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Automorphisms and forms of toric quotients of homogeneous spaces

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Abstract. We compute the automorphism group of the quotient of a generalized Grassmannian G/P by the action of a maximal torus of the semisimple group G. We classify the twisted forms of such quotients, that is, varieties isomorphic to these quotients over an algebraic closure of the base field. It is proved that all such forms are unirational.

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Introduction

Let G be a semisimple algebraic group defined over a field k of characteristic zero, and let P be a maximal parabolic subgroup of G. The main aim of this work is to find the automorphism group of the quotient of the projective homogeneous space G/P by the action of a maximal torus of the group G, and to classify the $\overline{k}/k$-forms of this quotient, that is, the varieties over k that are isomorphic to it over an algebraic closure $\overline{k}$. The proposed approach uses torsors whose structure group is a torus, so we start by recalling basic notions and constructions of this theory.

Let X be an algebraic variety over k and let T be an algebraic torus. A torsor (or principal homogeneous space) over X with structure group T is a variety $\mathscr{T}$ equipped with a morphism $\mathscr{T} \to X$ with a fibrewise action of T which is locally isomorphic in the étale topology to the direct product $X \times_k T$. Colliot-Thélène and Sansuc introduced a class of X-torsors, the so-called universal torsors; their structure group is the algebraic torus that is dual to the Picard group of $\overline{X} = X \times_k \overline{k}$ with its $\mathrm{Gal}(\overline{k}/k)$-module structure (under the assumption that the Picard group is torsion free). Numerous examples show that the investigation of rational points on a universal X-torsor is simpler than on the original variety X. Thus, if k is a number field, then one can interpret in these terms basic theorems of the descent theory of Colliot-Thélène and Sansuc (see [1], Ch. 3, [2], Ch. 6). But even in the case of an arbitrary field universal torsors are useful for the study of the R-equivalence of rational points. In recent years universal torsors have played an important role in analytic number theory, in problems of estimating the growth of the number of rational points of bounded height on projective varieties. We note that for many interesting varieties it is a nontrivial problem to describe universal torsors by explicit equations. In this paper we propose an alternative approach to this problem. Let Y be an affine cone over a smooth projective variety embedded

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by a T-linearized ample sheaf and let $U \subset Y$ be an open subset consisting of the points x such that the orbit Tx is closed and the natural map $T \to Tx$ is bijective. It is proved in geometric invariant theory that the quotient $X = U/T$ exists in the category of algebraic varieties. Here the morphism $U \to X$ is a torsor with structure group T (Lemma 1.4). Moreover, if the codimension of $Y \setminus U$ in Y is more than 1 and $\operatorname{Pic} \overline{Y} = 0$, then $U \to X$ is a universal torsor (Theorem 1.6, (i)).

Generalized Grassmannian varieties G/P, where P is a parabolic subgroup of a semisimple group G, form a natural class of smooth projective varieties admitting actions of a torus which can be taken to be any maximal torus of G. If P is a maximal parabolic subgroup, then $\operatorname{Pic} G/P$ is an infinite cyclic group generated by the class of an ample sheaf. Hence the affine cone over G/P has trivial Picard group, that is, this cone can be used for the construction of universal torsors. The simplest case where the complement to U has codimension greater than one is provided by the Grassmannian $G(2,5)$ with the action of a maximal torus of the group $G = \mathrm{SL}(5)$; in this case X is a del Pezzo surface of degree 5 (see [3], [4]). For this case we have the following results (see [2], Ch. 3):

(a) a description of universal torsors on X;
(b) a description of the automorphism group $\operatorname{Aut} \overline{X}$;
(c) a classification of $\overline{k}/k$-forms of X as quotients of $G(2,5)$ by actions of maximal tori of G;
(d) it follows from (c) that all the $\overline{k}/k$-forms of X are unirational; in particular, they contain a Zariski open subset of k-points.

Note that all del Pezzo surface of degree 5 are isomorphic over $\overline{k}$, so all these surfaces are twisted $\overline{k}/k$-forms of each other. Thus, as a consequence of (d), we have the classical theorem of Enriques asserting that any del Pezzo surface of degree 5 has a rational point over its field of definition (finite or infinite). For small values m and n, the quotients of Grassmannians $G(m,n)$ and their natural compactifications were extensively studied in the literature (see, for example, [3], [5] and references therein). For $n = 6$ and $m = 2$ the variety X is the smooth locus of the Segre cubic

$$\sum_{i=0}^{5} x_i = \sum_{i=0}^{5} x_i^3 = 0,$$

which is a cubic threefold having the maximal number of isolated singular points (this number is 10). For $n = 6$ and $m = 3$, the variety X is the smooth locus of a double cover of $\mathbf{P}^4_k$ branched along the Igusa threefold, that is, a compactification of the moduli space of Abelian surfaces with level-two structure (recall that we assume that the field k has characteristic zero). The Gelfand-MacPherson correspondence [5] represents X as the quotient of the variety of stable configurations of n points in $\mathbf{P}^{m-1}_k$ by the action of $\mathrm{GL}(m)$. Note that del Pezzo surfaces of degree $d = 2, 3, 4$ can be naturally embedded into quotients of generalized Grassmannians for the root systems E_7, E_6 and D_5, respectively (see [6], [7]).

In the present paper we generalize results (a), (b), (c) and (d), obtained for the case $G(2,5)$, to arbitrary generalized Grassmannians G/P such that the complement to the set of stable points with trivial stabilizer has codimension more than one. All cases where this condition is satisfied are listed in Proposition 2.1. A unified description of the automorphism group $\operatorname{Aut} \overline{X}$ can be obtained if we exclude the case of a maximal torus of the group of type B_n acting on the Grassmannian of n-dimensional isotropic subspaces of the $(2n+1)$-dimensional vector space with a nondegenerate symmetric form, and the case of a maximal torus acting on the quadric, the orbit of the highest weight vector in the 7-dimensional representation of the group of type G_2. We briefly sketch the proof. The quotient $X = U/T$ has a natural torsor $\mathscr{T}$ to which the automorphisms of X lift. Moreover, $\operatorname{Pic}\mathscr{T}$ is generated by the class of an ample sheaf, so that G/P is the Zariski closure of $\mathscr{T}$ in the corresponding projective embedding. A description of the automorphism group of X can be obtained from the description of the automorphism group of G/P given by Tits and Demazure [8] (Theorem 2.2). Further, the recent result of Gille [9] and Raghunathan [10] on maximal tori in quasi-split groups allows us to prove that any $\overline{k}/k$-form of X is a quotient of a homogeneous space of a quasi-split form of G by a maximal torus (Theorem 2.4). This implies that any $\overline{k}/k$-form of X is unirational; in particular, the set of rational points on it is dense in the Zariski topology. It would be interesting to answer the question about the rationality of these varieties over the field k.

§ 1. Torsors and twisted forms

1.1. Lifting automorphisms to a torsor. Let k be a field of characteristic zero with algebraic closure $\overline{k}$ and Galois group $\Gamma = \operatorname{Gal}(\overline{k}/k)$. We denote by $\overline{X}$ the variety obtained from X by the field extension from k to $\overline{k}$. Assume that X is geometrically integral and satisfies the condition $\overline{k}[X]^* = \overline{k}^*$. Let T be an algebraic torus defined over k. We denote by $\widehat{T}$ the group of characters of the torus T equipped with the natural structure of a Γ-module. There is the following exact sequence of Colliot-Thélène and Sansuc (see, for example, [2], formula (2.22)):

$$0 \to \mathrm{H}^1(k,T) \to \mathrm{H}^1(X,T) \xrightarrow{\chi} \operatorname{Hom}_\Gamma(\widehat{T}, \operatorname{Pic}\overline{X}) \xrightarrow{\partial} \mathrm{H}^2(k,T) \to \mathrm{H}^2(X,T). \quad (1)$$

The torsor $\mathscr{T} \to X$ with structure group T defines the class $[\mathscr{T}]$ in $\mathrm{H}^1(X,T)$. Its image $\chi([\mathscr{T}]) \in \operatorname{Hom}_\Gamma(\widehat{T}, \operatorname{Pic}\overline{X})$ is called the *type* of this torsor. If T is a k-split torus, that is, $T \simeq \mathbf{G}^n_{m,k}$, then (1) and Hilbert's Theorem 90 imply that a torsor with structure group T is determined by its type up to isomorphism. It follows from (1) that a torsor of type τ exists if and only if $\partial(\tau) = 0$. Moreover, one can see from (1) that the set of torsors of a given type is either empty or is in one-to-one (non-canonical) correspondence with $\mathrm{H}^1(X,T)$.

A torsor is called *universal* if its type is an isomorphism $\widehat{T} \simeq \operatorname{Pic}\overline{X}$.

There is another exact sequence introduced by Colliot-Thélène and Sansuc (see [1], formula (2.1.1)):

$$1 \to \overline{k}^* \to \overline{k}[\mathscr{T}]^* \to \widehat{T} \to \operatorname{Pic}\overline{X} \to \operatorname{Pic}\overline{\mathscr{T}} \to 0. \quad (2)$$

The homomorphism $\widehat{T} \to \operatorname{Pic}\overline{X}$ coincides up to sign with the type of the torsor $\mathscr{T} \to X$. It can be seen from (2) that the homomorphism defined by the type is injective if and only if $\overline{k}[\mathscr{T}]^* = \overline{k}^*$.

The following lemma is a variant of remarks given in [11] (Theorem 1.2 and Proposition 1.4). Let $\operatorname{Aut} X$ be the automorphism group of a k-variety X.

Lemma 1.1. *Let X be a geometrically integral variety over k such that $\overline{k}[X]^* = \overline{k}^*$. Let $f\colon \mathscr{T} \to X$ be a torsor whose structure group is a torus T and whose type $\lambda\colon \widehat{T} \to \operatorname{Pic}\overline{X}$ is injective. Denote by $\operatorname{Aut}(\overline{X},\lambda) \subset \operatorname{Aut}\overline{X}$ the set of automorphisms leaving the subgroup $\lambda(\widehat{T}) \subset \operatorname{Pic}\overline{X}$ invariant. Then there is an exact sequence of groups equipped with an action of the Galois group Γ, which is equivariant with respect to this action*:

$$1 \to T(\overline{k}) \to \mathrm{N}_{\operatorname{Aut}\overline{\mathscr{T}}}(T(\overline{k})) \to \operatorname{Aut}(\overline{X},\lambda) \to 1, \tag{3}$$

where $\mathrm{N}_{\operatorname{Aut}\overline{\mathscr{T}}}(T(\overline{k}))$ is the normalizer of $T(\overline{k})$ in $\operatorname{Aut}\overline{\mathscr{T}}$. Here $\mathrm{N}_{\operatorname{Aut}\overline{\mathscr{T}}}(T(\overline{k}))$ is the group of automorphisms of $\overline{\mathscr{T}}$ which preserve the fibres of the projection $\overline{\mathscr{T}} \to \overline{X}$.

Proof. The injectivity of the homomorphism $T(\overline{k}) \to \mathrm{N}_{\operatorname{Aut}\overline{\mathscr{T}}}(T(\overline{k}))$ is obvious. Also it is clear that any automorphism of the torsor $\overline{\mathscr{T}}$ normalizing the subgroup $T(\overline{k})$ descends to an automorphism of $\overline{X}$. The actions of the group $\mathrm{N}_{\operatorname{Aut}\overline{\mathscr{T}}}(T(\overline{k}))$ on $\overline{\mathscr{T}}$ and $\overline{X}$ are compatible, and this group also acts by conjugation on $\overline{T} = T(\overline{k})$. Under the simultaneous action of $\mathrm{N}_{\operatorname{Aut}\overline{\mathscr{T}}}(T(\overline{k}))$ on $\overline{X}$ and $\overline{T}$ the class $[\overline{\mathscr{T}}]$ in $\mathrm{H}^1(\overline{X},\overline{T})$ is stable. Therefore, it follows from the exact sequence (1) that this action preserves the type of this torsor $\chi([\mathscr{T}]) \in \operatorname{Hom}_\Gamma(\widehat{T},\operatorname{Pic}\overline{X})$, that is, the quotient group $\mathrm{N}_{\operatorname{Aut}\overline{\mathscr{T}}}(T(\overline{k}))$ by $T(\overline{k})$ is homomorphically mapped to $\operatorname{Aut}(\overline{X},\lambda)$.

Now we prove the injectivity of this homomorphism. If α lies in its kernel, then for any $x \in \mathscr{T}(\overline{k})$ we have $\alpha(x) = t(x)x$, where $t\colon \mathscr{T} \to T$ is a morphism of varieties. However, in view of the injectivity of λ, it follows from the exact sequence (2) that $\overline{k}[\mathscr{T}]^* = \overline{k}^*$, that is, any morphism from $\mathscr{T}$ to a torus contracts $\mathscr{T}$ to a point. Therefore, α is induced by an action of an element of $T(\overline{k})$, which proves the desired injectivity.

It remains to prove that the image of $\mathrm{N}_{\operatorname{Aut}\overline{\mathscr{T}}}(T(\overline{k}))$ in $\operatorname{Aut}(\overline{X},\lambda)$ coincides with the full group. In view of the injectivity of λ, the natural action of $\operatorname{Aut}\overline{X}$ on $\operatorname{Pic}\overline{X}$ determines an action of $\operatorname{Aut}(\overline{X},\lambda)$ on $\widehat{T}$. Suppose $h \in \operatorname{Aut}(\overline{X},\lambda)$ acts on $\widehat{T}$ by the automorphism $\widehat{\tau}_h$. We denote by τ_h the corresponding automorphism of the torus $\overline{T}$. Then $\lambda \circ \widehat{\tau}_h = h^* \circ \lambda$ for any $h \in \operatorname{Aut}(\overline{X},\lambda)$, where h^* is the natural action of h on $\operatorname{Pic}\overline{X}$. This implies that the $\overline{X}$-torsor with structure group $\overline{T}$ obtained from $f\colon \overline{\mathscr{T}} \to \overline{X}$ by the base change $h\colon \overline{X} \to \overline{X}$ has the same type as the $\overline{X}$-torsor obtained by the change of the structure group $\tau_h\colon \overline{T} \to \overline{T}$. Since $\overline{X}$ has no nonconstant invertible regular functions, we may use the sequence (1), which gives us that these two $\overline{X}$-torsors are isomorphic. Thus, any automorphism $h \in \operatorname{Aut}(\overline{X},\lambda)$ is obtained from some automorphism $\rho \in \operatorname{Aut}\overline{\mathscr{T}}$ acting fibrewise on $\overline{\mathscr{T}} \to \overline{X}$. Let us prove that any $\rho \in \operatorname{Aut}\overline{\mathscr{T}}$ acting fibrewise on $\overline{\mathscr{T}} \to \overline{X}$ normalizes $T(\overline{k})$. Let $t \in T(\overline{k})$. Then $\psi := \rho t \rho^{-1}$ preserves the fibres of the projection $\overline{\mathscr{T}} \to \overline{X}$. The canonical isomorphism $\mathscr{T} \times_X \mathscr{T} = \mathscr{T} \times_k T$ identifies the graph of ψ with the graph of some morphism $\overline{\mathscr{T}} \to \overline{T}$. By (2) the injectivity of λ implies that $\overline{k}[\mathscr{T}]^* = \overline{k}^*$, so the image of any morphism from $\overline{\mathscr{T}}$ to a torus consists of one point. Therefore, ψ acts on $\overline{\mathscr{T}}$ by the translation by an element of $T(\overline{k})$ (this argument is borrowed from [11], Lemma 1.1). This finishes the proof of our lemma.

In the case when $\mathscr{T} \to X$ is a universal torsor, (3) gives the exact sequence

$$1 \to T(\overline{k}) \to \mathrm{N}_{\operatorname{Aut}\overline{\mathscr{T}}}(T(\overline{k})) \to \operatorname{Aut}\overline{X} \to 1. \tag{4}$$

There exists a unique (up to isomorphism) universal $\overline{X}$-torsor over $\overline{k}$ (of fixed type). Therefore, any universal torsor over a $\overline{k}/k$-form of X is a $\overline{k}/k$-form of any universal X-torsor.

If G is a topological group equipped with a continuous action of Γ, then we denote by $\mathrm{Z}^1(\Gamma, G)$ the set of all continuous 1-cocycles of Γ with coefficients in G.

Lemma 1.2. *Under the assumptions of Lemma 1.1 let X' be a $\overline{k}/k$-form of X twisted by some 1-cocycle $\sigma \in \mathrm{Z}^1(\Gamma, \mathrm{Aut}(\overline{X}, \lambda))$. Then the set of X'-torsors of type λ coincides with the set of $\overline{k}/k$-forms of $\mathscr{T}$ twisted by a 1-cocycle from $\mathrm{Z}^1(\Gamma, \mathrm{N}_{\mathrm{Aut}\,\overline{\mathscr{T}}}(T(\overline{k})))$, which is a lifting of σ.*

We should explain that the type λ uniquely determines the structure group of an X'-torsor. Indeed, the action of Γ on $\mathrm{Pic}\,\overline{X'}$ determines an action of Γ on the submodule $\widehat{T} \simeq \lambda(\widehat{T})$. The latter determines uniquely the torus dual to this lattice.

Proof. Let $\mathscr{T}' \to X'$ be a torsor of type λ. Since $\overline{X}' \simeq \overline{X}$ and, under our assumptions, an $\overline{X}$-torsor of type λ is unique up to isomorphism, it follows that $\overline{\mathscr{T}}'$ and $\overline{\mathscr{T}}$ are isomorphic as $\overline{X}$-torsors. Therefore, $\mathscr{T}'$ is obtained from $\mathscr{T}$ by twisting by a 1-cocycle (which is a lifting of σ) with coefficients in the group of automorphisms of $\overline{\mathscr{T}}$, preserving the fibres of the map $\overline{\mathscr{T}} \to \overline{X}$. By Lemma 1.1 this group is $\mathrm{N}_{\mathrm{Aut}\,\overline{\mathscr{T}}}(T(\overline{k}))$. Conversely, if $\xi \in \mathrm{Z}^1(\Gamma, \mathrm{N}_{\mathrm{Aut}\,\overline{\mathscr{T}}}(T(\overline{k})))$ is a lifting of σ, then $\mathscr{T}' = \mathscr{T}_\xi$ is a torsor of type λ over $X' = X_\sigma$.

Assume that (3) is a sequence of groups of $\overline{k}$-points in the exact sequence of group k-schemes

$$1 \to T \to G \to H \to 1. \tag{5}$$

We denote by $\mathrm{Z}^1(k, G)$ the set of all continuous 1-cocycles of the Galois group Γ with coefficients in the group $G(\overline{k})$. Let $\sigma \in \mathrm{Z}^1(k, H)$ and let $X' = X_\sigma$ be the corresponding twisted form. Let T_σ be the form of T twisted by σ in the sense of the action of H on T by conjugation given by (5). It can be seen from the first part of the proof of Lemma 1.1 that the action of $H(\overline{k})$ on $\widehat{T} = \lambda(\widehat{T})$ determined above is compatible with the action of $H(\overline{k})$ on $\mathrm{Pic}\,\overline{X_\sigma}$. Hence T_σ is the structure group of X_σ-torsors of type λ, if there are any.

According to Lemma 1.2 X_σ-torsors of type λ exist if and only if the cohomology class $[\sigma] \in \mathrm{H}^1(k, H)$ is contained in the image of $\mathrm{H}^1(k, G)$. Recall that (5) determines the class $\Delta(\sigma) \in \mathrm{H}^2(k, T_\sigma)$, which is equal to zero if and only if the class $[\sigma]$ lies in the image of $\mathrm{H}^1(k, G)$ (see [12], Ch. I, § 5.6, Proposition 41). (Note that if we replace $\sigma = \sigma(\gamma)$, $\gamma \in \Gamma$, with a cohomologous cocycle $h^{-1} \cdot \sigma(\gamma) \cdot^{\gamma} h$, then the class of $\Delta(\sigma)$ will be replaced by $h^{-1} \cdot \Delta(\sigma) \cdot h$.)

Although we do not need this, we prove that $\Delta(\sigma)$ coincides with $\partial(\lambda)$, where ∂ is the map from the exact sequence (1) for X_σ and T_σ. In other words, two obstructions to the existence of X_σ-torsors of type λ coincide.

Proposition 1.3. *Under the assumptions of Lemma 1.1, let X_σ be a form of X obtained by twisting by a cocycle $\sigma \in \mathrm{Z}^1(k, H)$. Then $\Delta(\sigma) = \partial(\lambda)$.*

Proof. Let H_σ be a right k-torsor of the group H defined by σ, that is, a k-scheme with right action of H which becomes isomorphic to H with the natural right action on itself after the field extension from k to $\overline{k}$. Consider the gerbe $\mathscr{G}_1$ of liftings of H_σ

to a right torsor of G (see [13], Ch. IV, § 2.5, Proposition 8). This is a category fibred over the category of extensions $k \subset K \subset \overline{k}$. The fibre $\mathscr{G}_1(K)$ over K is the groupoid formed by those K-torsors of the group G which after the change of the structure group by the map $G \to H$ become isomorphic to $H_{\sigma,K} = H_\sigma \times_k K$. In other words, $\mathscr{G}_1(K)$ consists of the K-torsors D such that D/T is obtained from H_σ by the extension of the base field from k to K. The gerbe $\mathscr{G}_1$ is bound by the k-band (*Fr.* lien) defined by the k-torus T_σ acting on these torsors on the left (*ibid.*)

Further, let $\mathscr{G}_2$ be the gerbe of X_σ-torsors with structure group T_σ of type λ (see [13], Ch. V, § 3.1, Proposition 6), that is, the fibre $\mathscr{G}_2(K)$ consists of $X_{\sigma,K}$-torsors of type λ. According to [13], Ch. V, § 3.2, Proposition 1, (i), this gerbe is bound by the k-band $f_*f^*T_\sigma$, where $f\colon \mathscr{T} \to X$ is the structure morphism. It follows from the condition $\overline{k}[X]^* = \overline{k}^*$ that the canonical morphism $T_\sigma \to f_*f^*T_\sigma$ is an isomorphism. Therefore, $\mathscr{G}_2$ is also bound by the k-band, defined by the k-torus T_σ.

We associate with a right K-torsor D of the group G the form of the K-variety $\mathscr{T}_K$ twisted by D, that is, $\mathscr{T}_{K,D} = (D \times_K \mathscr{T}_K)/G$, where G acts from the right on D and from the left on $\mathscr{T}_K$. (Recall that $G(\overline{k}) = \mathrm{N}_{\mathrm{Aut}\,\overline{\mathscr{T}}}(T(\overline{k}))$.) The torus T_σ acts on the twisted variety $\mathscr{T}_{K,D}$ from the left endowing it with the structure of a $X_{\sigma,K}$-torsor of the group T_σ of type λ. The morphism of gerbes $\mathscr{G}_1 \to \mathscr{G}_2$ obtained here identifies the classes $[\mathscr{G}_1]$ and $[\mathscr{G}_2]$ in $\mathrm{H}^2(k, T_\sigma)$. However the class of $\mathscr{G}_1$ in $\mathrm{H}^2(k, T_\sigma)$ is $\Delta(\sigma)$ (see [13], Ch. IV, § 2.5, Remark 9 and § 3.5, Corollary 4), whereas the class of $\mathscr{G}_2$ is $\partial(\lambda)$ (see [2], Ch. 2, § 3 and Ch. 9, § 5, where the result from [13], Ch. V, § 3.2, Proposition 1 is used).

In particular, if $\Delta(\sigma) \neq 0$, then X_σ has no k-points: in the opposite case $\partial(\lambda) = 0$, because the last map in (1) has a section defined by a k-point X_σ.

1.2. The quotient of an affine variety by an action of a torus. Torsors appear naturally in the context of geometric invariant theory. Let V be a finite-dimensional vector space over a field k. We shall regard V as an affine space over k and we shall write $V(\overline{k})$ instead of $V \otimes_k \overline{k}$. Let G_1 be a connected reductive subgroup of $\mathrm{SL}(V)$, and let G be the smallest closed subgroup of $\mathrm{GL}(V)$ containing G_1 and the group of scalar matrices $\mathbf{G}_{m,k}$. Recall that a point $x \in V(\overline{k})$ is called *stable* if the orbit $\overline{G}_1 x$ is closed and its dimension is equal to $\dim G_1$ (see [14], p. 194). The set of all stable points $V^{\mathrm{s}} \subset V$ is open (it can be empty) and G-invariant. In geometric invariant theory one constructs a quasi-projective variety Z and an affine morphism $f\colon V^{\mathrm{s}} \to Z$ whose fibres are orbits of G (see [14], Theorem 1.10, (iii)). Let $V^{\mathrm{f}} \subset V$ be the subset of points whose stabilizers in G are trivial. Let $V^{\mathrm{sf}} = V^{\mathrm{s}} \cap V^{\mathrm{f}}$; this subset is also open and G-invariant.

Lemma 1.4. *Let $U = V^{\mathrm{sf}} \cap Y$, where Y is a closed G-invariant subvariety of V. Assume that U is smooth and let $X = f(U)$. Then the natural morphism $f\colon U \to X$ is a torsor with structure group G. Moreover, X is smooth.*

Proof. The statement is local with respect to X. Let $x \in X(\overline{k})$. We take on open affine neighbourhood W of the point x. Since f is an affine morphism, $f^{-1}(W)$ is an affine subset (see [15], Ch. II, § 5, Exercise 5.17). The stabilizers of all $\overline{k}$-points of $f^{-1}(W)$ are trivial. Hence, according to a corollary to Luna's étale slice theorem (see [14], p. 153), the natural morphism $f^{-1}(W) \to W$ is a torsor with group G.

This implies the same assertion for $f\colon U \to X$. By definition a torsor is locally trivial in the étale topology, so the smoothness of X follows from that of U.

We consider the case when G_1 is an algebraic torus. Recall that a torus is said to be split if the action of the Galois group on its group of characters is trivial. Let T_1 be a split torus in $\mathrm{SL}(V)$, let T be the smallest torus in $\mathrm{GL}(V)$ containing T_1 and the group of scalar matrices $\mathbf{G}_{m,k}$, and let $T_2 = T/\mathbf{G}_{m,k}$ be the image of T in $\mathrm{PGL}(V)$. Thus, $\widehat{T}_1$, $\widehat{T}$ and $\widehat{T}_2$ are trivial Γ-modules. The space V is decomposed into a direct sum of eigenspaces of the torus T:

$$V = \bigoplus_{\lambda\in\widehat{T}} V_\lambda,$$

where V_λ consists of vectors w such that $t\cdot w = \lambda(t)w$ for any $t \in T(\overline{k})$. Let Λ be the set of weights of the representation of T in V, that is, characters $\lambda \in \widehat{T}$ satisfying the condition $V_\lambda \neq 0$. Let Λ_1 be the set of weights of T_1 in V. It is obvious that the natural homomorphism $\widehat{T} \to \widehat{T}_1$ maps Λ isomorphically to Λ_1.

For $x \in V(\overline{k})$ we denote by $\mathrm{wt}_T(x)$ the set of weights $\lambda \in \Lambda$ such that the V_λ-component of x is not zero. We define $\mathrm{wt}_{T_1}(x) \subset \Lambda_1$ as the image of $\mathrm{wt}_T(x)$ under the surjection $\widehat{T} \to \widehat{T}_1$. The *weight polytope* of x is the convex hull $\mathrm{Conv}(\mathrm{wt}_{T_1}(x))$ of the set $\mathrm{wt}_{T_1}(x)$ in $\widehat{T}_1 \otimes \mathbf{R}$. The Hilbert-Mumford stability criterion asserts that a point x is stable with respect to the action of T_1 if and only if the zero point is contained in the interior of the weight polytope x (see [16], Theorem 9.2).

Lemma 1.5. *Let $Y \subset V$ be the complement to the origin in a geometrically integral T-invariant closed subvariety in V, where Y is smooth. Let $U = Y \cap V^{\mathrm{sf}}$. If* $\mathrm{codim}_Y(Y \setminus U) \geqslant 2$, *then $\overline{k}[X]^* = \overline{k}[U]^* = \overline{k}^*$, where $X = U/T$.*

Proof. We denote by $\mathbf{P}(Y)$ the image of Y in the projective space $\mathbf{P}(V)$. The natural morphism $Y \to \mathbf{P}(Y)$ is a torsor with group $\mathbf{G}_{m,k}$. Its type is an injective homomorphism $\mathbf{Z} \to \operatorname{Pic}\mathbf{P}(\overline{Y})$ sending the unit to the class of the sheaf $\mathscr{O}(1)$ (up to sign). It follows from (2) that $\overline{Y}$ has no nonconstant invertible regular functions. The assumption on the codimension of $Y \setminus U$ implies that the same holds for U and, therefore, for X.

Theorem 1.6. *Let $Y \subset V$ be the complement to the origin in a geometrically integral T-invariant closed subvariety which is not contained in a hyperplane. Assume that Y is smooth,* $\mathrm{codim}_Y(Y \setminus U) \geqslant 2$ *and* $\operatorname{Pic}\mathbf{P}(\overline{Y})$ *is generated by the class of the sheaf $\mathscr{O}(1)$. Then the following assertions hold.*

(i) *The variety X is smooth and satisfies the condition $\overline{k}[X]^* = \overline{k}^*$. The natural morphism $f\colon U \to X$ is a universal torsor; in particular, there exists a canonical isomorphism of trivial* Γ-*modules* $\operatorname{Pic}\overline{X} = \widehat{T}$.

(ii) *Let $K = \overline{k}(X)$ and let U_K be the generic fibre of the morphism $\overline{U} \to \overline{X}$. The Galois modules* $\operatorname{Div}\overline{X}$ *and $K[U_K]^*/\overline{k}^*$ are canonically isomorphic, Moreover, the semigroup of effective divisors on $\overline{X}$ is identified with $(K[U_K]^* \cap \overline{k}[Y])/\overline{k}^*$.*

(iii) $-\Lambda$ *is a unique minimal set of generators of the effective divisor class semigroup* $\operatorname{Pic}\overline{X} = \widehat{T} = K[U_K]^*/K^*$. *In particular, any automorphism $\overline{X}$ maps Λ to itself.*

(iv) *Let* $\operatorname{Pic}\overline{X} \to \mathbf{Z}$ *be the homomorphism dual to the embedding of the group of scalar matrices* $\mathbf{G}_{m,\overline{k}}$ *to* $\overline{T}$. *This homomorphism is equivariant with respect to the action of the group* $\operatorname{Aut}\overline{X}$ (*which is trivial on* $\mathbf{Z}$).

(v) *Any universal torsor on a* $\overline{k}/k$*-form of* X *having the same type as* $U \to X$ *is a* $\overline{k}/k$*-form of* U.

(vi) *Let* τ *be the embedding of* $\widehat{T}_2$ *into* $\operatorname{Pic}\overline{X}$ *as the kernel of the homomorphism* $\operatorname{Pic}\overline{X} \to \mathbf{Z}$ *from* (iv). *Any torsor of type* τ *on a* $\overline{k}/k$*-form of* X *is a dense open subset of the* $\overline{k}/k$*-form of* $\mathbf{P}(Y)$ *given by a cocycle with coefficients in the normalizer of* $T_2(\overline{k})$ *in* $\operatorname{Aut}\mathbf{P}(\overline{Y})$.

(vii) *The group* $\operatorname{Aut}\overline{X}$ *is canonically isomorphic to the quotient group of the normalizer of* $T_2(\overline{k})$ *in* $\operatorname{Aut}\mathbf{P}(\overline{Y})$ *by* $T_2(\overline{k})$.

Proof. (i) By Lemmas 1.4 and 1.5 it remains to check that the torsor $f\colon U \to X$ is universal. Since $\operatorname{Pic}\mathbf{P}(\overline{Y})$ is generated by the class of the sheaf $\mathscr{O}(1)$, it follows that $\operatorname{Pic}\overline{Y} = 0$ (the sequence (2) for the torsor $\overline{Y} \to \mathbf{P}(\overline{Y})$). Under our assumptions this implies that $\operatorname{Pic}\overline{U} = 0$. So from the exact sequence (2) for $f\colon U \to X$ we see that the type $\widehat{T} \to \operatorname{Pic}\overline{X}$ of the torsor f is an isomorphism of Abelian groups and therefore an isomorphism of Γ-modules as well.

(ii) The generic fibre U_K is a K-torsor with group $T_K = T \times_k K \simeq \mathbf{G}^n_{m,K}$. By Hilbert's Theorem 90 $\mathrm{H}^1(K, \mathbf{G}_{m,K}) = \{1\}$. Hence U_K is a trivial torsor, that is, $U_K \simeq T_K$. Rosenlicht's Lemma asserts that T acts on any invertible regular function on U_K by multiplication by a character. This defines a canonical isomorphism $K[U_K]^*/K^* = \widehat{T}$. Since $\overline{Y}$ is smooth and has no nonconstant invertible regular functions and, moreover, $\operatorname{Pic}\overline{Y} = 0$, we have $\operatorname{Div}\overline{U} = \operatorname{Div}\overline{Y} = \overline{k}(Y)^*/\overline{k}^*$. Similarly $\operatorname{Div} U_K = \overline{k}(Y)^*/K[U_K]^*$. Now the first assertion follows from the (split) exact sequence

$$0 \to \operatorname{Div}\overline{X} \to \operatorname{Div}\overline{U} \to \operatorname{Div} U_K \to 0. \tag{6}$$

A divisor $D \subset \overline{X}$ is effective if and only if the Zariski closure of the divisor $f^*(D)$ on $\overline{Y}$ is effective. Since $\operatorname{Pic}\overline{Y} = 0$, this closure is a principal divisor $\operatorname{div} g$, where g is a nonzero regular function on $\overline{Y}$. Hence, $g \in \overline{k}[Y]$. Conversely, any function $g \in K[U_K]^* \cap \overline{k}[Y] \subset K[U_K]$ determines an effective divisor on $\overline{U}$ which does not meet the generic fibre U_K. In view of the exactness of the sequence (6), it comes from a unique effective divisor on $\overline{X}$.

(iii) Since Y is the complement to a point in a closed affine subvariety $Y_c \subset V$ of dimension greater than one, the $\overline{k}$-algebra $\overline{k}[Y_c] = \overline{k}[Y] = \overline{k}[U]$ is generated as a vector space over $\overline{k}$ by monomials in coordinate functions. According to Rosenlicht's Lemma there is an isomorphism $K[U_K]^*/K^* \xrightarrow{\sim} \widehat{T}$ mapping the function g to a character χ such that $t \cdot g = \chi(t) g$ for any $t \in T(\overline{k})$. Coordinate hyperplanes do not meet the generic fibre U_K, that is, the coordinate functions are contained in $K[U_K]^*$. They are mapped to weights of the dual representation of T, that is, to the set $-\Lambda$. A linear combination of monomials lies in $K[U_K]^*$ if and only if all monomials with nonzero coefficients correspond to the same character which, in this case, must represent the class of this function in $\operatorname{Pic}\overline{X}$. The characters obtained in this way are linear combinations of weights $-\lambda$ with integer nonnegative coefficients. Therefore, $-\Lambda$ generates the semigroup E of classes of effective divisors in $\operatorname{Pic}\overline{X}$.

Let $\mathbf{G}_{m,\overline{k}} \to \overline{T}$ be the embedding of the subgroup of scalar matrices. The dual map $h\colon \widehat{T} \to \mathbf{Z}$ sends any element $\lambda \in \Lambda$ to the unit. Let $\alpha_1, \alpha_2, \dots$ be elements of E such that E is the set of their integral linear combinations with nonnegative coefficients. In particular, any $\lambda \in \Lambda$ can be written in the form $\lambda = \sum r_i \alpha_i$. Hence, $1 = h(\lambda) = \sum r_i h(\alpha_i)$. But since $h(\alpha_i)$ is a positive integer, $h(\alpha_i) = 1$ holds exactly for one index i and $h(\alpha_j) = 0$ for $j \neq i$. This implies that $\lambda = \alpha_i$, that is, $-\Lambda$ is contained in any system of generators E. So it is the unique minimal system of generators of E.

(iv) This follows from (iii).

(v) This follows from (i) and Lemma 1.2.

(vi) Let $\mathrm{N}(Y)$ (respectively, $\mathrm{N}(U)$) be the normalizer of $T_2(\overline{k})$ in $\operatorname{Aut}\mathbf{P}(\overline{Y})$ (respectively, in $\operatorname{Aut}\mathbf{P}(\overline{U})$). It follows by (iv) and Lemma 1.2 that any torsor of type τ on a $\overline{k}/k$-form of X is a $\overline{k}/k$-form of $\mathbf{P}(U)$ obtained from a cocycle with coefficients in $\mathrm{N}(U)$. Therefore, it remains to show that $\mathrm{N}(Y) = \mathrm{N}(U)$. The ample sheaf $\mathscr{O}(1)$ is invariant with respect to $\operatorname{Aut}\mathbf{P}(\overline{Y})$. This gives us an embedding $\operatorname{Aut}\mathbf{P}(\overline{Y}) \to \mathrm{PGL}(V \otimes_k \overline{k})$. Here $\mathrm{N}(Y)$ is mapped to the subgroup of the normalizer of $T_2(\overline{k})$ in $\mathrm{PGL}(V \otimes_k \overline{k})$ consisting of the elements stabilizing $\mathbf{P}(\overline{Y})$. This subgroup preserves the triviality of the stabilizer in T_2, as well as the stability with respect to the action of T_1. Hence $\mathrm{N}(Y)$ stabilizes the open set $\mathbf{P}(\overline{U}) \subset \mathbf{P}(\overline{Y})$. Thus, there is an embedding $\mathrm{N}(Y) \to \mathrm{N}(U)$. Similar arguments can be applied to $\mathbf{P}(\overline{U})$. Indeed, by the conditions of the theorem $\operatorname{Pic}\mathbf{P}(\overline{U}) = \operatorname{Pic}\mathbf{P}(\overline{Y})$ is generated by the class of $\mathscr{O}(1)$. So any automorphism of $\mathbf{P}(\overline{U})$ acts on the projectivization

$$(V \otimes_k \overline{k})^* = \mathrm{H}^0(\mathbf{P}(\overline{Y}), \mathscr{O}(1)) = \mathrm{H}^0(\mathbf{P}(\overline{U}), \mathscr{O}(1))$$

and, therefore, comes from an element of $\mathrm{PGL}(V \otimes_k \overline{k})$. A projective transformation stabilizing the set $\mathbf{P}(\overline{U})$ must stabilize its Zariski closure $\mathbf{P}(\overline{Y})$. This proves that $\mathrm{N}(Y) = \mathrm{N}(U)$.

(vii) This follows from (iv), Lemma 1.1 and the isomorphism $\mathrm{N}(Y) = \mathrm{N}(U)$.

Remark. 1. Let λ be a weight of multiplicity one, so that V^* has a unique (up to proportionality) eigenvector of weight $-\lambda$. We denote this vector by f_λ. Then the divisor on X given by the equation $f_\lambda = 0$ is a unique effective divisor in its class. Indeed, it follows from part (ii) of Theorem 1.6 that the linear system associated with a divisor of some weight is generated by all monomials of given weight. However, f_λ is a unique monomial of weight $-\lambda$.

2. Geometric invariant theory provides a canonical compactification of X, namely, the quotient of the set of semistable points. However, we point out that in most cases this compactification is singular.

§ 2. The quotient G/P by the action of a maximal torus

2.1. The automorphism group. Let G be a split simply connected simple algebraic group over k. By the definition of a split group, G possesses a k-split maximal torus $H \simeq \mathbf{G}_{m,k}^n$ and a Borel subgroup $B \supset H$ defined over k. Let R be a root system of G with respect to H and let $S = \{\alpha_1, \dots, \alpha_n\}$ be a basis of simple roots of R defined by B. We use the same indexing of roots as in [17] and [18]. (To avoid repetitions we shall assume that if R is of type B, C or D, then R is one of the root systems B_n, $n \geqslant 2$, C_n, $n \geqslant 3$, or D_n, $n \geqslant 4$.) Let $\{\omega_1, \ldots, \omega_n\}$ be the basis of fundamental weights dual to S. We shall use the following standard notation: $Q(\mathrm{R})$ is the lattice generated by the roots; $P(\mathrm{R})$ is the lattice generated by the fundamental weights; $W = W(\mathrm{R})$ is the Weyl group; $A(\mathrm{R})$ is the automorphism group of the root system R. It is known that $A(\mathrm{R}) = W(\mathrm{R}) \rtimes \operatorname{Aut} S$, where $\operatorname{Aut} S$ is the automorphism group of the system of simple roots S (or, equivalently, of the Dynkin diagram of R).

Let $Z(G)$ be the centre of G and let $G_{\mathrm{ad}} = G/Z(G)$ be the adjoint group of G. Put $H_{\mathrm{ad}} = H/Z(G)$. Recall that $\operatorname{Aut} G$ is an algebraic k-group $G_{\mathrm{ad}} \rtimes \operatorname{Aut} S$. Moreover, by the definition of a split group, the action of the Galois group Γ on $\operatorname{Aut} S$ is trivial. The finite group $\operatorname{Aut} S$ is identified with the group of outer automorphisms of G and also with the subgroup of $\operatorname{Aut} G$ stabilizing H, B and the set of one-parameter unipotent root subgroups of G defined by simple roots. The subgroup $\operatorname{Aut}(G, H) \subset \operatorname{Aut} G$ that consists of automorphisms stabilizing H is decomposed into a semidirect product $\mathrm{N}_{G_{\mathrm{ad}}}(H_{\mathrm{ad}}) \rtimes \operatorname{Aut} S$, where $\mathrm{N}_{G_{\mathrm{ad}}}(H_{\mathrm{ad}})$ is the normalizer of H_{ad} in G_{ad}. Thus, there exists the following commutative diagram of algebraic groups with exact rows and split exact columns:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H_{\mathrm{ad}} & \longrightarrow & \mathrm{N}_{G_{\mathrm{ad}}}(H_{\mathrm{ad}}) & \longrightarrow & W & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H_{\mathrm{ad}} & \longrightarrow & \operatorname{Aut}(G, H) & \longrightarrow & A(\mathrm{R}) & \longrightarrow & 1 \\
 & & & & \uparrow\downarrow & & \uparrow\downarrow & & \\
 & & & & \operatorname{Aut} S & = & \operatorname{Aut} S & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{7}$$

Here the action of the Galois group Γ on $A(\mathrm{R})$ is trivial because G is split.

Let $\rho\colon G \to \mathrm{GL}(V)$ be an irreducible finite-dimensional representation of G with highest weight ω_i. We set $T_1 = \rho(H)$. Let $v \in V$ be a vector of highest weight. The stabilizer of the line $kv \in \mathbf{P}(V)$ is a maximal parabolic subgroup $P \subset G$. It is defined by the closed subset of R equal to the union of the set of positive roots R^+ and the set of integer linear combinations of all simple roots except α_i. The homogeneous space G/P is a closed subvariety of $\mathbf{P}(V)$. Moreover, it is a unique closed orbit of G in $\mathbf{P}(V)$. The Picard group $\operatorname{Pic} G/P \simeq \mathbf{Z}$ is generated by the class of the very ample sheaf $\mathscr{O}(1)$.

Let $Y \subset V \backslash \{0\}$ be the affine cone over G/P with removed origin. Let $T \subset \mathrm{GL}(V)$ be a torus, generated by $T_1 = \rho(H)$ and the group of scalar matrices $\mathbf{G}_{m,k}$, and let $T_2 = H_{\mathrm{ad}} = T_1/\rho(Z(G))$. By Schur's lemma $\rho(Z(G)) = \rho(G) \cap \mathbf{G}_{m,k}$. We are in the situation described before Theorem 1.6. We are interested in the quotient X of an open set $U = Y \cap V^{\mathrm{sf}} \subset Y$ by the action of T and, in particular, in its automorphism group $\operatorname{Aut} X$.

Remark. A smooth variety X is projective, if $\mathrm{R} = \mathrm{A}_n$ and i is coprime with $n+1$. In this case G/P is the Grassmannian $G(i, n+1)$, whose set of stable points $G(i, n+1)^{\mathrm{s}}$ coincides with the set of semistable points $G(i, n+1)^{\mathrm{ss}}$ (see [14], Proposition 4.3 or [19]), and T_2 acts freely on this set. It was proved in [19] that $(G/P)^{\mathrm{s}} \neq (G/P)^{\mathrm{ss}}$ for all the remaining pairs (R, α_i), so in all these cases X is not a projective variety.

Proposition 2.1. *The condition* $\operatorname{codim}_Y(Y \setminus U) \geqslant 2$ *fails for a pair* (R, α) *if and only if it is in the following list*: (R_n, α_1), (A_n, α_n), (A_3, α_2), (B_2, α_2), (D_4, α_3) *or* (D_4, α_4), *where* $\mathrm{R}_n = \mathrm{A}_n$, B_n, C_n *or* D_n.

Proof. We take a basis of eigenvectors of the torus T in V and denote by $\mathscr{U}$ the open subset of the affine cone over G/P consisting of the points with at most one vanishing coordinate. By Lemma 2.1 of [6] the complement to $\mathbf{P}(\mathscr{U})$ in G/P has codimension two. It was proved in Proposition 2.4 of [6] that for pairs (R, α) which are not in the above list we have $\mathscr{U} \subset V^{\mathrm{s}}$. Also it can be seen from that proof that for pairs in the above list Y contains a coordinate hyperplane of nonstable points. If R is of type A_n, D_n or E_n, then it follows from Corollary 2.3 of [6] that the codimension of $Y^{\mathrm{s}} \setminus U$ is greater than one. Therefore, to complete the proof it remains to check that if (R, α_i) is not in our list and R is of type B_n, C_n, F_4 or G_2, then $\mathscr{U} \subset U$, that is, T acts freely on $\mathscr{U}$.

An equivalent assertion is that T_2 acts freely on $\mathbf{P}(\mathscr{U})$. This is obvious for points with all coordinates distinct from zero. Thus, from now on we assume that precisely one weight coordinate, say of weight λ, is equal to zero. Points with trivial stabilizer in T_2 are characterized by the property that the set $\mathrm{wt}_{T_1}(x) - \mathrm{wt}_{T_1}(x)$ generates $Q(\mathrm{R}) = \widehat{T}_2$. In particular, $Q(\mathrm{R})$ is generated by the set $\Lambda_1 - \Lambda_1$, where Λ_1 is the collection of weights of T_1. If the multiplicity of the weight λ is more than one, then $\mathrm{wt}_{T_1}(x) = \Lambda_1$, so the stabilizer of x in T_2 is trivial. Thus, we may assume that the multiplicity of the weight λ is equal to one. We must verify that $(\Lambda_1 \setminus \{\lambda\}) - (\Lambda_1 \setminus \{\lambda\})$ generates $Q(\mathrm{R})$.

First we consider the case $\lambda = 0$. If $\mathrm{R} = \mathrm{B}_n$ or C_n, then a direct examination using the description of fundamental representations in [17], [18] shows that the multiplicity of the weight $\lambda = 0$ is more than one except for the pair (B_n, α_1) from our list. The weights of fundamental representations in the case F_4 were described in [18], Ch. VIII, § 9, Exercise 16. Further, ω_1 and ω_4 are weights of the representation with highest weight ω_i for $i = 1, 2, 3$. But $\omega_1 - \omega_4$ is a short root and the roots $w(\omega_1 - \omega_4)$, $w \in W(\mathrm{F}_4)$ generate $Q(\mathrm{F}_4)$. Finally, $\lambda = 0$ is a weight of multiplicity more than one for the representation with highest weight ω_4. In the case $\mathrm{R} = \mathrm{G}_2$ the desired assertion is easily verified by using Table X in [17], Ch. VI.

Now assume that $\lambda \neq 0$. All the representations determined by the pair (R, α_i), where R is of type B_n, C_n, F_4 or G_2, are self-dual (see Table 1 in [18], Ch. VIII). Hence $\Lambda_1 = -\Lambda_1$. For any $\mu \in \Lambda_1 \setminus \{\lambda, -\lambda\}$ the obvious equalities $\lambda - \mu = (-\mu) - (-\lambda)$ and $\lambda - (-\lambda) = \mu - (-\lambda) + (-\mu) - (-\lambda)$ prove that

$$(\Lambda_1 \setminus \{\lambda\}) - (\Lambda_1 \setminus \{\lambda\}) = \Lambda_1 - \Lambda_1.$$

This completes the proof.

Remark. In general there are stable points on which the action of T is not free. The following example has been pointed out to the author by Vera Serganova. Let

$(\mathrm{R},\alpha)=(\mathrm{C}_n,\alpha_n)$, $n\geqslant 3$. Consider a vector space K with basis $e_1,f_1,\dots,e_n,f_n$ and a skew-symmetric form such that $(e_i,e_j)=(f_i,f_j)=0$ and $(e_i,f_j)=\delta_{ij}$ for all i and j. Let G/P be the Lagrangian Grassmannian of isotropic n-dimensional subspaces of K. Here $G=\mathrm{Sp}(2n)$ is the subgroup of $\mathrm{SL}(2n)$ stabilizing the skew-symmetric form and P is the stabilizer of the linear span of $e_1,\dots,e_n$. The subgroup of diagonal matrices in G is a maximal torus $H\simeq \mathbf{G}_{m,k}^n$. A point of G/P defined by the isotropic subspace spanned by vectors e_i+f_i, $i=1,\dots,n$, is stable because its weight polytope is the cube with edge of length two whose centre is at the origin. However, the stabilizer of this point in H contains the element $(-1,-1,1,\dots,1)$, which does not belong to the centre of G.

We denote by $\mathrm{Aut}(S,\alpha)$ the subgroup of $\mathrm{Aut}\,S$ consisting of the elements stabilizing a simple root α.

Theorem 2.2. *Assume that the pair* (R,α) *satisfies the following two conditions*:

1) *The complement in* G/P *to the set of stable points with trivial stabilizers in* T_2 *has codimension greater than one, so that* (R,α) *is not in the list from Proposition* 2.1;

2) $(\mathrm{R},\alpha)\neq(\mathrm{B}_n,\alpha_n)$, $(\mathrm{R},\alpha)\neq(\mathrm{G}_2,\alpha_1)$.

Then there exists a canonical isomorphism of k-groups with trivial action of the Galois group Γ

$$\mathrm{Aut}\,\overline{X}=W\rtimes\mathrm{Aut}(S,\alpha_i).$$

Also there is a natural bijection between the set of $\overline{k}/k$-forms of X considered up to isomorphism and the set of homomorphisms $\Gamma\to W\rtimes\mathrm{Aut}(S,\alpha_i)$, *considered up to conjugation.*

Proof. By Theorem 1.6, (vii) the group $\mathrm{Aut}\,\overline{X}$ is canonically isomorphic to the quotient group of the normalizer of $T_2(\overline{k})$ in $\mathrm{Aut}\,\overline{G/P}$ by the subgroup $T_2(\overline{k})$. Since the pair (R,α_i) is neither (B_n,α_n) nor (G_2,α_1), we can apply a theorem of Tits and Demazure [8] which asserts that the group $\mathrm{Aut}\,\overline{G/P}$ is a subgroup of $\mathrm{Aut}\,\overline{G}$ consisting of all elements stabilizing the conjugacy class of P. Since $\mathrm{Aut}\,\overline{G}=G_{\mathrm{ad}}(\overline{k})\rtimes\mathrm{Aut}\,S$, it follows that $\mathrm{Aut}\,\overline{G/P}=G_{\mathrm{ad}}(\overline{k})\rtimes\mathrm{Aut}(S,\alpha_i)$. The normalizer of $T_2(\overline{k})=H_{\mathrm{ad}}(\overline{k})$ in $\mathrm{Aut}\,\overline{G}$ is an extension of $\mathrm{Aut}\,S$ by $\mathrm{N}_{G_{\mathrm{ad}}}(H_{\mathrm{ad}})$. Hence the normalizer of $T_2(\overline{k})$ in $\mathrm{Aut}\,\overline{G/P}$ factored by $T_2(\overline{k})$ is $W\rtimes\mathrm{Aut}(S,\alpha_i)$.

Since Γ acts trivially on $W\rtimes\mathrm{Aut}\,S$, Γ also acts trivially on $\mathrm{Aut}\,\overline{X}=W\rtimes\mathrm{Aut}(S,\alpha_i)$. It is well known that the $\overline{k}/k$-forms of a quasiprojective variety X are in one-to-one correspondence with the elements of the Galois cohomology set $\mathrm{H}^1(k,\mathrm{Aut}\,\overline{X})$. This implies the last assertion of the theorem.

Remark. The pair (A_3,α_2) is contained in the list from Proposition 2.1 because the set of nonstable points of the Grassmannian $G(2,4)$ coincides with the set of points having at least one coordinate equal to zero. In this case X is $\mathbf{P}^1$ with three points removed and the automorphism group $\mathrm{Aut}\,\overline{X}\simeq S_3$ is not isomorphic to $W\rtimes\mathrm{Aut}(S,\alpha_i)\simeq S_4\rtimes\mathbf{Z}/2$. This shows that condition 1) in Theorem 2.2 is necessary. Condition 2) is also necessary because in these cases the theorem of Tits and Demazure does not hold: in the case of pair (B_n,α_n) the automorphism group $\mathrm{Aut}\,\overline{G/P}$ contains the adjoint group of type D_{n+1}, and in the case of the pair (G_2,α_1) it contains the adjoint group of type B_3 (see [8], p. 181). Therefore, a connected component of $\mathrm{Aut}\,\overline{X}$ contains a one-dimensional torus.

If $\mathrm{R} = \mathrm{A}_n$, then the Weyl group is isomorphic to the symmetric group S_{n+1}. If the marked root is α_i, then this group acts on $G/P = G(i, n+1)$ by permutations of coordinates.

The unique nontrivial element of $\operatorname{Aut} S = \mathbf{Z}/2$ induces an isomorphism $G(i, n+1) \simeq G(n+1-i, n+1)$ and an involution on $G(i, n+1)$, if $2i = n+1$. The Gelfand-MacPherson correspondence [5] represents X in the form of the quotient of the set of stable configurations of $n+1$ points in $\mathbf{P}_k^{i-1}$ by the action of $\mathrm{GL}(i)$. For example, in the case of $n = 5$ and $i = 3$ we must consider six-tuples of points on the plane. The involution on X defined by the nontrivial element of $\operatorname{Aut} S$ corresponds to the involution of X regarded as a double cover of $\mathbf{P}_k^4$ branched along the Igusa variety. In the classical literature this involution is known as *association*; a six-tuple of points is invariant with respect to association if and only if all the six points lie on a conic (for details see [5]).

2.2. Forms and rational points. The forms of a group G are obtained by twisting by cocycles of the Galois group Γ with coefficients in $\operatorname{Aut} G$ (for details see [20], Ch. II, § 2). In the particular case when the cocycle takes value in $\operatorname{Aut} S$ regarded as a subgroup of $\operatorname{Aut} G$ stabilizing H, B and the set of unipotent root subgroups of G defined by simple roots, the form of G is called *quasi-split*. A quasi-split group has a Borel subgroup defined over k.

Recall that the action of $\operatorname{Aut}(G, H)$ on H determines an action of the automorphism group of the root system $A(\mathrm{R}) = W \rtimes \operatorname{Aut} S$ on H (see (7)).

Proposition 2.3. *Let σ be a homomorphism $\Gamma \to A(\mathrm{R})$ and let θ be the composition of σ and the surjection $A(\mathrm{R}) \to \operatorname{Aut} S$. We denote by G_θ a quasi-split form of G given by θ. Let H_σ be a form of a k-split maximal torus $H \subset G$ twisted by a cocycle σ. Then the torus H_σ is isomorphic to a maximal torus in G_θ.*

Proof. The right column of (7) is a split exact sequence

$$1 \to W \to A(\mathrm{R}) \to \operatorname{Aut} S \to 1.$$

The section $\operatorname{Aut} S \to A(\mathrm{R})$ allows us to regard θ as a cocycle with coefficients in $A(\mathrm{R})$. This also allows us to define twisted forms $(\operatorname{Aut} S)_\theta$ and $A(\mathrm{R})_\theta$. The action of $A(\mathrm{R})$ by conjugation on W allows us to define a twisted form W_θ. In this case we get the following exact sequence of group schemes over k:

$$1 \to W_\theta \to A(\mathrm{R})_\theta \to (\operatorname{Aut} S)_\theta \to 1.$$

Recall that there is a canonical bijection between the sets $\mathrm{Z}^1(k, A(\mathrm{R}))$ and $\mathrm{Z}^1(k, A(\mathrm{R})_\theta)$, as well as between $\mathrm{H}^1(k, A(\mathrm{R}))$ and $\mathrm{H}^1(k, A(\mathrm{R})_\theta)$. Here the class $[\theta]$ in $\mathrm{H}^1(k, A(\mathrm{R}))$ corresponds to the distinguished element of $\mathrm{H}^1(k, A(\mathrm{R})_\theta)$ (see [12], Ch. I, § 5.3, Proposition 35). This bijection identifies the inverse image of $[\theta] \in \mathrm{H}^1(k, \operatorname{Aut} S)$ in $\mathrm{H}^1(k, A(\mathrm{R}))$ with the image of $\mathrm{H}^1(k, W_\theta)$ in $\mathrm{H}^1(k, A(\mathrm{R})_\theta)$ (see [12], Ch. I, § 5.5, Corollary 2 or [20], Ch. I, § 3). Let $\tilde{\sigma} \in \mathrm{Z}^1(k, W_\theta)$ be a cocycle such that the image of the class $[\tilde{\sigma}] \in \mathrm{H}^1(k, W_\theta)$ in $\mathrm{H}^1(k, A(\mathrm{R})_\theta)$ corresponds to σ under this identification.

It is obvious that the twisted form of the torus H by θ is a maximal torus $H_\theta \subset G_\theta$. We denote by N_θ the normalizer of H_θ in G_θ. The group k-scheme W_θ is isomorphic to N_θ / H_θ, so that W_θ acts naturally on H_θ. The twisting of H_θ

by the cocycle $\tilde{\sigma}$ is isomorphic to H_σ. Since G_θ is a quasi-split group, we can apply the Gille-Raghunathan theorem (see [9], Theorem 5.1, (b) and [10], Theorem 1.1). According to this theorem the torus obtained by twisting of H_θ by any 1-cocycle Γ with coefficients in W_θ can be embedded into G_θ as a maximal torus of the same group. In particular, this construction can be applied to H_σ.

Under the realization of $\mathrm{Aut}(S, \alpha_i)$ as a subgroup of $\mathrm{Aut}\, G$ preserving H and B, the parabolic subgroup P is stable under the action of $\mathrm{Aut}(S, \alpha_i)$ because P is the unique group in its conjugacy class that contains B. Therefore, if $\theta(\Gamma) \subset \mathrm{Aut}(S, \alpha_i)$, then the quasi-split group G_θ contains a twisted form P_θ of the group P.

Theorem 2.4. *Let σ be a homomorphism $\Gamma \to W \rtimes \mathrm{Aut}(S, \alpha_i) \subset A(\mathrm{R})$ and let θ be the composition of σ and the surjection $W \rtimes \mathrm{Aut}(S, \alpha_i) \to \mathrm{Aut}(S, \alpha_i)$. Let $H_\sigma \subset G_\theta$ be an embedding of H_σ as a maximal torus in the quasi-split group G_θ. Assume that conditions* 1) *and* 2) *of Theorem 2.2 are satisfied. Then the form of X twisted by σ is isomorphic to the quotient by the group H_σ of the set of those stable points G_θ/P_θ with respect to the action of H_σ whose stabilizers in $H_\sigma/Z(G_\theta)$ are trivial.*

Proof. Let $(G_\theta/P_\theta)^{\mathrm{sf},\theta}$ be the open subset of the homogeneous space (G_θ/P_θ) consisting of the points stable with respect to the action of H_θ with trivial stabilizers in the torus $H_\theta/Z(G_\theta)$. Since the actions of the group $\mathrm{Aut}(S, \alpha_i)$ on G, P, H, $(G/P)^{\mathrm{sf}}$ and $X = H\backslash(G/P)^{\mathrm{sf}}$ are compatible, the twisted form of X by the cocycle θ is the quotient of $X_\theta = H_\theta\backslash(G_\theta/P_\theta)^{\mathrm{sf},\theta}$.

The twisted form of X by the cocycle σ is obtained by twisting X_θ by the cocycle $\tilde{\sigma} \in \mathrm{Z}^1(k, W_\theta)$ (see the proof of Proposition 2.3), that is, $X_\sigma = (X_\theta)_{\tilde{\sigma}}$. Since the maximal tori H_θ and H_σ of the group G_θ are conjugate over $\overline{k}$, there exists an element $g \in G_\theta(\overline{k})$ such that $g \cdot H_\theta \cdot g^{-1} = H_\sigma$. Then $\rho(\gamma) = g^{-1} \cdot {}^\gamma g \in \mathrm{Z}^1(k, N_\theta)$, $\gamma \in \Gamma$, is a 1-cocycle of the Galois group with coefficients in N_θ which is also a lifting of $\tilde{\sigma} \in \mathrm{Z}^1(k, W_\theta)$. Here the image of the class $[\rho]$ in $\mathrm{H}^1(k, G)$ is trivial. The actions of the group N_θ on the homogeneous space G_θ/P_θ, on the torus H_θ and on the quotient $X_\theta = H_\theta\backslash(G_\theta/P_\theta)^{\mathrm{sf},\theta}$ are compatible. So the twisted form $X_\sigma = (X_\theta)_{\tilde{\sigma}}$ is the quotient of the corresponding twisted form $(G_\theta/P_\theta)^{\mathrm{sf},\theta}$ by the action of $H_\sigma = (H_\theta)_{\tilde{\sigma}}$. Since the image of $[\rho]$ in $\mathrm{H}^1(k, G)$ is trivial and since the closedness of orbits and the triviality of stabilizers are conditions over $\overline{k}$, the twisted form $(G_\theta/P_\theta)^{\mathrm{sf},\theta}$ mentioned above is an open subset of G_θ/P_θ. More precisely, this is the subset of stable points with respect to the action of H_σ having trivial stabilizers in $H_\sigma/Z(G_\theta)$.

Corollary 2.5. *Assume that conditions* 1) *and* 2) *of Theorem 2.2 are satisfied. Then any $\overline{k}/k$-form of X is unirational. In particular, the set of its k-points is Zariski dense.*

Proof. By Theorem 2.4 it is sufficient to show that G_θ/P_θ is rational over k. In this case the set of k-points is dense because the field k is infinite (recall that $\mathrm{char}(k) = 0$). Let P_θ^- be the parabolic subgroup opposite to P_θ, and let $R(P_\theta^-)$ be its unipotent radical. The affine space $R(P_\theta^-)$ is a dense open subset of G_θ/P_θ (the open Schubert cell).

Remark. It is well known that any del Pezzo surface of degree 5 is rational over its field of definition. It is interesting to find out if this property can be generalized to arbitrary $\overline{k}/k$-forms of quotients G/P by an action of a maximal torus.

Theorem 2.2 for the case of a root system of type A_n was proved by Neil Fitzgerald, who considered the quotient of $(\mathbf{P}_k^{m-1})^n$ by the action of $\mathrm{PGL}(m)$ (unpublished). The author is grateful to Philippe Gille and Vera Serganova for useful discussions and to the anonymous referee for careful reading and detailed comments. The work on this paper started at the Centre de Recherches Mathématiques de l'Université de Montréal. I should like to thank Irina Vedenova and Viktor Perelmouter for their hospitality.

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