Algebraic number theory

Problems sheet 5

March 11, 2011

Notation. In this sheet $K = \mathbb{Q}(\sqrt{d})$, where d is a square-free integer. We write $A(d)$ for $\mathcal{O}_K$.

1. Let $d > 1$ be divisible by a prime congruent to -1 modulo 4. Show that every unit in $A(d)$ has norm 1 (and never -1 !).

In the rest of this sheet use the following theorem proved in lectures: every class of ideals of $A(d)$ contains an integral ideal I of norm $||I|| < \lambda(d)$, where

$\lambda(d) = \sqrt{d}$ if $d > 0$ is congruent to 2 or 3 modulo 4,
$\lambda(d) = \sqrt{d}/2$ if $d > 0$ is congruent to 1 modulo 4,
$\lambda(d) = 4\sqrt{|d|}/\pi$ if $d < 0$ is congruent to 2 or 3 modulo 4,
$\lambda(d) = 2\sqrt{|d|}/\pi$ if $d < 0$ is congruent to 1 modulo 4.

2. Compute the class groups of $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{6})$.

3. Compute the class group of $\mathbb{Q}(\sqrt{-163})$. [*Hint*: prove that all prime ideals of small norm are principal.]

4. Prove that the class numbers of quadratic fields can be arbitrary large.

5. Prove that the class group of $\mathbb{Q}(\sqrt{-21})$ is the product of two cyclic groups of order 2. [*Hint*: Show that the prime ideals over 2 and 3 are not principal, and their product is not principal either.]