Algebraic number theory

Problem sheet 1

January 25, 2011

1. (i) Let $\alpha = \sqrt[3]{5} \in \mathbb{R}$. Work out a system of generators of the $\mathbb{Z}$-module $\mathbb{Z}[\alpha]$, hence check that it is a free $\mathbb{Z}$-module of rank 3. Working as in the proof of Thm. 3.2 find a monic polynomial $f(x)$ with coefficients in $\mathbb{Z}$ such that $f(\alpha + \alpha^2) = 0$.

(ii) The same question as in (i) for $\alpha = \sqrt[4]{5}$.

2. Let d be a square-free integer congruent to 1 modulo 4. Prove that $\mathbb{Z}[\sqrt{d}]$ is not integrally closed by exhibiting an element of $\mathbb{Q}[\sqrt{d}]$ not in $\mathbb{Z}[\sqrt{d}]$, which is integral over $\mathbb{Z}$.

3. Let $A \subset R$ be rings (as always, commutative, with 1) such that R is integral over A. Prove that if R is a field, then so is A.

Let F be a field, and $F[x]$ the ring of polynomials with coefficients in F.

4. Prove that $F[x]$ is integrally closed. (Hint: imitate the proof that $\mathbb{Z}$ is integrally closed which was given in class. You can use the theorem that every polynomial is uniquely written as a product of irreducibles, up to re-ordering of factors and multiplication by non-zero constants.)

5. Consider the extension of rings $F[x^2] \subset F[x]$.

(i) Prove that $F[x]$ is integral over $F[x^2]$.

(ii) Let $f(x) \in F[x]$. Find an integral dependence relation for $f(x)$ over $F[x^2]$.

6. Let $A \subset F[x]$ be the set of polynomials without linear term, i.e., of the form $a_n x^n + \ldots + a_2 x^2 + a_0$, $a_i \in F$. Prove that A is a subring of $F[x]$, then prove that A is not integrally closed. Using Q4, prove that $F[x]$ is the integral closure of A.

7. Let $F \subset K$ be fields. In this case the integral closure $\tilde{F}$ of F in K is also called the *algebraic closure* of F in K. Prove that $\tilde{F}$ is a field.