fix also a vector $\mathbf{d} \in (\mathcal{O}_F)^{ns}$. We are interested in the number of solutions

$$N(\mathcal{B}, P) = |\{\mathbf{x} \in (P\mathcal{B}) \cap \mathfrak{n}^{ns} : f_i(\mathbf{x} + \mathbf{d}) = 0 \text{ for } 1 \leqslant i \leqslant r\}|,$$

where P is large. Theorem 1.3 is a corollary of the following result.

THEOREM 1.4. *Let $\{\xi_1, \ldots, \xi_n\} \subset \mathcal{O}_K$ be a basis of K as an F-vector space, and let B be a matrix that satisfies Condition* II. *If*

$$\operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}(\mathbf{x})\right) = r$$

for any $\mathbf{x} \in \mathcal{B}$, then

$$N(\mathcal{B}, P) = \mu(\mathcal{B})P^{mn(r+1)} + O(P^{mn(r+1)-\eta})$$

for some $\eta > 0$, where $\mu(\mathcal{B})$ is the product of local densities given explicitly in equation (3.16). *Moreover, if the system of equations $f_i(\mathbf{x} + \mathbf{d}) = 0$ has a non-singular solution in $\mathfrak{n}_\nu^{ns}$ for all finite places ν of F, and the system of equations $f_i(\mathbf{x}) = 0$ has a non-singular solution in $\mathcal{B}$, then $\mu(\mathcal{B}) > 0$.*

Here, $\mathfrak{n}_\nu = \mathfrak{n}\mathcal{O}_\nu$, where $\mathcal{O}_\nu$ is the ring of integers of F_ν. We specify the condition on the box $\mathcal{B}$ to simplify the treatment of the singular integral.

Theorem 1.4 is of interest because, on the one hand, the number of variables in (1.3) is linear in the number of the equations and their degrees. On the other hand, the catalogue of examples in which the circle method has been applied to number fields with conclusions independent of the degree of the field, is extremely small (see, for example, [**2**, **13**]). Our approach relies on the work of Birch, Davenport and Lewis [**3**] and of Heath-Brown and one of the authors [**10**]. For our system of linear equations with variables replaced by norm forms we obtain an asymptotic formula without weights, in contrast to [**16**].

The paper is organized as follows. In Section 2, we describe vertical torsors for the variety over a field F whose F-points bijectively correspond to the representations of the values of an arbitrary polynomial in several variables by a norm from a finite extension K/F. We apply descent and deduce our main Theorem 1.1 from Theorem 1.3, and prove Corollary 1.2. In Section 3, we set up the circle method over number fields and prove Theorems 1.3 and 1.4.

2. *Descent*

We begin by proving a slightly more general descent statement than the one needed to deduce Theorem 1.1 from Theorem 1.3.

Let F be a field of characteristic zero with an algebraic closure $\bar{F}$ and the Galois group $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$. When X is an F-variety we write $\bar{X} = X \times_F \bar{F}$. We denote the smooth locus of X by X_{sm}.

Let $N(\mathbf{z})$ be a norm form attached to a field extension K/F of degree n. Define a hypersurface $X \subset \mathbb{A}_F^{r+n}$ by the equation $P(\mathbf{t}) = N(\mathbf{z})$, where $P(\mathbf{t})$ is a non-constant polynomial in $F[\mathbf{t}] = F[t_1, \ldots, t_r]$. The closed subset $Y \subset \mathbb{A}_F^r$ given by $P(\mathbf{t}) = 0$ is the union of irreducible components $Y = Y_1 \cup \cdots \cup Y_d$. For each $i = 1, \ldots, d$ choose a geometrically irreducible component $Y_i' \subset \bar{Y}_i$, and let $F_i \subset \bar{F}$ be the invariant subfield of the stabilizer of Y_i' in Γ_F. Let $P_i(\mathbf{t}) \in F_i[\mathbf{t}]$ be an absolutely irreducible polynomial in $\mathbf{t} = (t_1, \ldots, t_r)$ such that Y_i' is given by $P_i(\mathbf{t}) = 0$. Let us use $N_{F_i/F}$ as an abbreviation for the norm $N_{F_i(\mathbf{t})/F(\mathbf{t})}$. Then $N_{F_i/F}(P_i(\mathbf{t}))$ is an irreducible polynomial in $F[\mathbf{t}]$ such that Y_i is given by $N_{F_i/F}(P_i(\mathbf{t})) = 0$. Thus, the hypersurface $X \subset \mathbb{A}_F^{r+n}$ can be given by

$$\prod_{i=1}^{d} N_{F_i/F}(P_i(\mathbf{t}))^{e_i} = cN(\mathbf{z}), \tag{2.1}$$

where $c \in F^*$ and $e_1, \ldots, e_d$ are positive integers.

Let $\mathbf{y}_i$ be a variable with values in $K \otimes_F F_i$ for $i = 1, \ldots, d$. Consider the quasi-affine subvariety $V \subset \mathbb{A}_F^r \times \prod_{i=1}^d R_{K\otimes_F F_i/F}(\mathbb{A}^1)$ defined by

$$P_i(\mathbf{t}) = \varrho_i N_{K\otimes_F F_i/F_i}(\mathbf{y}_i) \neq 0, \tag{2.2}$$

where $\varrho_i \in F_i^*$ and $i = 1, \ldots, d$.

THEOREM 2.1. *Let F be a number field. Suppose that for any $\varrho_i \in F_i^*$, $i = 1, \ldots, d$, the variety V satisfies the Hasse principle and weak approximation. Then the Brauer–Manin obstruction is the only obstruction to the Hasse principle and weak approximation on any smooth and proper model of the affine hypersurface X.*

Proof. Let $\pi : X \to \mathbb{A}_F^r$ be the morphism defined by the projection to coordinates $t_1, \ldots, t_r$. Define $U_0 \subset \mathbb{A}_F^r$ as the open subset given by $N(\mathbf{z}) \neq 0$, and $U = \pi^{-1}(U_0)$. We note that $\bar{U} \cong \bar{U}_0 \times \mathbb{G}_{m,\bar{F}}^{n-1}$, and this implies $\mathrm{Pic}(\bar{U}) = 0$.

We can write $N(\mathbf{z})$ as the product $\prod_{i=1}^n u_i(\mathbf{z})$ of linearly independent linear forms with coefficients in $\bar{F}$. It is easy to check that the complement to the union of closed subsets given by $u_i = u_j = 0$ for all $i \neq j$, is smooth. Thus, $\pi(X_{\mathrm{sm}}) = \mathbb{A}_F^r$ and $U \subset X_{\mathrm{sm}}$.

Recall that $R_{K/F}(\mathbb{G}_{m,K})$ is a torus over F defined as the Weil restriction of the multiplicative group $\mathbb{G}_{m,K}$. The module of characters of $R_{K/F}(\mathbb{G}_{m,K})$ is the induced Γ_F-module $\mathbb{Z}[\Gamma_F/\Gamma_K]$ that will be denoted by $\mathbb{Z}[K/F]$. The norm torus T is the kernel of the surjective homomorphism $R_{K/F}(\mathbb{G}_{m,K}) \to \mathbb{G}_{m,F}$ given by the norm $N_{K/F}$, so T is the affine hyperplane $N(\mathbf{z}) = 1$. The module of characters $\hat{T}$ fits into the exact sequence of Γ_F-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[K/F] \longrightarrow \hat{T} \longrightarrow 0,$$

where $1 \in \mathbb{Z}$ goes to the sum of canonical generators of $\mathbb{Z}[K/F]$.

It is known (see, for example, [**6**]) that T, like any other torus, has a smooth equivariant compactification. This is a smooth, projective and geometrically integral variety T^c over F with an action of T that contains an open T-orbit isomorphic to T. The contracted product $U^c = U \times^T T^c$ can be defined as the quotient of $U \times T^c$ by the simultaneous action of T on both factors. Thus, the morphism $\pi : U \to U_0$ extends to a smooth and proper morphism $U^c \to U_0$, and U is open and dense in U^c. Moreover, each geometric fibre of $U^c \to U_0$ is a smooth compactification of T. Let X' be the scheme over F obtained by gluing X_{sm} and U^c along U. The argument in [**5**, p. 71] shows that X' is separated, hence X' is a variety. We denote the natural morphism $X' \to \mathbb{A}_F^r$ also by π. Since the generic fibre X_η' of this morphism is projective and geometrically integral, by restricting an invertible regular function f on $\bar{X}'$ to X_η' we see that $f \in \bar{F}(\mathbb{A}_F^r)$. However, the morphism $\pi : X' \to \mathbb{A}_F^r$ is surjective, hence if the divisor of f in $\mathbb{A}^r_F$ is non-zero, then the divisor of f in $\bar{X}'$ is non-zero too. We conclude that $\bar{F}[X']^* = \bar{F}^*$, that is, $\bar{X}'$ has no non-constant invertible regular functions.

It is clear that the geometrically irreducible components of the hypersurface $Y_i \subset \mathbb{A}^r_F$ form a Γ_F-stable $\mathbb{Z}$-basis of the free abelian group $\mathbb{Z}[F_i/F]$. We thus have a natural isomorphism of Γ_F-modules

$$\bar{F}[U_0]^*/\bar{F}^* = \bigoplus\nolimits_{i=1}^d \mathbb{Z}[F_i/F].$$

The geometrically irreducible components of $X' \setminus U^c$ form a Γ_F-stable $\mathbb{Z}$-basis of the free abelian group of divisors on $\bar{X}'$ with support outside of $\bar{U}^c$:

$$\mathrm{Div}_{\bar{X}'\setminus\bar{U}^c}(\bar{X}') = \mathbb{Z}[K/F] \otimes \left(\bigoplus\nolimits_{i=1}^d \mathbb{Z}[F_i/F]\right).$$

We call an irreducible divisor $D \subset \bar{X}'$ horizontal if π induces a dominant map $D \to \mathbb{A}^r_{\bar{F}}$. The subgroup of $\mathrm{Div}_{\bar{X}'\setminus\bar{U}}(\bar{X}')$ generated by horizontal divisors is $\mathrm{Div}_{\bar{U}^c\setminus\bar{U}}(\bar{U}^c)$, which is isomorphic to $\mathrm{Div}_{\bar{T}^c\setminus\bar{T}}(\bar{T}^c)$ as a Γ_F-module; see [**5**, Lemma 2.1]. We obtain a direct sum decomposition of Γ_F-modules:

$$\mathrm{Div}_{\bar{X}'\setminus\bar{U}}(\bar{X}') = \mathrm{Div}_{\bar{T}^c\setminus\bar{T}}(\bar{T}^c) \oplus \mathrm{Div}_{\bar{X}'\setminus\bar{U}^c}(\bar{X}').$$

There is a commutative diagram of Γ_F-modules with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bar{F}[U_0]^*/\bar{F}^* & \to & \mathrm{Div}_{\bar{X}'\setminus\bar{U}^c}(\bar{X}') & \to & \hat{T} \otimes \left(\bigoplus_{i=1}^d \mathbb{Z}[F_i/F]\right) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bar{F}[U]^*/\bar{F}^* & \to & \mathrm{Div}_{\bar{X}'\setminus\bar{U}}(\bar{X}') & \to & \mathrm{Pic}(\bar{X}') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \hat{T} & \to & \mathrm{Div}_{\bar{T}^c\setminus\bar{T}}(\bar{T}^c) & \to & \mathrm{Pic}(\bar{T}^c) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.3}$$

constructed in the same way as the diagram in [**5**, Proposition 2.2]. The injective maps in the top and middle rows are induced by the map $\mathrm{div}_{\bar{X}'}$ sending a function to its divisor in $\bar{X}'$. The middle row is exact because $\bar{F}[X']^* = \bar{F}^*$ and $\mathrm{Pic}(\bar{U}) = 0$. The vertical maps from the middle row to the bottom row are given by the restriction to the generic fibre of $\pi : \bar{X}' \to \mathbb{A}^r_{\bar{F}}$. We refer to [**5**, Proposition 2.2] for the identification of the modules and the maps in the bottom row. The smooth and projective variety $\bar{T}^c$ is rational, hence $\mathrm{Pic}(\bar{T}^c)$ is torsion-free. From the exactness of the right-hand column of (2.3) we see that $\mathrm{Pic}(\bar{X}')$ is torsion-free.

We refer to [**14**, Section 2] for more details on torsors, in particular, for the definition of the type of a torsor under a torus. Let λ be the injective map of Γ_F-modules from the right-hand column of (2.3). We shall call a torsor $\mathcal{T} \to X'$ of type λ a *vertical* torsor. Let $\mathcal{T}_U$ be the restriction of $\mathcal{T}$ to $U \subset X'$.

LEMMA 2.2. *Vertical X'-torsors exist. For each such torsor $\mathcal{T}$ there exist a principal homogeneous space E of the torus T, and $\varrho_i \in F_i^*$, $i = 1, \ldots, d$, such that $\mathcal{T}_U = E \times V$, where V is defined in* (2.2).

Proof. Recall that $\mathrm{Pic}(\bar{U}) = 0$, and take the two upper rows of our diagram (2.3) as the diagram of [**14**, (4.21)]. An immediate application of the local description of torsors [**14**, Theorem 4.3.1] shows that $\mathcal{T}_U$ is given by (2.1) together with (2.2). Let E be the principal homogeneous space of T with the equation

$$\prod_{i=1}^{d} N_{F_i/F}(\varrho_i)^{e_i} = cN(\mathbf{z}).$$

Multiplying $\mathbf{z}$ by $\prod_{i=1}^{d} N_{K\otimes_F F_i/K}(\mathbf{y}_i)^{e_i}$, we get an isomorphism $\mathcal{T}_U = E \times V$. □

We resume the proof of Theorem 2.1.

Recall that $\mathrm{Br}_0(X)$ is the image of the natural map $\mathrm{Br}(F) \to \mathrm{Br}(X)$, and $\mathrm{Br}_1(X)$ is the kernel of the natural map $\mathrm{Br}(X) \to \mathrm{Br}(\bar{X})$.

It suffices to consider one smooth and proper model of X over F. We can take it to be a smooth and projective variety X^c that contains X' as a dense open subset. (Since F has characteristic zero, such a variety exists by Hironaka's theorem.)

Let $\mathbf{A}$ be the ring of adèles of the number field F. Let $(M_\nu) \in X^c(\mathbf{A})^{\mathrm{Br}_1(X^c)}$ be a collection of local points $M_\nu \in X^c(F_\nu)$, one for each place ν of F, orthogonal to $\mathrm{Br}_1(X^c)$. By a theorem of Grothendieck, $\mathrm{Br}_1(X^c)$ is naturally a subgroup of $\mathrm{Br}_1(X')$. We have seen that $\bar{F}[X']^* = \bar{F}^*$ and $\mathrm{Pic}(\bar{X}')$ is torsion-free. It is well known that this implies that $\mathrm{Br}_1(X')/\mathrm{Br}_0(X')$ is a subgroup of $H^1(F, \mathrm{Pic}(\bar{X}'))$, and hence is finite. Thus, we can use [**8**, Proposition 1.1] (a consequence of Harari's 'formal lemma') which says that the natural injective map of topological spaces

$$X'(\mathbf{A})^{\mathrm{Br}_1(X')} \longrightarrow X^c(\mathbf{A})^{\mathrm{Br}_1(X^c)} = \left(\prod_{\nu} X^c(F_\nu)\right)^{\mathrm{Br}_1(X^c)}$$

has a dense image. Thus, we can assume $(M_\nu) \in X'(\mathbf{A})^{\mathrm{Br}_1(X')}$. Furthermore, using the finiteness of $\mathrm{Br}_1(X')/\mathrm{Br}_0(X')$ and the fact that the value of an element of $\mathrm{Br}_1(X')$ at a point of $X'(F_\nu)$ is locally constant in the topology of F_ν, we can assume without loss of generality that $M_\nu \in U(F_\nu)$ for all ν.

The main theorem of the descent theory of Colliot-Thélène and Sansuc states that every point in $X'(\mathbf{A})^{\mathrm{Br}_1(X')}$ is in the image of the map $\mathcal{T}_0(\mathbf{A}) \to X'(\mathbf{A})$, where $\mathcal{T}_0 \to X'$ is a universal torsor (see [**7**, Section 3; **14**, Theorem 6.1.2(a)]). Thus, we can find a point $(N_\nu) \in \mathcal{T}_0(\mathbf{A})$ such that the image of N_ν in X' is M_ν for all ν.

The structure group of $\mathcal{T}_0 \to X'$ is the Néron–Severi torus T_0 defined by the property $\hat{T}_0 = \mathrm{Pic}(\bar{X}')$. The right-hand column of (2.3) gives rise to the dual exact sequence of tori

$$1 \longrightarrow T_1 \longrightarrow T_0 \longrightarrow T_2 \longrightarrow 1,$$

which is the definition of T_1 and T_2. The quotient $\mathcal{T} = \mathcal{T}_0/T_1$ is an X'-torsor with the structure group T_2. The type of $\mathcal{T} \to X'$ is the natural map

$$\hat{T}_2 = \hat{T} \otimes \left(\bigoplus\nolimits_{i=1}^{d} \mathbb{Z}[F_i/F]\right) \longrightarrow \mathrm{Pic}(\bar{X}'),$$

so $\mathcal{T}$ is a vertical torsor. Since $\mathcal{T}_0$ is a universal torsor, we have $\bar{F}[\mathcal{T}_0]^* = \bar{F}^*$ and $\mathrm{Pic}(\bar{\mathcal{T}}_0) = 0$, hence $\mathrm{Br}_1(\mathcal{T}_0) = \mathrm{Br}_0(\mathcal{T}_0)$.

Let $(P_\nu) \in \mathcal{T}(\mathbf{A})$ be the image of (N_ν). By the functoriality of the Brauer–Manin pairing, we see that $(P_\nu) \in \mathcal{T}(\mathbf{A})^{\mathrm{Br}_1(\mathcal{T})}$. By Lemma 2.2, the restriction of $\mathcal{T}$ to U is isomorphic to $E \times V$, where V is given by (2.2). Let S be a finite set of places of F containing all the places where we need to approximate. By assumption we can find an F-point in V close to the image of P_ν in $V(F_\nu)$ for $\nu \in S$.

The argument in [**5**, p. 85] shows that there is an F-point in E close to the image of P_ν in $E(F_\nu)$ for $\nu \in S$. We reproduce this argument for the convenience of the reader. Let E^c be a smooth compactification of E. The projection $\mathcal{T}_U = E \times V \to E$ extends to a rational map f from the smooth variety $\mathcal{T}$ to the projective variety E^c. By a standard result of algebraic geometry there is an open subset $W \subset \mathcal{T}$ with complement $\mathcal{T} \setminus W$ of codimension at least 2 in $\mathcal{T}$ such that f is a morphism $W \to E^c$. By Grothendieck's purity theorem the natural restriction maps $\mathrm{Br}(\mathcal{T}) \to \mathrm{Br}(W)$ and $\mathrm{Br}(\bar{\mathcal{T}}) \to \mathrm{Br}(\bar{W})$ are isomorphisms. Hence, $\mathrm{Br}_1(\mathcal{T}) \to \mathrm{Br}_1(W)$ is also an isomorphism. Thus, $f^*\mathrm{Br}_1(E^c) \subset \mathrm{Br}_1(W)$ is contained in $\mathrm{Br}_1(\mathcal{T})$, and so the image of (P_ν) in E^c belongs to $E^c(\mathbf{A})^{\mathrm{Br}_1(E^c)}$. By Sansuc's theorem, $E(F)$ is a dense subset of $E^c(\mathbf{A})^{\mathrm{Br}_1(E^c)}$.

We conclude that there is a point in $\mathcal{T}(F)$ which is arbitrarily close to P_ν for $\nu \in S$. The image of this point in $X^c(F)$ approximates (M_ν). This finishes the proof of Theorem 2.1. $\square$

Proof of Theorem 1.1. Consider the particular case of (2.1) where for each $i = 1, \ldots, d$, we have $F_i = F$ and $P_i(\mathbf{t})$ is a linear polynomial $L_i(\mathbf{t})$. Then the natural projection $\mathbb{A}^r_F \times \prod_{i=1}^d R_{K/F}(\mathbb{A}^1_K) \to \prod_{i=1}^d R_{K/F}(\mathbb{A}^1_K)$ defines an isomorphism $V = V_0 \times \mathbb{A}^g_F$, where g is the dimension of the kernel of the linear map $F^r \to F^d$ given by the homogeneous parts of $L_1, \ldots, L_d$, and V_0 is defined as follows. For some $\varrho_1, \ldots, \varrho_d \in F^*$ the variety V_0 is given by the equations in K-variables $\mathbf{y}_1, \ldots, \mathbf{y}_d$:

$$\sum_{i=1}^{d} \lambda_i \varrho_i N(\mathbf{y}_i) + \lambda_{d+1} = 0, \quad N(\mathbf{y}_i) \neq 0,\ i = 1, \ldots, d, \tag{2.4}$$

for all vectors $(\lambda_1, \ldots, \lambda_{d+1}) \in F^{d+1}$ satisfying

$$\sum_{i=1}^{d} \lambda_i L_i(\mathbf{t}) + \lambda_{d+1} = 0 \tag{2.5}$$

identically in $\mathbf{t} = (t_1, \ldots, t_r)$.

In the situation of Theorem 1.1 we have $d = 2r$, and the linear polynomials $L_1, \ldots, L_{2r}, L_{2r+1} = 1$ satisfy Condition I. In particular, their linear span has dimension $r+1$ and we have $V = V_0$. Thus, the vectors $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{2r+1})$ satisfying equation (2.5) form an r-dimensional subspace $\Lambda \subset F^{2r+1}$. Let $\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(r)}$ be a basis of Λ. Set $a_{ij} = \lambda_j^{(i)} \rho_j$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant 2r+1$, where we set $\rho_{2r+1} = 1$. Thus, V is a dense open subset of the affine variety given by the system of equations (1.2).

An easy linear algebra argument shows that if $L_1, \ldots, L_{2r+1}$ satisfy Condition I, then the matrix (a_{ij}) satisfies Condition II. Indeed, take any j_0 from 1 to $2r+1$ and write $\{L_1, \ldots, L_{2r+1}\}$ as the union of subsets of linearly independent functions $\mathcal{A} = \{L_{j_0}, L_{j_1}, \ldots, L_{j_r}\}$ and $\mathcal{B} = \{L_{j_0}, L_{j_{r+1}}, \ldots, L_{j_{2r}}\}$. Since the elements of $\mathcal{B}$ are linearly independent, there exists a unique vector in Λ whose coordinates with subscripts $j_1, \ldots, j_r$ are arbitrary elements of F. It follows that the matrix $(a_{ij_l})_{1\leqslant i\leqslant r, 1\leqslant l\leqslant r}$ has full rank. The elements of $\mathcal{A}$ are also linearly independent, so the matrix $(a_{ij_{r+l}})_{1\leqslant i\leqslant r, 1\leqslant l\leqslant r}$ has full rank too. Hence, the matrix (a_{ij}) satisfies Condition II, and now the result follows from Theorems 2.1 and 1.3. $\square$

Proof of Corollary 1.2. The arguments in the proof of [**5**, Corollary 2.7] apply *verbatim* in our situation, establishing $\mathrm{Br}(X'_c) = \mathrm{Br}_0(X'_c)$ in case (i). The proof of the statement in the example [**5**, pp. 77–78] gives the same conclusion in case (ii). $\square$

3. *Circle method*

3.1. *Preliminaries*

We write $\mathrm{Tr} = \mathrm{Tr}_{F/\mathbb{Q}}$ for the trace from F to $\mathbb{Q}$. Let

$$\mathcal{C} = \{x \in F : \mathrm{Tr}(xy) \in \mathbb{Z} \text{ for all } y \in \mathcal{O}_F\}$$

be the inverse different. We extend Tr to a linear form $V = F \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}$. Choose a $\mathbb{Z}$-basis

$\zeta_1,\dots,\zeta_m$ of $\mathcal{O}_F$. Let $\rho_1,\dots,\rho_m$ be the dual basis of $\mathcal{C}$ defined by the property that the matrix with entries $\mathrm{Tr}(\zeta_i\rho_j)$ is the identity matrix. Then any $x\in F$ can be written as

$$x=\sum_{i=1}^m \mathrm{Tr}(x\rho_i)\zeta_i. \tag{3.1}$$

We set

$$f_{ik}(\mathbf{x})=\mathrm{Tr}(\rho_k f_i(\mathbf{x})).$$

Let $(c_{ik})_{1\leqslant i\leqslant r,1\leqslant k\leqslant q}$ be an $(r\times q)$-matrix with entries in the $\mathbb{R}$-algebra V. We shall say that the rank of this matrix is r if it defines a surjective linear map $V^q\to V^r$. Recall that $\omega_1,\dots,\omega_m$ is a $\mathbb{Z}$-basis of the ideal $\mathfrak{n}\subset\mathcal{O}_F$. We attach to (c_{ik}) the $(rm\times qm)$-matrix with real entries $(\mathrm{Tr}(c_{ik}\rho_j\omega_l))_{(i,j),(k,l)}$, where we use the lexicographic ordering of the pairs (i,j), $1\leqslant i\leqslant r$, $1\leqslant j\leqslant m$, and the pairs (k,l), $1\leqslant k\leqslant q$, $1\leqslant l\leqslant m$.

The following observation will be often used in this paper.

Lemma 3.1. *The matrix (c_{ik}) with entries in V has rank r if and only if the matrix $(\mathrm{Tr}(c_{ik}\rho_j\omega_l))_{(i,j),(k,l)}$ with entries in $\mathbb{R}$ has rank mr.*

Proof. Take any $d_1,\dots,d_r\in V$ and write $d_i=d_{i1}\omega_1+\dots+d_{im}\omega_m$. If there exist $\mu_1,\dots,\mu_q\in V$ such that $c_{i1}\mu_1+\dots+c_{iq}\mu_q=d_i$ for $1\leqslant i\leqslant r$, we write $\mu_k=\mu_{k1}\omega_1+\dots+\mu_{km}\omega_m$ and then obtain

$$\sum_{k=1}^q\sum_{l=1}^m \mu_{kl}\mathrm{Tr}(c_{ik}\rho_j\omega_l)=\sum_{p=1}^m d_{ip}\mathrm{Tr}(\omega_p\rho_j) \tag{3.2}$$

for $1\leqslant i\leqslant r$ and $1\leqslant j\leqslant m$. The $(m\times m)$-matrix $\mathrm{Tr}(\omega_p\rho_j)$ is invertible by the non-degeneracy of the bilinear form $\mathrm{Tr}(xy): F\times F\to\mathbb{Q}$. Thus, the rank of $(\mathrm{Tr}(c_{ik}\rho_j\omega_l))$ is mr. Conversely, from (3.2) using (3.1) we deduce

$$\sum_{k=1}^q c_{ik}\sum_{l=1}^m \mu_{kl}\omega_l=d_i.$$

This finishes the proof of the lemma. □

Let Z be the affine variety over F defined by the system of equations (1.3). The Weil restriction $R_{F/\mathbb{Q}}(Z)$ is the variety over $\mathbb{Q}$ defined by the system of equations $f_{ij}(\mathbf{x})=0$ for $1\leqslant i\leqslant r$ and $1\leqslant j\leqslant m$. For any $\mathbb{Q}$-algebra S there is a natural bijection of points $R_{F/\mathbb{Q}}(Z)(S)=Z(S\otimes_{\mathbb{Q}}F)$. A useful consequence of Lemma 3.1 is the observation that a V-point of Z is singular if and only if the corresponding $\mathbb{R}$-point on the variety $R_{F/\mathbb{Q}}(Z)$ is singular. Indeed, by the chain rule we have

$$\frac{\partial f_{ij}}{\partial x_{kl}}(\mathbf{x})=\mathrm{Tr}\left(\rho_j\frac{\partial}{\partial x_{kl}}f_i(\mathbf{x})\right)=\mathrm{Tr}\left(\rho_j\omega_l\frac{\partial f_i}{\partial x_k}(\mathbf{x})\right), \tag{3.3}$$

and the statement follows from Lemma 3.1 with $q=ns$ and $c_{ik}=\partial f_i/\partial x_k$.

3.2. *Exponential sums*

For $1\leqslant j\leqslant s$, we define $\mathcal{B}_j$ to be the set

$$\mathcal{B}_j=\{\mathbf{x}_j\in V^n:|u_{ik}-x_{ik}|\leqslant\kappa \text{ for } n(j-1)+1\leqslant i\leqslant nj \text{ and } 1\leqslant k\leqslant m\}.$$

Then we have

$$\mathcal{B}=\mathcal{B}_1\times\dots\times\mathcal{B}_s.$$

Now, we introduce the exponential sums

$$S_j(\boldsymbol{\beta}) = \sum_{\mathbf{x}_j \in (P\mathcal{B}_j)\cap \mathfrak{n}^n} e(\mathrm{Tr}(\boldsymbol{\beta} N(\mathbf{x}_j + \mathbf{d}_j))), \quad 1 \leqslant j \leqslant s,$$

where we write $\boldsymbol{\beta} = \beta_1\rho_1 + \cdots + \beta_m\rho_m$, and identify $\boldsymbol{\beta}$ with the vector $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$. Consider the linear forms

$$\boldsymbol{\lambda}_j = \sum_{i=1}^{r} b_{ij}\boldsymbol{\alpha}_i, \quad 1 \leqslant j \leqslant s.$$

For $\boldsymbol{\lambda}_j$ and $\boldsymbol{\alpha}_i$ we use the same conventions as for $\boldsymbol{\beta}$. By orthogonality, we have

$$N(\mathcal{B}, P) = \int_{[0,1]^{mr}} S_1(\boldsymbol{\lambda}_1)\ldots S_s(\boldsymbol{\lambda}_s)\,\mathrm{d}\boldsymbol{\alpha}, \tag{3.4}$$

where we write $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r)$ and $\mathrm{d}\boldsymbol{\alpha} = \mathrm{d}\alpha_{11}\ldots \mathrm{d}\alpha_{rm}$.

Next, we turn towards a form of Weyl's inequality for the exponential sums $S_j(\boldsymbol{\beta})$, which we deduce from Birch's work [1].

LEMMA 3.2. *Let Δ and θ be positive integers satisfying $2^{n-1}\Delta < \theta$. Let j be an integer such that $1 \leqslant j \leqslant s$. Then either of the following two conditions holds.*

(i) *We have $|S_j(\boldsymbol{\beta})| \ll P^{mn-\Delta}$, or*

(ii) *There is an integer q such that $1 \leqslant q \leqslant P^{m(n-1)\theta}$, and integers $a_1, \ldots, a_m$ such that $\gcd(a_1, \ldots, a_m, q) = 1$ and*

$$2|q\beta_i - a_i| \leqslant P^{-n+m(n-1)\theta}, \quad 1 \leqslant i \leqslant m.$$

Proof. Since j is fixed, we drop it from the notation. Recall that $\mathbf{z} = (z_1, \ldots, z_n)$, where $z_k \in V$, so we can write $z_k = z_{k1}\omega_1 + \cdots + z_{km}\omega_m$. Let $Y \subset \mathbb{C}^{mn}$ be the Zariski closed subset given by

$$\mathrm{rk}\left(\frac{\partial \mathrm{Tr}(\rho_i N(\mathbf{z}))}{\partial z_{kl}}\right)_{i,(k,l)} < m, \tag{3.5}$$

where $1 \leqslant i \leqslant m$, and the pairs (k, l), where $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant m$, are ordered lexicographically as in the previous section. By Lemma 3.1 and (3.3) this is equivalent to

$$\frac{\partial N(\mathbf{z})}{\partial z_k} = 0, \quad 1 \leqslant k \leqslant n.$$

However, by Euler's formula for homogeneous polynomials we have

$$nN(\mathbf{z}) = \sum_{k=1}^{n} z_k \frac{\partial N(\mathbf{z})}{\partial z_k}.$$

Since $N(\mathbf{z})$ is not identically zero, we see that $\dim(Y) \leqslant mn - 1$.

We have

$$S_1(\boldsymbol{\beta}) = \sum_{\mathbf{z}\in(P\mathcal{B}_1)\cap\mathfrak{n}^n} e\left(\sum_{i=1}^{m} \beta_i \mathrm{Tr}(\rho_i N(\mathbf{z} + \mathbf{d}))\right).$$

Applying [1, Lemmas 3.2 and 3.3] to the system of equations $\mathrm{Tr}(\rho_i N(\mathbf{z} + \mathbf{d})) = 0$ in mn variables z_{kl} we obtain that either one of the alternatives of our lemma holds, or we have

$$(n-2)mn + \dim(Y) \geqslant (n-1)mn - 2^{n-1}\Delta/\theta - \varepsilon$$

for some $\varepsilon > 0$. This is impossible since $2^{n-1}\Delta < \theta$. $\square$

In the following, we always choose $\Delta > 0$ small enough so that the condition of Lemma 3.2 is satisfied.

3.3. *The circle method*

We start this section with choosing appropriate major and minor arcs. For a positive real number θ, integers q and $a_{11}, \ldots, a_{rm}$ define the major arc $\mathfrak{M}_{\mathbf{a},q}(\theta)$ to be the set of $\boldsymbol{\alpha} \in [0,1]^{mr}$ such that

$$|q\alpha_{ij} - a_{ij}| \leqslant qP^{-n+mr(n-1)\theta}, \quad 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant m.$$

Then $\mathfrak{M}(\theta)$ is the union of the major arcs

$$\mathfrak{M}(\theta) = \bigcup_{1 \leqslant q \leqslant P^{mr(n-1)\theta}} \bigcup_{\mathbf{a}} \mathfrak{M}_{\mathbf{a},q}(\theta),$$

where the second union is over all vectors $\mathbf{a}$ satisfying $\gcd(a_{11}, \ldots, a_{rm}, q) = 1$ and $0 \leqslant a_{ij} < q$. We choose θ sufficiently small such that all the major arcs in the union of $\mathfrak{M}(\theta)$ are disjoint, which is possible by [**1**, Lemma 4.1]. As usual, we define the minor arcs $\mathfrak{m}(\theta)$ as the complement $\mathfrak{m}(\theta) = [0,1]^{mr} \setminus \mathfrak{M}(\theta)$ to the major arcs.

Let us now treat the contribution of the minor arcs to the integral (3.4). For this we need the following lemma, which appeared in a similar way in the work of Birch, Davenport and Lewis [**3**].

LEMMA 3.3. *For any* $\varepsilon > 0$, *we have*

$$\int_{[0,1]^m} |S_j(\boldsymbol{\beta})|^2 \,\mathrm{d}\boldsymbol{\beta} \ll P^{mn+\varepsilon}, \quad 1 \leqslant j \leqslant s.$$

Proof. By orthogonality, we see that this integral is equal to the number of solutions $\mathbf{z}_1, \mathbf{z}_2 \in (P\mathcal{B}_j) \cap \mathfrak{n}^n$ of the equation

$$N(\mathbf{z}_1 + \mathbf{d}) = N(\mathbf{z}_2 + \mathbf{d}).$$

Write $N_{K/\mathbb{Q}}$ for the norm from K to $\mathbb{Q}$. For $z \in K$ we denote by $z^{(l)}$, where $1 \leqslant l \leqslant mn$, the conjugates of z. By the transitivity of norm, the number above is bounded by the number of solutions $z_1, z_2 \in \mathcal{O}_K$ of

$$N_{K/\mathbb{Q}}(z_1) = N_{K/\mathbb{Q}}(z_2), \quad \max_{1 \leqslant l \leqslant mn} |z_1^{(l)}| \leqslant C_1 P, \quad \max_{1 \leqslant l \leqslant mn} |z_2^{(l)}| \leqslant C_1 P,$$

for some constant C_1. For an integer u let $r(u)$ be the number of $z \in \mathcal{O}_K$ such that

$$N_{K/\mathbb{Q}}(z) = u, \quad \max_{1 \leqslant l \leqslant mn} |z^{(l)}| \leqslant C_1 P. \tag{3.6}$$

Now the integral in the lemma is bounded by

$$\sum_{|u| \leqslant C_2 P^{mn}} r(u)^2$$

for some large enough C_2. To prove the lemma it is enough to show that for u in this sum we have $r(u) \ll P^{\varepsilon}$. One can find this result in [**11**, Lemma 4.3]. For convenience, we repeat a proof here. Group together the solutions z of (3.6) that generate the same principal ideal (z). Let us denote the norm of an integral ideal $\mathfrak{a}$ by $\mathrm{Nm}(\mathfrak{a})$. By the unique factorization of prime ideals the number of integral ideals $\mathfrak{a}$ of norm $\mathrm{Nm}(\mathfrak{a}) = u > 0$ is bounded by some constant times u^{ε}. Now, we fix a solution z of (3.6), if it exists, and consider the number $A(u, z)$ of solutions $\tilde{z}$ of (3.6) such that z and $\tilde{z}$ differ by a unit. Let $\iota_1, \ldots, \iota_T$ be fundamental units of K.

For some integers $v_1, \ldots, v_T$, we have

$$\tilde{z} = \zeta \iota_1^{v_1} \ldots \iota_T^{v_T} z, \tag{3.7}$$

where ζ is a root of unity. Furthermore, we have

$$\sum_{l=1}^{mn} \log|\tilde{z}^{(l)}| = \log|u| \ll \log P,$$

and $\log|\tilde{z}^{(l)}| \leqslant \log(C_1 P)$ for all l. Therefore, there is a constant C_3 such that for large values of P we have the bound

$$|\log|\tilde{z}^{(l)}|| \leqslant C_3 \log P, \quad 1 \leqslant l \leqslant mn.$$

The same estimate is true for $|\log|z^{(l)}||$. Using (3.7) we see that

$$\left|\sum_{i=1}^{T} v_i \log|\iota_i^{(l)}|\right| \ll \log P, \quad 1 \leqslant l \leqslant mn.$$

By Dirichlet's unit theorem the rank of the matrix $(\iota_i^{(l)})$, where $1 \leqslant i \leqslant T$ and $1 \leqslant l \leqslant mn$, is T, and hence $A(u,z) \ll (\log P)^T$. This implies $r(u) \ll P^{\varepsilon}$. □

LEMMA 3.4. *There exists $\eta > 0$ such that we have*

$$\int_{\mathfrak{m}(\theta)} |S_1(\boldsymbol{\lambda}_1) \ldots S_s(\boldsymbol{\lambda}_s)| \,\mathrm{d}\boldsymbol{\alpha} = O(P^{mn(r+1)-\eta}).$$

Proof. In the first part of the proof we show that if $\boldsymbol{\alpha}$ is of minor arc type, then so is one of the $\boldsymbol{\lambda}_i$ for some possibly different parameter θ'. For this let $\mathfrak{m}_j(\theta)$ be the set of $\boldsymbol{\alpha} \in \mathfrak{m}(\theta)$ such that $|S_j(\boldsymbol{\lambda}_j)| \ll P^{mn-\Delta}$. Assume $\boldsymbol{\alpha} \notin \bigcup_{j=1}^{s} \mathfrak{m}_j(\theta)$ and choose $\theta' < \theta$ such that we still have $2^{n-1}\Delta < \theta'$. Then we apply Lemma 3.2 and find integers $1 \leqslant q_j \leqslant P^{m(n-1)\theta'}$ and a_{jl} for $1 \leqslant j \leqslant s$ and $1 \leqslant l \leqslant m$ with the property

$$2|q_j \lambda_{jl} - a_{jl}| \leqslant P^{-n+m(n-1)\theta'}$$

for all j and l. For simplicity of notation, we assume next that the matrix $(b_{ij})_{1 \leqslant i,j \leqslant r}$ has full rank, which is possible after renaming indices since the matrix B has full rank by assumption. Thus, there are $c_{ij} \in k$ such that

$$\boldsymbol{\alpha}_i = \sum_{j=1}^{r} c_{ij} \boldsymbol{\lambda}_j$$

for $1 \leqslant i \leqslant r$. Next, we define $\tilde{\boldsymbol{\lambda}}_j = q_j^{-1}(a_{j1}\rho_1 + \cdots + a_{jm}\rho_m)$ and

$$\tilde{a}_{ik} = \mathrm{Tr}\left(\zeta_k \sum_{j=1}^{r} c_{ij} \tilde{\boldsymbol{\lambda}}_j\right), \quad 1 \leqslant i \leqslant r,\ 1 \leqslant k \leqslant m.$$

By construction there is an integer $q \ll P^{mr(n-1)\theta'}$ such that $q\tilde{a}_{ik} \in \mathbb{Z}$ for all i and k. We can estimate

$$|\alpha_{ik} - \tilde{a}_{ik}| = |\mathrm{Tr}(\zeta_k \sum c_{ij}(\boldsymbol{\lambda}_j - \tilde{\boldsymbol{\lambda}}_j))| \ll \max_{j,l}(|q_j^{-1} a_{jl} - \lambda_{jl}|) \ll P^{-n+m(n-1)\theta'}.$$

It follows that $\boldsymbol{\alpha} \in \mathfrak{M}(\theta)$, and hence $\mathfrak{m}(\theta) = \cup_j \mathfrak{m}_j(\theta)$.

We estimate the contribution from the sets $\mathfrak{m}_j(\theta)$ to the integral in the lemma separately. For simplicity of notation, we assume $|S_s(\boldsymbol{\lambda}_s)| \ll P^{mn-\Delta}$ and that both the first r columns and the next r columns of the matrix B form submatrices of full rank, which we can do without loss of generality by Condition II. Using the Cauchy–Schwarz inequality we estimate

$$\int_{\mathfrak{m}_s(\theta)} |S_1(\boldsymbol{\lambda}_1)\dots S_s(\boldsymbol{\lambda}_s)|\,\mathrm{d}\boldsymbol{\alpha} \ll P^{mn-\Delta}\int_{\mathfrak{m}_s(\theta)} |S_1(\boldsymbol{\lambda}_1)\dots S_{2r}(\boldsymbol{\lambda}_{2r})|\,\mathrm{d}\boldsymbol{\alpha} \\ \ll P^{mn-\Delta} I_1^{1/2} I_2^{1/2},$$

where

$$I_1 = \int_{[0,1]^{mr}} |S_1(\boldsymbol{\lambda}_1)\dots S_r(\boldsymbol{\lambda}_r)|^2\,\mathrm{d}\boldsymbol{\alpha},$$

and I_2 is of analogous form. We now perform a change of variables in the integral I_1. For this note that

$$\lambda_{jl} = \mathrm{Tr}\left(\zeta_l \sum_{i=1}^r b_{ij}\boldsymbol{\alpha}_i\right) = \sum_{i=1}^r\sum_{k=1}^m \alpha_{ik}\mathrm{Tr}(b_{ij}\rho_k\zeta_l).$$

Order the pairs (i,k) and (j,l) lexicographically, and let M be the $(mr\times mr)$-matrix with entries $\mathrm{Tr}(b_{ij}\rho_k\zeta_l)$. Then M has full rank and $\mathrm{Tr}(b_{ij}\rho_k\zeta_l)\in\mathbb{Z}$ for all i,j,k and l. Write $\mathrm{d}\boldsymbol{\lambda}$ for the Lebesgue measure $\mathrm{d}\lambda_{11}\dots\mathrm{d}\lambda_{rm}$. By 1-periodicity of our exponential sums we have

$$I_1 = \frac{1}{\det M}\int_{M[0,1]^{mr}} |S_1(\boldsymbol{\lambda}_1)\dots S_r(\boldsymbol{\lambda}_r)|^2\,\mathrm{d}\boldsymbol{\lambda} \ll \int_{[0,1]^{mr}} |S_1(\boldsymbol{\lambda}_1)\dots S_r(\boldsymbol{\lambda}_r)|^2\,\mathrm{d}\boldsymbol{\lambda}.$$

The integral I_2 can be treated in the very same way as I_1. Our lemma now follows from Lemma 3.3. □

We analyse the major arcs following Birch's approach in [1]. Let us write

$$S(\boldsymbol{\alpha}) = S_1(\boldsymbol{\lambda}_1)\dots S_s(\boldsymbol{\lambda}_s) = \sum_{\mathbf{x}\in(P\mathcal{B})\cap\mathfrak{n}^{ns}} e(\alpha_{11}f_{11}(\mathbf{x}+\mathbf{d})+\dots+\alpha_{rm}f_{rm}(\mathbf{x}+\mathbf{d})),$$

where the exponential sums $S_i(\boldsymbol{\lambda}_i)$ were defined at the beginning of Section 3.2. We define the exponential sums

$$S_{\mathbf{a},q} = \sum_{\mathbf{x}\in(\mathbb{Z}/q)^{mns}} e\left(\sum_{i=1}^r\sum_{j=1}^m a_{ij}f_{ij}(\mathbf{x}+\mathbf{d})/q\right).$$

For $\boldsymbol{\gamma}\in\mathbb{R}^{mr}$, we define

$$I(\boldsymbol{\gamma}) = \int_{\mathbf{t}\in\mathcal{B}} e\left(\sum_{i=1}^r\sum_{j=1}^m \gamma_{ij}f_{ij}(\mathbf{t})\right)\mathrm{d}\mathbf{t},\quad J(P) = \int_{|\boldsymbol{\gamma}|\leqslant P} I(\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\gamma}.$$

In this last integral, we use the notation $|\boldsymbol{\gamma}| = \max_{ij}|\gamma_{ij}|$. For the vector $\mathbf{t}$, we use the same conventions as were adopted, in the introduction, for the vector $\mathbf{x}$.

LEMMA 3.5. *For a small enough $\theta>0$ there exists $\eta>0$ such that we have*

$$\int_{\mathfrak{M}(\theta)} S(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha} = \mathfrak{S}(P)J(P^{mr(n-1)\theta})P^{mn(r+1)} + O(P^{mn(r+1)-\eta}),$$

where

$$\mathfrak{S}(P) = \sum_{q\leqslant P^{mr(n-1)\theta}} q^{-mns}\sum_{\mathbf{a}} S_{\mathbf{a},q},$$

and $\mathbf{a}$ ranges over all vectors with $0\leqslant a_{ij}<q$ and $\gcd(a_{11},\dots,a_{rm},q)=1$.

Proof. This is a combination of [**1**, Lemmas 5.1 and 5.5] together with the argument of [**12**, Section 9] which ensures that the error introduced by replacing $f_{ij}(\mathbf{x}+\mathbf{d})$ by $f_{ij}(\mathbf{x})$ is small enough. □

3.4. *Singular integral*

By assumption, the system of equations $f_i(\mathbf{t}) = 0$ has no singularities in the box $\mathcal{B}$. By Lemma 3.1 and the remark following it, the corresponding system $f_{ij} = 0$ is also non-singular on $\mathcal{B}$. Splitting the box into smaller ones if necessary we may assume that the same $(mr \times mr)$-minor of the Jacobian matrix of the f_{ij} has full rank on the whole box. For simplicity of notation, we assume furthermore that it is the minor C given by $\partial f_{ij}/\partial t_{nk,l}$ for $1 \leqslant i, k \leqslant r$ and $1 \leqslant j, l \leqslant m$. Here again, we order the pairs (i, j) and (k, l) lexicographically. After splitting the box $\mathcal{B}$ into even smaller boxes, so that on each of them the inverse function theorem becomes applicable, we can perform a coordinate transformation in the integral $I(\boldsymbol{\gamma})$ introduced in the last section, as follows. Set $u_{ij} = f_{ij}(\mathbf{t})$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant m$, and write $\mathbf{u}$ for the vector $(u_{11}, \dots, u_{rm})$. After renaming the indices of the vector $\mathbf{t}$ we can write $\mathbf{t} = (\mathbf{t}', \mathbf{t}'')$ with $\mathbf{t}' \in \mathbb{R}^{m(ns-r)}$ and $\mathbf{t}'' \in \mathbb{R}^{mr}$, so that $\mathbf{t}''$ consists of all the coordinates of $\mathbf{t}$ of the form $t_{nk,l}$ for $1 \leqslant k \leqslant r$ and $1 \leqslant l \leqslant m$. Let $V(\mathbf{u})$ be the set of all $\mathbf{t}' \in \mathbb{R}^{m(ns-r)}$ such that there is some $\mathbf{t}'' \in \mathbb{R}^{mr}$ with the corresponding $\mathbf{t} = (\mathbf{t}', \mathbf{t}'') \in \mathcal{B}$ and $u_{ij} = f_{ij}(\mathbf{t}'', \mathbf{t}')$ for all $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant m$. Define

$$\psi(\mathbf{u}) = \int_{\mathbf{t}' \in V(\mathbf{u})} |\det C(\mathbf{t})|^{-1} \, \mathrm{d}\mathbf{t}',$$

where $\mathbf{t}$ is implicitly given by $\mathbf{u}$ and $\mathbf{t}'$. Then we obtain

$$I(\boldsymbol{\gamma}) = \int_{\mathbb{R}^{rm}} \psi(\mathbf{u}) e(\boldsymbol{\gamma} \cdot \mathbf{u}) \, \mathrm{d}\mathbf{u},$$

where we write $\boldsymbol{\gamma} \cdot \mathbf{u}$ for the scalar product $\sum_{i=1}^{r} \sum_{j=1}^{m} \gamma_{ij} u_{ij}$.

Our next goal is to show, using the Fourier inversion theorem, that $J(P)$ absolutely converges to $\psi(0)$ when $P \to \infty$. First, we need a lemma.

Lemma 3.6. *Let $\mathcal{A}$ be a rectangular box in $\mathbb{R}^D$. For $1 \leqslant i \leqslant m$ let $F_i(\mathbf{z}) \in \mathbb{R}[\mathbf{z}]$ be polynomials with real coefficients in $\mathbf{z} = (z_1, \dots, z_D)$. Let l be an integer such that $0 \leqslant l \leqslant D - m$. Assume that all $(m \times m)$-minors of the matrix*

$$\left(\frac{\partial F_i}{\partial z_j}\right)_{1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant m+l}$$

have full rank on some open subset $\mathcal{U} \supset \mathcal{A}$. Let $G : \mathcal{U} \to \mathbb{R}$ be a smooth function. Then for any $\beta_1, \dots, \beta_m \in \mathbb{R}$ one has

$$\left| \int_{\mathcal{A}} G(\mathbf{z}) e(\beta_1 F_1(\mathbf{z}) + \dots + \beta_m F_m(\mathbf{z})) \, \mathrm{d}z_1 \dots \mathrm{d}z_D \right| \ll \left(\max_i |\beta_i| \right)^{-l-1},$$

where the implied constant depends only on $\mathcal{A}$ and the functions F_i and G.

Proof. Write $\boldsymbol{\beta}\mathbf{F}(\mathbf{z}) = \beta_1 F_1(\mathbf{z}) + \dots + \beta_m F_m(\mathbf{z})$. Consider the differential D-form on $\mathcal{U}$:

$$\omega = G(\mathbf{z}) e(\boldsymbol{\beta}\mathbf{F}(\mathbf{z})) \, \mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_D.$$

For any smooth functions $\phi_i(\mathbf{z})$ on $\mathcal{U}$ we define the $(D-1)$-form

$$\mu = \sum_{i=1}^{m} G(\mathbf{z})\phi_i(\mathbf{z})e(\boldsymbol{\beta}\mathbf{F}(\mathbf{z}))\,\mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}z_i} \wedge \cdots \wedge \mathrm{d}z_D,$$

where $\widehat{\mathrm{d}z_i}$ means that $\mathrm{d}z_i$ is omitted. Then $\mathrm{d}\mu = \omega_1 + \omega_2$, where

$$\omega_1 = \sum_{i=1}^{m} (-1)^{i+1} \frac{\partial}{\partial z_i}(G(\mathbf{z})\phi_i(\mathbf{z}))e(\boldsymbol{\beta}\mathbf{F}(\mathbf{z}))\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_D$$

and

$$\omega_2 = \sum_{i=1}^{m} (-1)^{i+1} G(\mathbf{z})\phi_i(\mathbf{z}) \frac{\partial}{\partial z_i} e(\boldsymbol{\beta}\mathbf{F}(\mathbf{z}))\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_D.$$

Without loss of generality, we assume $|\beta_1| = \max_i |\beta_i|$. We claim that the functions $\phi_i(\mathbf{z})$ can be chosen so that $\omega_2 = \beta_1\omega$ for all $\beta_1, \ldots, \beta_m$. For this, we have to solve

$$2\pi\sqrt{-1}\sum_{i=1}^{m}(-1)^{i+1}\phi_i(\mathbf{z})\left(\beta_1\frac{\partial F_1}{\partial z_i}(\mathbf{z}) + \cdots + \beta_m\frac{\partial F_m}{\partial z_i}(\mathbf{z})\right) = \beta_1,$$

where $\mathbf{z} \in \mathcal{U}$. The $(m \times m)$-matrix

$$J = \left(\frac{\partial F_i}{\partial z_j}\right)_{1 \leqslant i,j \leqslant m}$$

is invertible by assumption, hence we can choose the functions $\phi_i(\mathbf{z})$ to be the functions defined by the following equality of row vectors:

$$2\pi\sqrt{-1}((-1)^{i+1}\phi_i(\mathbf{z})) = (1, 0, \ldots, 0)J^{-1}.$$

Now, we have $\mathrm{d}\mu = \omega_1 + \beta_1\omega$ on $\mathcal{U}$, and the Stokes theorem gives

$$\int_{\mathcal{A}} \omega = \frac{1}{\beta_1}\left(\int_{\partial\mathcal{A}} \mu - \int_{\mathcal{A}} \omega_1\right), \tag{3.8}$$

where $\partial\mathcal{A}$ is the boundary of $\mathcal{A}$. The integrals on the right-hand side of (3.8) have the same form as the integral we started with. Thus, we can iterate the above procedure l times for each occurring term. In the end, we estimate each integral by its L^1-bound using the trivial estimate $|e(\boldsymbol{\beta}\mathbf{F}(\mathbf{z}))| \leqslant 1$. This produces the desired inequality. □

Now, we can prove that the integral $J(P)$ is absolutely convergent. Define $\gamma_i = \gamma_{i1}\rho_1 + \cdots + \gamma_{im}\rho_m$. From the definition of $I(\boldsymbol{\gamma})$, we have

$$I(\boldsymbol{\gamma}) = \int_{\mathbf{t}\in\mathcal{B}} e\left(\sum_{i=1}^{r}\mathrm{Tr}(\gamma_i f_i(\mathbf{t}))\right)\mathrm{d}\mathbf{t}.$$

This can be rewritten as

$$\int_{\mathbf{t}\in\mathcal{B}} e\left(\sum_{i=1}^{r}\mathrm{Tr}\left(\gamma_i\sum_{j=1}^{s}b_{ij}N(\mathbf{t}_j)\right)\right)\mathrm{d}\mathbf{t} = \prod_{j=1}^{s}\nu_j(\boldsymbol{\gamma}),$$

where

$$\nu_j(\boldsymbol{\gamma}) = \int_{\mathbf{t}_j\in\mathcal{B}_j} e\left(\mathrm{Tr}\left(\sum_{i=1}^{r}\gamma_i b_{ij}N(\mathbf{t}_j)\right)\right)\mathrm{d}\mathbf{t}_j.$$

In Theorem 1.4, we have assumed

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial x_k}\right) = r$$

on the box $\mathcal{B}$. Without loss of generality, we assume that the matrix $(b_{ij})_{1\leqslant i,j\leqslant r}$ has full rank, and after possibly dissecting the box $\mathcal{B}$ into smaller ones, we assume

$$\frac{\partial N(\mathbf{t}_j)}{\partial t_{n(j-1)+1}} \neq 0$$

on $\mathcal{B}_j$, for all $1 \leqslant j \leqslant r$.

Next, we note that

$$\sum_{i=1}^{r} \gamma_i b_{ij} = \sum_{k=1}^{m} \rho_k \operatorname{Tr}\left(\zeta_k \sum_{i=1}^{r} \gamma_i b_{ij}\right) = \sum_{k=1}^{m} \rho_k \operatorname{Tr}\left(\sum_{i=1}^{r}\sum_{l=1}^{m} \gamma_{il} b_{ij}\rho_l\zeta_k\right).$$

Since the matrix $(b_{ij})_{1\leqslant i,j\leqslant r}$ with entries in F has full rank, we have

$$\det\left(\operatorname{Tr}(b_{ij}\rho_l\zeta_k)_{\substack{(1,1)\leqslant(i,l)\leqslant(r,m)\\(1,1)\leqslant(j,k)\leqslant(r,m)}}\right) \neq 0,$$

which follows from Lemma 3.1. Thus, we have the relation

$$|\boldsymbol{\gamma}| \asymp \max_{(j,k)}\left|\sum_{i=1}^{r}\sum_{l=1}^{m} \gamma_{il}\operatorname{Tr}(b_{ij}\rho_l\zeta_k)\right|,$$

where the implied constants only depend on the numbers b_{ij} and the bases ζ_k and ρ_l. Next, we choose j_0 where the maximum is attained, and assume $j_0 = 1$ for simplicity of notation.

Now we apply Lemma 3.6 to the integral $\nu_1(\boldsymbol{\gamma})$. For this, we set

$$F_k(\mathbf{t}_1) = \operatorname{Tr}(\rho_k N(\mathbf{t}_1))$$

for $1 \leqslant k \leqslant m$. By the above assumptions and Lemma 3.1 we have

$$\det\left(\frac{\partial F_k(\mathbf{t}_1)}{\partial t_{1l}}\right)_{1\leqslant k,l\leqslant m} \neq 0$$

on the box $\mathcal{B}_1$. Lemma 3.6 implies the bound

$$\left|\int_{\mathcal{B}_1} e\left(\operatorname{Tr}\left(\left(\sum_{i=1}^{r}\gamma_i b_{ij}\right) N(\mathbf{t}_1)\right)\right) \mathrm{d}\mathbf{t}_1\right| \ll |\boldsymbol{\gamma}|^{-1}.$$

Since the matrix B satisfies Condition II, we can assume that the matrices $(b_{ij})_{\substack{1\leqslant i\leqslant r\\2\leqslant j\leqslant r+1}}$ and $(b_{ij})_{\substack{1\leqslant i\leqslant r\\r+2\leqslant j\leqslant s}}$ have full rank, possibly after renaming the indices. For a large real number T, we obtain the estimate

$$\begin{aligned}\int_{T<|\boldsymbol{\gamma}|\leqslant 2T} |I(\boldsymbol{\gamma})|\,\mathrm{d}\boldsymbol{\gamma} &\ll \sup_{T<|\boldsymbol{\gamma}|\leqslant 2T} |\nu_1(\boldsymbol{\gamma})| \int_{|\boldsymbol{\gamma}|\leqslant 2T} \prod_{j=2}^{s} |\nu_j(\boldsymbol{\gamma})|\,\mathrm{d}\boldsymbol{\gamma}\\ &\ll T^{-1} J_1(2T)^{1/2} J_2(2T)^{1/2},\end{aligned}$$

where

$$J_1(T) = \int_{|\boldsymbol{\gamma}|\leqslant T} \prod_{j=2}^{r+1} |\nu_j(\boldsymbol{\gamma})|^2\,\mathrm{d}\boldsymbol{\gamma},$$

and similarly for $J_2(T)$. To establish the absolute convergence of $J(P)$ it is now sufficient to show $J_i(T) \ll T^{\varepsilon}$ for $i = 1, 2$.

For this, we consider the exponential sum

$$|S_2(\boldsymbol{\lambda}_2)\ldots S_{r+1}(\boldsymbol{\lambda}_{r+1})|^2,$$

and perform the circle method analysis of the preceding sections with respect to this exponential sum instead of $S(\boldsymbol{\alpha})$. The second part of the proof of Lemma 3.4 gives the estimate

$$\begin{aligned}\int_{\mathfrak{M}(\theta)} |S_2(\boldsymbol{\lambda}_2)\dots S_{r+1}(\boldsymbol{\lambda}_{r+1})|^2 \,\mathrm{d}\boldsymbol{\alpha} &\ll \int_{[0,1]^{mr}} |S_2(\boldsymbol{\lambda}_2)\dots S_{r+1}(\boldsymbol{\lambda}_{r+1})|^2 \,\mathrm{d}\boldsymbol{\alpha}\\ &\ll T^{mnr+\varepsilon}.\end{aligned}$$

Furthermore, we have

$$\int_{\mathfrak{M}(\theta)} |S_2(\boldsymbol{\lambda}_2)\dots S_{r+1}(\boldsymbol{\lambda}_{r+1})|^2 \,\mathrm{d}\boldsymbol{\alpha} = \tilde{\mathfrak{S}}(T)J_1(T^{mr(n-1)\theta})T^{mnr} + O(T^{mnr-\eta})$$

for some $\eta > 0$, where the singular series is

$$\tilde{\mathfrak{S}}(T) = \sum_{q \leqslant T^{mr(n-1)\theta}} q^{-2mnr} \sum_{\mathbf{a}} |S^{(2)}_{\mathbf{a},q}\dots S^{(r+1)}_{\mathbf{a},q}|^2.$$

The term $q = 1$ and $a_{11} = \dots = a_{rm} = 0$ produces the lower bound $\tilde{\mathfrak{S}}(T) \geqslant 1$. Thus, we have $J_1(T^{mr(n-1)\theta}) \ll T^{\varepsilon}$, as desired. Since the same arguments apply also to J_2, we see that $J(P)$ is absolutely convergent. Indeed, we have $|J(P) - \lim_{P\to\infty} J(P)| \ll P^{-1+\varepsilon}$ for some $\varepsilon > 0$.

Lemma 3.7. *There exists $\varepsilon > 0$ such that $J(P) = \psi(0) + O(P^{-1+\varepsilon})$ when $P \to \infty$. Moreover, if the system of equations $f_i(\mathbf{x}) = 0$ has a non-singular solution in $\mathcal{B}$, simultaneously for all the infinite places of F, then $\psi(0) > 0$.*

Proof. The first statement follows from a form of the Fourier inversion theorem [**15**, Corollary 1.21], which can be applied since $I(\boldsymbol{\gamma})$ is integrable, and the continuity of $\psi(\mathbf{u})$ which is explained, for example, in [**1**, Section 6]. Next, let $\tau_1, \dots, \tau_m$ be the m different embeddings $F \to \bar{\mathbb{Q}}$. Assume that we are given a solution $\mathbf{t} \in \mathcal{B}$ of the system of equations $\tau_l(f_i(\mathbf{t})) = 0$ for all $1 \leqslant i \leqslant r$ and $1 \leqslant l \leqslant m$. Then $\mathbf{t}$ is a solution of $f_i(\mathbf{x}) = 0$ in V since $\det(\tau_l(\omega_j)) \neq 0$, and thus a non-singular solution of the system $f_{ij}(\mathbf{x}) = 0$. Therefore, $\psi(0)$ is positive as the integral of a positive integrand over a domain of positive measure. □

3.5. *Singular series*

Our next goal is to establish the absolute convergence of the singular series $\mathfrak{S}(P)$ for $P \to \infty$. This is done using a method similar to that of [**10**].

As usual, we order the pairs (l, j) lexicographically. We claim that no $(m \times m)$-minor of the matrix

$$\left(\frac{\partial}{\partial t_{lj}} \mathrm{Tr}(\rho_i N(\mathbf{t}_1))\right)_{\substack{1\leqslant i\leqslant m\\ (1,1)\leqslant(l,j)\leqslant(2,m)}} \tag{3.9}$$

has determinant zero. In the opposite case, we can find integers $1 \leqslant j_1 < \dots < j_q \leqslant m$ and $1 \leqslant j_{q+1} < \dots < j_m \leqslant m$, and rational numbers $c_1, \dots, c_m$, not all of them zero, such that

$$\sum_{k=1}^{q} c_k \frac{\partial \mathrm{Tr}(\rho_i N(\mathbf{t}_1))}{\partial t_{1,j_k}} + \sum_{k=q+1}^{m} c_k \frac{\partial \mathrm{Tr}(\rho_i N(\mathbf{t}_1))}{\partial t_{2,j_k}} = 0$$

for all $1 \leqslant i \leqslant m$. From (3.3) and the non-degeneracy of the trace we deduce

$$\sum_{k=1}^{q} c_k \omega_{j_k} \frac{\partial N(\mathbf{t}_1)}{\partial t_1} + \sum_{k=q+1}^{m} c_k \omega_{j_k} \frac{\partial N(\mathbf{t}_1)}{\partial t_2} = 0$$

identically in $\mathbf{t}_1$. Now, we set

$$t_1 = \sum_{k=1}^{q} c_k \omega_{j_k}, \quad t_2 = \sum_{k=q+1}^{m} c_k \omega_{j_k}, \quad t_3 = \cdots = t_n = 0,$$

and obtain

$$t_1 \frac{\partial}{\partial t_1} N(\mathbf{t}_1) + t_2 \frac{\partial}{\partial t_2} N(\mathbf{t}_1) = 0.$$

By Euler's identity, the left-hand side is equal to $mN(t_1, t_2, 0, \ldots, 0)$. This is a contradiction since not all of the $c_i \in \mathbb{Q}$ are zero, and thus $N(t_1, t_2, 0, \ldots, 0) \neq 0$. This proves the above claim, which we use in the proof of our next lemma.

Lemma 3.8. *The series*

$$\mathfrak{S} = \lim_{P \to \infty} \mathfrak{S}(P) = \sum_{q=1}^{\infty} q^{-mns} \sum_{\mathbf{a}} S_{\mathbf{a},q}$$

is absolutely convergent, and we have $|\mathfrak{S} - \mathfrak{S}(P)| \ll P^{-\eta}$ *for some* $\eta > 0$.

Proof. First, we choose a box $\mathcal{B} = \mathcal{B}_1 \times \cdots \times \mathcal{B}_s$ in V^{ns} in such a way that each $(m \times m)$-minor of the matrix (3.9) has full rank on $\mathcal{B}_1$, and similarly for all the other $\mathcal{B}_i$. We choose the $\mathcal{B}_i$ as cubes of products of half-open and half-closed intervals. Stretching $\mathcal{B}$ by a suitable factor we can assume that each box has side length 1. Note that this does not change the non-vanishing of the $(m \times m)$-minors of (3.9). We have $S_{\mathbf{a},q} = S(\boldsymbol{\alpha})$, where $P = q$ and $\alpha_{ij} = a_{ij}/q$, and then we obtain $S_{\mathbf{a},q} = S^{(1)}_{\mathbf{a},q} \ldots S^{(s)}_{\mathbf{a},q}$, where $S^{(j)}_{\mathbf{a},q} = S_j(\boldsymbol{\lambda}_j)$. By the first part of the proof of Lemma 3.4, for every $\mathbf{a}$ satisfying $\gcd(q, a_{11}, \ldots, a_{rm}) = 1$, there exists j such that

$$|S^{(j)}_{\mathbf{a},q}| \ll q^{mn-\Delta} \tag{3.10}$$

for some $\Delta > 0$. For simplicity, we assume $j = s$, since the other contributions can be estimated in exactly the same way. We assume as before that the submatrix of B formed by the first r columns, as well as that formed by the next r columns, both have full rank. We now apply the circle method as before to the exponential sum

$$|S_1(\boldsymbol{\lambda}_1) \ldots S_r(\boldsymbol{\lambda}_r)|^2,$$

instead of $S(\boldsymbol{\alpha})$. By the second part of the proof of Lemma 3.4 we have

$$\int_{\mathfrak{m}(\theta)} |S_1(\boldsymbol{\lambda}_1) \ldots S_r(\boldsymbol{\lambda}_r)|^2 \, \mathrm{d}\boldsymbol{\alpha} \ll \int_{[0,1]^{mr}} |S_1(\boldsymbol{\lambda}_1) \ldots S_r(\boldsymbol{\lambda}_r)|^2 \, \mathrm{d}\boldsymbol{\alpha} \ll P^{mnr+\varepsilon}.$$

Next, the major arc analysis gives us

$$\int_{\mathfrak{M}(\theta)} |S_1(\boldsymbol{\lambda}_1) \ldots S_r(\boldsymbol{\lambda}_r)|^2 \, \mathrm{d}\boldsymbol{\alpha} = \mathfrak{S}'(P) J'(P^{mr(n-1)\theta}) P^{mnr} + O(P^{mnr-\eta})$$

for some $\eta > 0$. Here, the singular series is

$$\mathfrak{S}'(P) = \sum_{q \leqslant P^{mr(n-1)\theta}} q^{-2mnr} \sum_{\mathbf{a}} |S^{(1)}_{\mathbf{a},q} \ldots S^{(r)}_{\mathbf{a},q}|^2,$$

and the singular integral is

$$J'(P) = \int_{|\boldsymbol{\gamma}| \leqslant P} \prod_{j=1}^{r} |\nu_j(\boldsymbol{\gamma})|^2 \, \mathrm{d}\boldsymbol{\gamma}.$$

By Lemma 3.6, applied with $l = m - 1$, and the choice of our boxes $\mathcal{B}_j$, we have

$$|\nu_j(\boldsymbol{\gamma})| \ll \prod_{k=1}^{m} \left(1 + \left|\mathrm{Tr}\left(\zeta_k \sum_{i=1}^{r} \gamma_i b_{ij}\right)\right|\right)^{-1}$$

for all j.

This gives the estimate

$$J'(P) \ll \int_{|\boldsymbol{\gamma}| \leqslant P} \prod_{j=1}^{r} \prod_{k=1}^{m} \left(1 + \left|\mathrm{Tr}\left(\zeta_k \sum_{i=1}^{r} \gamma_i b_{ij}\right)\right|\right)^{-2} \mathrm{d}\boldsymbol{\gamma}.$$

Next, we use the coordinate transformation $\boldsymbol{\gamma}' = M\boldsymbol{\gamma}$ with the matrix $M = (\mathrm{Tr}(b_{ij}\rho_l\zeta_k))_{(i,l),(j,k)}$, and obtain

$$J'(P) \ll \int_{|\boldsymbol{\gamma}'| \leqslant CP} \prod_{j=1}^{r} \prod_{k=1}^{m} (1 + |\gamma'_{jk}|)^{-2} \, \mathrm{d}\boldsymbol{\gamma}'$$

for some constant C. Note that M has full rank by Lemma 3.1 since $(b_{ij})_{1 \leqslant i,j \leqslant r}$ is assumed to have full rank. The above equation shows that $J'(P)$ is absolutely convergent, and thus the same arguments as in Lemma 3.7 imply $J'(P) = c_0 + o(1)$. Here, $c_0 > 0$ since the diagonal solutions ensure the existence of non-singular solutions in Lemma 3.7. We deduce

$$\mathfrak{S}'(P) \ll P^{\varepsilon}. \tag{3.11}$$

We come back to our main argument and consider

$$\mathfrak{S}_R = \sum_{R < q \leqslant 2R} q^{-mns} \sum_{\mathbf{a}} |S_{\mathbf{a},q}|.$$

Using equation (3.10) and the Cauchy–Schwarz inequality, we obtain

$$\mathfrak{S}_R \ll R^{-\Delta} (\mathfrak{S}_R^{(1)})^{1/2} (\mathfrak{S}_R^{(2)})^{1/2},$$

where

$$\mathfrak{S}_R^{(1)} = \sum_{R < q \leqslant 2R} q^{-2mnr} \sum_{\mathbf{a}} |S_{\mathbf{a},q}^{(1)} \dots S_{\mathbf{a},q}^{(r)}|^2,$$

and $\mathfrak{S}_R^{(2)}$ is of analogous form. Thus, equation (3.11) gives us $\mathfrak{S}_R \ll R^{-\Delta+\varepsilon}$, which proves the lemma for ε small enough. □

As usual, the singular series factorizes as $\mathfrak{S} = \prod_p c_p$, where the product is taken over all rational primes p, and the local factors are

$$c_p = \sum_{l=1}^{\infty} p^{-lmns} \sum_{\mathbf{a}} S_{\mathbf{a},p^l}.$$

Note that our polynomials $f_{ij}(\mathbf{x} + \mathbf{d})$ have coefficients in $\mathbb{Z}$ since all the entries of the matrix B are in $\mathcal{O}_F$ and also we assumed $\{\xi_1, \dots, \xi_n\} \subset \mathcal{O}_K$ in Theorem 1.4. By standard arguments, we see that the constants c_p can be written as

$$c_p = \lim_{l \to \infty} p^{-l(mns - mr)} C(p, l).$$

Here, $C(p, l)$ is the number of solutions to the simultaneous congruences

$$f_{ij}(\mathbf{x} + \mathbf{d}) \equiv 0 \bmod p^l, \tag{3.12}$$

for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant m$, where all the components x_{kl} of $\mathbf{x}$ run through a complete set of residues modulo p^l.

Next, we factorize the local densities c_p further to obtain an interpretation in terms of the number field F and the original system of equations $f_i(\mathbf{x}+\mathbf{d})=0$. For this let $\mathfrak{p}_1,\dots,\mathfrak{p}_t$ be the primes of F which lie above p and let $(p)=\prod_{i=1}^t \mathfrak{p}_i^{e_i}$ be the prime ideal factorization of the principal ideal (p). Note that for any $z\in F$, we have $z=\sum_{j=1}^m \zeta_j \mathrm{Tr}(\rho_j z)$, and $z\in (p^l)$ if and only if $\mathrm{Tr}(\rho_j z)\equiv 0 \bmod p^l$ for $1\leqslant j\leqslant m$. Therefore, we see that for fixed i and $1\leqslant j\leqslant m$ the system of equations (3.12) is equivalent to

$$f_i(\mathbf{x}+\mathbf{d})\equiv 0 \bmod (p^l). \tag{3.13}$$

Note that for some fixed i a full set of residues $x_{i1},\dots,x_{im}$ in $(\mathbb{Z}/p^l)^m$ corresponds to a full set of residues $x_i\in\mathfrak{n}$ modulo the ideal $(p^l)\mathfrak{n}$ under the identification $x_i=x_{i1}\omega_1+\cdots+x_{im}\omega_m$. Therefore, $C(p,l)$ is equal to the number of solutions of the system of congruences (3.13) for $1\leqslant i\leqslant r$, where $x_j\in\mathfrak{n}$ run through a complete set of residues modulo $(p^l)\mathfrak{n}$ for $1\leqslant j\leqslant ns$. Next, write $\mathfrak{n}=\mathfrak{n}'\prod_{i=1}^t\mathfrak{p}_i^{n_i}$ with $n_i\in\mathbb{N}$ such that $\mathfrak{n}'$ is coprime to (p). By a slightly modified Chinese remainder theorem, we have an isomorphism

$$\mathfrak{n}/(p^l)\mathfrak{n}\longrightarrow \textstyle\bigoplus_{i=1}^t \mathfrak{p}_i^{n_i}/\mathfrak{p}_i^{le_i+n_i}. \tag{3.14}$$

Hence, $C(p,l)=\prod_{k=1}^t D(\mathfrak{p}_k,le_k)$, where $D(\mathfrak{p}_k,l)$ is the number of solutions of the system $f_i(\mathbf{x}+\mathbf{d})\equiv 0 \bmod \mathfrak{p}_k^l$ for $1\leqslant i\leqslant r$, where we count solutions $x_j\in\mathfrak{p}_k^{n_k}$ modulo $\mathfrak{p}_k^{l+n_k}$ for all $1\leqslant j\leqslant ns$.

Lemma 3.9. *The singular series factorizes as $\mathfrak{S}=\prod_{\mathfrak{p}}\sigma_{\mathfrak{p}}$, where the product is taken over all primes $\mathfrak{p}$ of $\mathcal{O}_F$, and the corresponding factors are given by*

$$\sigma_{\mathfrak{p}}=\lim_{l\to\infty}\mathrm{Nm}(\mathfrak{p})^{-l(ns-r)}D(\mathfrak{p},l).$$

Moreover, $\mathfrak{S}>0$ if the system of equations $f_i(\mathbf{x}+\mathbf{d})=0$ has a non-singular solution in $\mathfrak{n}_\nu^{ns}$ for all finite places of k.

Proof. By the above discussion and the multiplicativity of the norm of ideals, for the first part of the lemma it is enough to show that the limits in the definition of $\sigma_{\mathfrak{p}}$ exist. For this, we identify $\mathrm{Nm}(\mathfrak{p})^{-l(ns-r)}D(\mathfrak{p},l)$ with a subseries of $\mathfrak{S}$. For some ideal $\mathfrak{a}$, let $\hat{\mathfrak{a}}$ be the dual given by

$$\hat{\mathfrak{a}}=\{y\in F:\mathrm{Tr}(yz)\in\mathbb{Z}\text{ for all }z\in\mathfrak{a}\},$$

and note that $\hat{\mathfrak{a}}=\mathfrak{a}^{-1}\mathcal{C}$. For some $z\in\mathcal{O}_F$, we consider the character $e(\mathrm{Tr}(yz))$ for $y\in\hat{\mathfrak{a}}$. This is trivial if and only if $z\in\mathfrak{a}$, since $\hat{\hat{\mathfrak{a}}}=\mathfrak{a}$. Therefore, we have the orthogonality relation

$$\sum_y e(\mathrm{Tr}(yz))=\begin{cases}\mathrm{Nm}(\mathfrak{a}) & \text{for } z\in\mathfrak{a},\\ 0 & \text{otherwise,}\end{cases}$$

where the sum is taken over a complete set of residues $y\in\hat{\mathfrak{a}}$ modulo $\mathcal{C}$. Note that the index of $\mathcal{C}$ in $\hat{\mathfrak{a}}$ is just $\mathrm{Nm}(\mathfrak{a})$. Using this relation r times we see that

$$\mathrm{Nm}(\mathfrak{p})^{-l(ns-r)}D(\mathfrak{p},l)=\mathrm{Nm}(\mathfrak{p})^{-lns}\sum_{\mathbf{y}}\sum_{\mathbf{x}}e\left(\sum_{i=1}^r\mathrm{Tr}(y_i f_i(\mathbf{x}+\mathbf{d}))\right),$$

where the first sum is over all $\mathbf{y}\in\left(\hat{\mathfrak{p}^l}\right)^r$ modulo $\mathcal{C}$, and the second sum is over all $\mathbf{x}$ with $x_j\in\mathfrak{p}^{n_{\mathfrak{p}}}$ modulo $\mathfrak{p}^{l+n_{\mathfrak{p}}}$. We write here $n_{\mathfrak{p}}$ for the power to which $\mathfrak{p}$ occurs in $\mathfrak{n}$ as we did in the analysis preceding this lemma. Putting $y_i=y_{i1}\rho_1+\cdots+y_{im}\rho_m$ with $y_{ij}=a_{ij}/q$ for some integers a_{ij} and q, and using equation (3.14) for extending the summation over $\mathbf{x}$ to several sets of representatives, we can identify $\mathrm{Nm}(\mathfrak{p})^{-l(ns-r)}D(\mathfrak{p},l)$ with a subseries of $\mathfrak{S}$ as claimed above.

We turn to the second part of the lemma. Since $\mathfrak{S}$ is absolutely convergent, it is enough to show that $\sigma_{\mathfrak{p}}$ is positive if the system of equations $f_i(\mathbf{x}+\mathbf{d})=0$ has a non-singular solution in $\mathfrak{n}_{\mathfrak{p}}^{ns}$ for a fixed prime $\mathfrak{p}$. Suppose that $\mathbf{y}\in(\mathfrak{n}(\mathcal{O}_F)_{\mathfrak{p}})^{ns}$ is such a non-singular solution, and assume for simplicity of notation that the leading minor of the corresponding Jacobian matrix has full rank. Set

$$\delta=\nu_{\mathfrak{p}}\left(\det\left(\frac{\partial f_i}{\partial x_j}(\mathbf{y}+\mathbf{d})\right)_{1\leqslant i,j\leqslant r}\right),$$

where we write $\nu_{\mathfrak{p}}$ for the $\mathfrak{p}$-adic valuation. Now set $u=2\delta+n_{\mathfrak{p}}+1$, and choose $x_{r+1},\ldots,x_{ns}\in\mathcal{O}_F$ with

$$x_i\equiv y_i \bmod \mathfrak{p}^u. \tag{3.15}$$

Then we have

$$f_i(y_1+d_1,\ldots,y_r+d_r,x_{r+1}+d_{r+1},\ldots,x_{ns}+d_{ns})\equiv 0 \bmod \mathfrak{p}^u$$

for $1\leqslant i\leqslant r$. From a slightly modified version of [**9**, Proposition 5.20], a form of Hensel's lemma, we obtain $x_1,\ldots,x_r$ with

$$f_i(\mathbf{x}+\mathbf{d})\equiv 0 \bmod \mathfrak{p}^l,$$

and $x_j\equiv y_j \bmod \mathfrak{p}^{\delta+n_{\mathfrak{p}}+1}$ for $1\leqslant j\leqslant r$. If we restrict ourselves in equation (3.15) to a complete set of residues modulo $\mathfrak{p}^{l+n_{\mathfrak{p}}}$ for each x_i, then there are $\mathrm{Nm}(\mathfrak{p})^{(l-2\delta-1)(ns-r)}$ choices. This shows

$$D(\mathfrak{p},l)\geqslant \mathrm{Nm}(\mathfrak{p})^{(l-2\delta-1)(ns-r)}$$

for l large enough, which proves the lemma. □

3.6. *Proofs of Theorems* 1.3 *and* 1.4

Proof of Theorem 1.4. We note that

$$N(\mathcal{B},P)=\int_{\mathfrak{M}(\theta)}S(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha}+\int_{\mathfrak{m}(\theta)}S(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha}.$$

Therefore, this theorem is a consequence of Lemma 3.4 for the minor arc part, and Lemma 3.5 together with Lemmas 3.7, 3.8 and 3.9 insofar as the main term is concerned. In particular, we get

$$\mu(\mathcal{B})=\psi(0)\prod_{\mathfrak{p}}\sigma_{\mathfrak{p}}, \tag{3.16}$$

where the product is again taken over all primes of k. □

Now we deduce Theorem 1.3 from Theorem 1.4 using an argument as in Skinner's paper (see [**13**, proof of Corollary 1 in Section 5]).

Proof of Theorem 1.3. Assume that we are given some $\varepsilon>0$ and a finite set of places S of F, which we can assume to contain all infinite places. Furthermore, assume that we are given solutions

$$(\mathbf{x}_1^{(\nu)},\ldots,\mathbf{x}_{2r}^{(\nu)})\in Y_{\mathrm{sm}}(F_\nu)$$

for all $\nu\in S$. We want to find $(\mathbf{x}_1,\ldots,\mathbf{x}_{2r})\in Y_{\mathrm{sm}}(F)$ such that

$$|x_j-x_j^{(\nu)}|_\nu<\varepsilon$$

for all $1\leqslant j\leqslant 2nr$ and all $\nu\in S$.

First we write

$$x^{(\nu)}_{n(j-1)+1}\xi_1 + \cdots + x^{(\nu)}_{nj}\xi_n = \frac{y^{(\nu)}_{n(j-1)+1}\xi_1 + \cdots + y^{(\nu)}_{nj}\xi_n}{y^{(\nu)}_{2nr+1}\xi_1 + \cdots + y^{(\nu)}_{ns}\xi_n},$$

with $\mathrm{ord}_\nu(y^{(\nu)}_j) \geqslant 0$ for all finite $\nu \in S$ and $1 \leqslant j \leqslant ns$. Using the Chinese remainder theorem we can find $\mathbf{d} \in \mathcal{O}_F^{ns}$ with $|d_j - y^{(\nu)}_j|_\nu < \tilde{\varepsilon}$ for all j and all finite places $\nu \in S_f$. As in [**13**], if $\mathfrak{p}$ is the prime corresponding to a finite place ν, then we write

$$n_\mathfrak{p} = \min_j \mathrm{ord}_\nu(d_j - y^{(\nu)}_j), \quad \mathfrak{n} = \prod_{\mathfrak{p}\in S_f} \mathfrak{p}^{n_\mathfrak{p}}.$$

We turn to infinite places, and note that there is a unique $\mathbf{u} \in V^{ns}$ such that $\tau_\nu(\mathbf{u}) = \mathbf{y}^{(\nu)}$ for all infinite places ν. Here, we write τ_ν for the embedding corresponding to the infinite place ν. Then we have

$$\sum_{j=1}^{2r} a_{ij}N(\mathbf{u}_j) + a_{i,2r+1}N(\mathbf{u}_{2r+1}) = 0$$

for $1 \leqslant i \leqslant r$. We note that there is $z \in \mathcal{O}_F$ such that $z\xi_i$ is integral for all i. Multiplying the above equation by a power of z, we can assume $\xi_1, \dots, \xi_n$ to be integral. Next, set $b_{ij} = \tilde{a}a_{ij}$ for $j \leqslant s$ so that we have $b_{ij} \in \mathcal{O}_F$ for all i and j. By construction, $\mathbf{u}$ is then a non-singular solution of the system of equations $f_i(\mathbf{x}) = 0$. Choosing $\mathcal{B}$ sufficiently small around $\mathbf{u}$, we can assume that for any $\mathbf{x} \in \mathcal{B}$ we have

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial x_j}(\mathbf{x})\right) = r.$$

Now, we can apply Theorem 1.4, and obtain

$$N(\mathcal{B}, P) = \mu(\mathcal{B})P^{mn(r+1)} + o(P^{mn(r+1)}),$$

with some positive constant $\mu(\mathcal{B})$, since $Y_{\mathrm{sm}}(F_\nu) \neq \emptyset$ for all places ν. Let P and t be large integers with $P \equiv 1 \bmod \mathfrak{n}^t$. For P sufficiently large, we then get a solution $\mathbf{z} \neq 0$ to the system of equations $f_i(\mathbf{z}) = 0$ with $\mathbf{z} - \mathbf{d} \in (P\mathcal{B}) \cap \mathfrak{n}^{ns}$. We define $\mathbf{x}$ by

$$x_{n(j-1)+1}\xi_1 + \cdots + x_{nj}\xi_n = \frac{z_{n(j-1)+1}\xi_1 + \cdots + z_{nj}\xi_n}{z_{2nr+1}\xi_1 + \cdots + z_{ns}\xi_n}$$

for $1 \leqslant j \leqslant 2r$. Then the x_i are rational functions in $\mathbf{z}$ and hence they are continuous in $\mathbf{z}$. For an infinite place ν, we estimate

$$\max_j \left|\frac{1}{P}z_j - y^{(\nu)}_j\right|_\nu \ll \frac{1}{P} + \kappa.$$

For finite places $\nu \in S$ we have

$$\max_j \left|\frac{1}{P}z_j - y^{(\nu)}_j\right|_\nu \ll \max_j \left(\left|\frac{1}{P}z_j - z_j\right|_\nu + |z_j - d_j|_\nu + |d_j - y^{(\nu)}_j|_\nu\right) \ll \tilde{\varepsilon}$$

for some $P \equiv 1 \bmod \mathfrak{n}^t$ with t sufficiently large. By choosing κ and $\tilde{\varepsilon}$ sufficiently small and P sufficiently large, we finally obtain

$$|x_j - x^{(\nu)}_j|_\nu < \varepsilon$$

for all $1 \leqslant j \leqslant 2nr$ and all $\nu \in S$ as required. □

Acknowledgements. We are grateful to Professor T. D. Wooley for suggesting this problem to us.

References

1. B. J. Birch, 'Forms in many variables', *Proc. R. Soc. Ser. A* 265 (1961) 245–263.
2. B. J. Birch, 'Waring's problem in algebraic number fields', *Proc. Cambridge Phil. Soc.* 57 (1961) 449–459.
3. B. J. Birch, H. Davenport and D. J. Lewis, 'The addition of norm forms', *Mathematika* 9 (1962) 75–82.
4. J.-L. Colliot-Thélène, 'Points rationnels sur les fibrations', *Higher dimensional varieties and rational points* Budapest, 2001 (ed. K. Böröczky, J. Kollár and T. Szamuely; Springer, Bolyai Society Colloquium Publications, 2003, Berlin).
5. J.-L. Colliot-Thélène, D. Harari and A. N. Skorobogatov, 'Valeurs d'un polynôme à une variable représentées par une norme', *Number theory and algebraic geometry*, London Mathematical Society Lecture Note Series 303 (Cambridge University Press, 2003) 69–89.
6. J.-L. Colliot-Thélène, D. Harari and A. N. Skorobogatov, 'Compactification équivariante d'un tore (d'après Brylinski et Künnemann)', *Expo. Math.* 23 (2005) 161–170.
7. J.-L. Colliot-Thélène and J.-J. Sansuc, 'La descente sur les variétés rationnelles. II', *Duke Math. J.* 54 (1987) 375–492.
8. J-L. Colliot-Thélène and A. N. Skorobogatov, 'Descent on fibrations over $\mathbb{P}^1_k$ revisited', *Math. Proc. Cambridge Phil. Soc.* 128 (2000) 383–393.
9. M. J. Greenberg, *Lectures on forms in many variables* (W. A. Benjamin, 1969).
10. R. Heath-Brown and A. Skorobogatov, 'Rational solutions of certain equations involving norms', *Acta Math.* 189 (2002) 161–177.
11. P. A. B. Pleasants, 'Cubic polynomials over algebraic number fields', *J. Number Theory* 7 (1975) 310–344.
12. W. M. Schmidt, 'The density of integer points on homogeneous varieties', *Acta Math.* 154 (1985) 243–296.
13. C. M. Skinner, 'Forms over number fields and weak approximation', *Comp. Math.* 106 (1997) 11–29.
14. A. N. Skorobogatov, *Torsors and rational points* (Cambridge University Press, Cambridge, 2001).
15. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series 32 (Princeton University Press, Princeton, NJ, 1971).
16. M. Swarbrick Jones, 'A note on a theorem of Heath-Brown and Skorobogatov', *Quat. J. Math.* 64 (2013) 1239–1251.
17. D. Wei, 'On the equation $N_{K/k}(\Theta) = P(t)$', Preprint, 2012, arXiv:1202.4115.
18. D. Wei, 'The unramified Brauer group of norm one tori', Preprint, 2012, arXiv:1202.4714.

Damaris Schindler
Hausdorff Center for Mathematics
Endenicher Allee 62
53115 Bonn
Germany

damaris.schindler@hcm.uni-bonn.de

Alexei Skorobogatov
Department of Mathematics
South Kensington Campus
Imperial College London
SW7 2BZ
United Kingdom

a.skorobogatov@imperial.ac.uk