Supplement to the proof of Proposition 5.4

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Proposition 5.4 of [OS18] concerns a smooth, proper and geometrically integral variety X over a field k of characteristic 0. We can assume that k can be embedded into $\mathbb{C}$; let us fix such an embedding and define $H = \mathrm{H}^2(X_\mathbb{C}, \mathbb{Z}(1))/_{\mathrm{tors}}$. For a prime ℓ define $H_\ell = \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))/_{\mathrm{tors}}$. The group $\mathrm{Aut}(\overline{X})$ acts on H via the map $\mathrm{Aut}(\overline{X}) \to \mathrm{Aut}(X_\mathbb{C})$. Define $A(X)$ as the image of this action. The comparison theorem between Betti and ℓ-adic étale cohomology gives an isomorphism $H \otimes \mathbb{Z}_\ell \tilde{\longrightarrow} H_\ell$, which is $\mathrm{Aut}(\overline{X})$-equivariant. Thus the image of the natural action of $\mathrm{Aut}(\overline{X})$ on H_ℓ is canonically isomorphic to $A(X)$. The action of $\mathrm{Aut}(\overline{X})$ on H_ℓ is compatible with the action of the Galois group $\Gamma_k = \mathrm{Gal}(\bar{k}/k)$, hence Γ_k acts naturally on $A(X)$.

In the third paragraph of the proof of [OS18, Prop. 5.4] we stated that the action of Γ_k on $A(X)$ factors through a finite image of Γ_k. As was pointed out by Bjorn Poonen, our justification of this claim is insufficient. Let us prove this claim.

Since Γ_k acts continuously on H_ℓ, the image of Γ_k is a compact subgroup $\mathfrak{G}_\ell$ of $\mathrm{Aut}_{\mathbb{Z}_\ell}(H_\ell)$. Let G_ℓ be the Zariski closure of $\mathfrak{G}_\ell$ in $\mathrm{Aut}_{\mathbb{Z}_\ell}(H_\ell)$. The Γ_k-orbits in $\mathrm{Aut}(\overline{X})$ are finite, hence the $\mathfrak{G}_\ell$-orbits in $A(X)$ are finite. For any $x \in A(X)$ the inclusion $\mathfrak{G}_\ell(x) \subset G_\ell(x)$ of subsets of $\mathrm{Aut}_{\mathbb{Z}_\ell}(H_\ell)$ is an equality, because $\mathfrak{G}_\ell(x)$ is finite, hence Zariski closed in $G_\ell(x)$. Thus G_ℓ preserves $A(X) \subset \mathrm{Aut}_{\mathbb{Z}_\ell}(H_\ell)$.

Since $\mathfrak{G}_\ell$ is compact, G_ℓ has only finitely many connected components in the Zariski topology. The connected component of the identity $G_\ell^\circ \subset G_\ell$ acts on $A(X)$ with finite and connected, hence trivial orbits. Thus $\mathfrak{G}_\ell \cap G_\ell^\circ$ is a subgroup of $\mathfrak{G}_\ell$ of finite index that acts trivially on $A(X)$.

References

[OS18] M. Orr and A.N. Skorobogatov. Finiteness theorems for K3 surfaces and abelian varieties of CM type. *Compos. Math.* **154** (2018) 1571–1592.