Algebraic number theory

Problem sheet 3

February 23, 2011

1. (a) Prove that the following abelian groups are not finitely generated: $(\mathbb{Q}, +)$, $(\mathbb{Q}^*, \times)$, $(\mathbb{Q}/\mathbb{Z}, +)$.

(b) Prove that any subgroup and any factor group of a finitely generated abelian group are also finitely generated.

2. Let $K = \mathbb{Q}(\alpha)$, where α is a root of $t^2 - 3t + 5 = 0$. Find all primes ramified in K, and give a criterion for other primes to be split or inert. Find all the prime ideals over $p = 47$.

3. (a) Explain why for every quadratic field K there exists at least one prime that ramifies in K.

(b) Prove that for every non-empty finite set S of primes there exists a quadratic field K which is ramified exactly at the primes of S. Find the number of such fields, for every S.

4. Find all elements of trace 0 in $\mathbb{Q}(\sqrt[3]{d})$, where d is a cube-free integer. The same question for the biquadratic field $\mathbb{Q}(\sqrt{a}, \sqrt{b})$, where a and b are distinct square-free integers.

5. Let z be an element of a quadratic field K of norm $N_K(z) = 1$. Prove that there exists $a \in K^*$ such that $z = a/\bar{a}$.

6. Prove that the ideal $I = (2, 1+\sqrt{-5})$ of the ring of integers of $\mathbb{Q}(\sqrt{-5})$ is not principal, but its square I^2 is principal.

7. Deduce from Q4 of Sheet 2 that a prime p can be written as $a^2 + b^2$ for some integers a and b if and only if $p = 2$, or p is 1 modulo 4. Similarly, $p = a^2+2b^2$ if and only if $p = 2$, or p is 1 or 3 modulo 8. Finally, $p = a^2+ab+b^2$ if and only if $p = 3$, or p is 1 modulo 3.