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Non-abelian Cohomology and Rational Points

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Abstract. Using non-abelian cohomology we introduce new obstructions to the Hasse principle. In particular, we generalize the classical descent formalism to principal homogeneous spaces under noncommutative algebraic groups and give explicit examples of application.

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0. Introduction

This work grew out of our attempt to understand the rôle of non-abelian unramified coverings in the second author's counterexample to the Hasse principle which could not be explained by the Manin obstruction [Sk]. Suppose X is a variety over a number field k which has points in all the completions of k. It has become clear that given a torsor $Y \to X$ under a possibly non-abelian algebraic group G, one can still apply the same descent procedure as in the classical case of elliptic curves. It consists of 'twisting' $Y \to X$ by a cocycle $\sigma \in Z^1(k, G)$, and determining the set of cohomology classes $[\sigma] \in H^1(k, G)$ such that the twisted torsor Y^σ has points everywhere locally. This set is a generalization of the Selmer group related to an isogeny of elliptic curves. If X is proper and G linear, then this 'Selmer set' is contained in an explicitly computable finite subset of $H^1(k, G)$. If the Selmer set is empty, then X has no k-rational point. We shall refer to this as the *descent obstruction* given by the torsor $Y \to X$. One difference with the case of abelian groups considered in the descent theory of Colliot-Thélène and Sansuc [CS] is that Y^σ is a torsor under an inner form G^σ of G, and not under G itself (Subsection 4.1). The example mentioned above can be explained in this framework with a certain finite nilpotent group G (Subsection 4.3).

The connection of this obstruction for G abelian with the Manin obstruction to the Hasse principle which uses the Brauer–Grothendieck group $\mathrm{Br}\, X = H^2_{\text{ét}}(X, \mathbf{G}_m)$, was also studied by Colliot-Thélène and Sansuc in [CS]. Generalizing one of the main results of [CS], one proves that the abelian descent obstruction is equivalent to the algebraic part of the Manin obstruction, that is, the obstruction given by the subgroup of $\mathrm{Br}\, X$ consisting of elements killed over an algebraic closure of k

(see [Sk], Theorem 3, and Theorem 4.9 below). It is known that elements which survive can produce a nontrivial obstruction [Ha96]. We prove that even such a 'transcendental' Manin obstruction can be realized by non-abelian torsors (assuming the standard conjecture that $\mathrm{Br}\, X$ coincides with the group of similarity classes of Azumaya algebras on X, see Theorem 4.10). Thus the obstruction to the Hasse principle on X related to all possible X-torsors should indeed be stronger than the Manin obstruction.

The situation with weak approximation is similar. A smooth and proper k-variety X with a k-point and non-abelian geometric fundamental group, under some conditions (for example, if $H^1(X, \mathcal{O}) = H^2(X, \mathcal{O}) = 0$) always has adelic points satisfying the Brauer–Manin conditions but which are not in the closure of the set of k-points [Ha99]. Here again it is non-abelian torsors that provide finer conditions for an adelic point to be in the closure of $X(k)$ (Subsections 4.2 and 5.2).

So far we have discussed the obstructions to the Hasse principle and weak approximation for a variety over a number field k related to an already given torsor. In the literature one also finds obstructions to the existence of rational points over arbitrary fields; they can be realized as classes in the second abelian or non-abelian Galois cohomology sets, or as obstructions to descending an $\overline{X}$-torsor, where $\overline{X} = X \times_k \bar{k}$, to an X-torsor. In [CS] one encounters the obstruction for the existence of an abelian torsor of given 'type'; in fact one can regard it as the obstruction for a 'descent datum' on an $\overline{X}$-torsor to come from an X-torsor. (If $H^0(\overline{X}, \mathbf{G}_m) = \bar{k}^*$ and $\mathrm{Pic}\, \overline{X}$ is of finite type, then the finest of these is the obstruction for the existence of a universal torsor.) When X is a homogeneous space under a k-group G with geometric stabilizer $\overline{H} \subset \overline{G}$, Springer [Sp] constructs a class in the second Galois cohomology set with coefficients in an appropriate lien on $\overline{H}(\bar{k})$ which is the obstruction to lifting X to a k-torsor under G. Historically, the first obstruction of this kind known to us is given by the exact sequence of the étale fundamental group of X constructed by Grothendieck in [Gr]. This sequence splits if $X(k) \neq \emptyset$, which can be interpreted as the neutrality of the corresponding 2-cocycle. We formulate a uniform approach to such obstructions in terms of group extensions of the Galois group of k (Section 2), and study their interrelations. The splitting of the exact sequence of fundamental group implies that Springer's obstruction disappears ([Sp], Theorem 3.8). We prove that going over to an open subset of X we have a similar implication for the abelian obstruction of Colliot-Thélène and Sansuc (see Section 3).

The second Galois cohomology set can also be used to deal with the homogeneous spaces constructed by Borovoi and Kunyavskiĭ (Subsection 5.3).

1. Preliminaries

1.1. NOTATION AND CONVENTIONS

In this paper k is a field of characteristic zero with an algebraic closure $\bar{k}$. Let $\Gamma := \mathrm{Gal}\,(\bar{k}/k)$ be the absolute Galois group of k. A k-variety is a separated k-scheme of finite type. The group Γ is equipped with its profinite group topology and the set of $\bar{k}$-points of a $\bar{k}$-variety is equipped with the discrete topology.

An *algebraic k-group* is a k-group scheme which is a k-variety. An algebraic group G over k is *linear* if it is affine as a k-variety. It is *of multiplicative type* if the $\bar{k}$-group scheme $\overline{G} := G \times_k \bar{k}$ is a subgroup of $\mathbf{G}_m^n$ for some $n > 0$, where $\mathbf{G}_m$ is the multiplicative group.

For any connected scheme X equipped with a geometric base point $\bar{x}$, we let $\pi_1(X, \bar{x})$ be the associated étale fundamental group (up to inner automorphisms, it is independent of $\bar{x}$). Let $\pi_1^{\mathrm{ab}}(X)$ be the abelianization of $\pi_1(X, \bar{x})$ in the category of profinite groups. It is independent of the base point. We write $\bar{k}[X]^* := H^0(X \times_k \bar{k}, \mathbf{G}_m)$ for the group of invertible functions on $X \times_k \bar{k}$. If Y is an X-scheme (resp. an X-group scheme), we shall denote by $\mathtt{Aut}\,(Y/X)$ (resp. $\mathtt{Aut}^{\mathrm{gr}}(Y/X)$) the group of X-automorphisms of Y (resp. the group of X-automorphisms of Y which are compatible with the group scheme structure).

CONVENTION. We write $\overline{Y}$, $\overline{Z}$, $\overline{G}, \ldots$ for $\bar{k}$-varieties which are not necessarily obtained by extension of scalars from varieties over k, though this may be the case.

When k is a number field we denote by $\mathbf{A}_k$ the ring of adèles of k, by k_v the completion of k at the place v, and by $\mathcal{O}_v$ the ring of integers of k_v. Let Ω be the set of all places of k. If X is a k-variety, we write $X(\mathbf{A}_k)$ for the set of adelic points of X. If we further assume that X is proper, then $X(\mathbf{A}_k)$ is just $\prod_{v \in \Omega} X(k_v)$ equipped with the product of v-adic topologies.

Let G be an abstract group. An element $g \in G$ induces an inner automorphism of G defined by $(\operatorname{int} g)(h) = ghg^{-1}$. We let $\operatorname{Out} G$ be the quotient of the group $\operatorname{Aut} G$ of automorphisms of G by the inner automorphisms.

1.2. SEMILINEAR AUTOMORPHISMS

DEFINITION 1.1. Let $\bar{f}: \overline{Y} \to \operatorname{Spec} \bar{k}$ be a $\bar{k}$-variety. We denote by $\mathtt{SAut}\,(\overline{Y}/k) \subset \mathtt{Aut}\,(\overline{Y}/k)$ the subgroup of *semilinear* k-automorphisms of $\overline{Y}$. These are the elements φ of $\mathtt{Aut}\,(\overline{Y}/k)$ such that $\bar{f} \circ \varphi = (g^*)^{-1} \circ \bar{f}$ for some $g \in \Gamma$ (such a g is then unique). We let $q: \mathtt{SAut}\,(\overline{Y}/k) \to \Gamma$ denote the homomorphism which sends $\varphi \in \mathtt{SAut}\,(\overline{Y}/k)$ to the element g such that $\bar{f} \circ \varphi = (g^*)^{-1} \circ \bar{f}$.

Remark 1.2. When $\overline{Y}$ is connected and reduced, the integral closure of k in $\bar{k}[\overline{Y}]$ is $\bar{k}$, so any k-automorphism of $\overline{Y}$ induces an automorphism of $\bar{k}$ over k, hence is semilinear. In this case $\mathtt{SAut}\,(\overline{Y}/k) = \mathtt{Aut}\,(\overline{Y}/k)$. This equality is not true in general, e.g. for $\overline{Y} = \operatorname{Spec}(\bar{k} \oplus \bar{k})$.

DEFINITION 1.3. Let X be a k-variety, and $\overline{Y}$ a $\bar{k}$-variety with a morphism $\overline{Y} \to \overline{X}$. We define $\mathtt{SAut}\,(\overline{Y}/X) := \mathtt{Aut}\,(\overline{Y}/X) \cap \mathtt{SAut}\,(\overline{Y}/k)$. If $\overline{Y}$ is an $\overline{X}$-group scheme, we denote by $\mathtt{SAut}^{\mathrm{gr}}(\overline{Y}/X)$ the subgroup of $\mathtt{SAut}\,(\overline{Y}/X)$ consisting of the elements which are compatible with the group scheme structure. All those groups are equipped with the weak topology associated to the discrete topology on $\overline{Y}(\bar{k})$, that is the coarsest topology for which the map $\varphi \mapsto \varphi(\bar{m})$ is continuous for any $\bar{m} \in \overline{Y}(\bar{k})$. We write $\mathtt{Aut}\,(\overline{Y})$ instead of $\mathtt{Aut}\,(\overline{Y}/k)$, and $\mathtt{SAut}\,(\overline{Y})$ instead of $\mathtt{SAut}\,(\overline{Y}/k)$, if no confusion is possible (and similarly for $\mathtt{Aut}^{\mathrm{gr}}$ and $\mathtt{SAut}^{\mathrm{gr}}$). Note that in general the map $q\colon \mathtt{SAut}\,(\overline{Y}) \to \Gamma$ is not continuous.

LEMMA 1.4. *Let $\overline{Y}$ be a quasi-projective $\bar{k}$-variety (resp. an algebraic $\bar{k}$-group). Then the k-forms of $\overline{Y}$ are in natural bijection with the continuous homomorphic sections of the map $q\colon \mathtt{SAut}\,(\overline{Y}) \to \Gamma$ (resp. $q\colon \mathtt{SAut}^{\mathrm{gr}}(\overline{Y}) \to \Gamma$).*

Proof. Let j be a continuous homomorphic section of q. Then j induces an action* of Γ on $\overline{Y}(\bar{k})$ such that the stabilizer of any point is open. Now apply [BS], 2.12 and [Se59], V.20. We obtain a k-form Y of $\overline{Y}$ as the quotient of $\overline{Y}$ by the action of Γ.

Conversely, if Y is a k-form of $\overline{Y}$, the Galois group Γ acts on $\overline{Y} = Y \times_k \bar{k}$ via the second factor and this action defines a section j of q. Note that the constructions of the form Y and of the section j are inverse to each other. □

1.3. TORSORS

Let X be a scheme, and G_X a smooth X-group scheme.

DEFINITION 1.5. An *X-torsor* under G_X (or G_X-torsor over X, or principal homogeneous space of G_X over X) is a faithfully flat X-scheme Y equipped with a right action of G_X such that the associated map $(m, s) \mapsto (m, m.s)\colon Y \times_X G_X \to Y \times_X Y$ is an isomorphism.

We denote by $H^1(X, G_X)$ the pointed set of isomorphism classes of right torsors over X under G_X. The distinguished point of $H^1(X, G_X)$ is the class of the trivial torsor G_X. For any X-torsor Y under G_X we let $[Y]$ be the class of Y in $H^1(X, G_X)$.

Remark 1.6. If G_X is an affine X-group scheme, then we can compute $H^1(X, G_X)$ as the Čech cohomology set $\check{H}^1(X, G_X)$ ([Mi80], III.4.3 and III.4.7). If, moreover, G_X is abelian, $H^1(X, G_X)$ is identified with the étale cohomology group of the sheaf of abelian groups represented by G_X. If G is an algebraic k-group, then $H^1(k, G)$ is just the usual first Galois cohomology set $H^1(k, G(\bar{k}))$, which is a quotient of the set $Z^1(k, G(\bar{k}))$ of 1-cocycles ([Se94], I.5.1 and I.5.2).

DEFINITION 1.7. Let G be an algebraic k-group, $\overline{G} = G \times_k \bar{k}$. The Galois group Γ acts on $\mathtt{Aut}^{gr}(\overline{G})$. The *twist of the group G* by $\sigma \in Z^1(k, \mathtt{Aut}^{gr}(\overline{G}))$ is the quotient G^σ of $\overline{G}$ by the twisted action of Γ, which is: $(g, \bar{s}) \mapsto \sigma_g(g(\bar{s}))$, $g \in \Gamma$, $\bar{s} \in \overline{G}(k)$ ([Se94],

*In this paper, an *action* of a Galois group on a scheme always means its left action.

III.1.3). The homomorphism $\mathrm{int}\colon \overline{G}(\bar{k}) \to \mathtt{Aut}^{gr}(\overline{G})$ induces a map $Z^1(k, G(\bar{k})) \to Z^1(k, \mathtt{Aut}^{gr}(\overline{G}))$. If σ is in the image of this map, G^σ is called an *inner form* of G.

If an algebraic $\bar{k}$-group $\overline{G}$ admits a k-form G, then any k-form G' of G is obtained by twisting G with an element of $Z^1(k, \mathtt{Aut}^{gr}(\bar{G}))$ ([Se94], III.1.3).

DEFINITION 1.8. Let X be a k-variety, $f\colon Y \to X$ a right torsor under G which is a quasi-projective k-variety, and $\sigma \in Z^1(k, G)$. The twist of Y by σ is defined as the quotient Y^σ of $\overline{Y}$ by the twisted action of Γ, which is $(g, \bar{y}) \mapsto (g(\bar{y})) \cdot \sigma_g^{-1}$.

The twist Y^σ is equipped with a map $f^\sigma\colon Y^\sigma \to X$ which makes it a right torsor under the inner form G^σ. For example if G is abelian, then $G^\sigma = G$ and $[Y^\sigma] = [Y] - [\sigma]$. If σ and σ' are cohomologous cocycles, then Y^σ and $Y^{\sigma'}$ are isomorphic but in general not canonically.

1.4. LIENS

We recall some known facts about liens and non-abelian H^2. A convenient down-to-earth introduction is ([FSS], Section 1). Definitions of liens and non-abelian H^2 in a very general context can be found in [Gi], IV. The non-abelian H^2 naturally appears in the classification of group extensions [Mc].

Let $\overline{G}$ be an algebraic $\bar{k}$-group with unit element $\bar{e}$. We have an exact sequence of topological groups*

$$1 \to \mathtt{Aut}^{gr}(\overline{G}) \to \mathtt{SAut}^{gr}(\overline{G}) \to \Gamma \tag{1}$$

Let $\mathtt{Inn}\,\overline{G}$ be the group of inner automorphisms of the algebraic $\bar{k}$-group $\overline{G}$. We set $\mathtt{Out}\,\overline{G} = \mathtt{Aut}^{gr}(\overline{G})/\mathtt{Inn}\,\overline{G}$ (resp. $\mathtt{SOut}\,(\overline{G}) = \mathtt{SAut}^{gr}(\overline{G})/\mathtt{Inn}\,\overline{G}$).

The natural action of $\mathtt{SAut}^{gr}(\overline{G})$ on $\overline{G}(\bar{k})$ induces a canonical map $\mathtt{SOut}\,(\overline{G}) \to \mathrm{Out}\,(\overline{G}(\bar{k}))$. By Lemma 1.4, the k-forms of $\overline{G}$ are in natural bijection with the continuous splittings $\Gamma \to \mathtt{SAut}^{gr}(\overline{G})$ of (1).

The sequence (1) modulo $\mathtt{Inn}\,(\overline{G})$ gives rise to

$$1 \to \mathtt{Out}\,(\overline{G}) \to \mathtt{SOut}\,(\overline{G}) \to \Gamma \tag{2}$$

DEFINITION 1 ([FSS]). A *k-lien*** on $\overline{G}$ is a splitting $\kappa\colon \Gamma \to \mathtt{SOut}\,(\overline{G})$ of (2), which lifts to a continuous map (not necessarily a homomorphism) $\Gamma \to \mathtt{SAut}^{gr}(\overline{G})$. The lien κ is called *trivial*‡ if κ lifts to a continuous homomorphic map $\Gamma \to \mathtt{SAut}^{gr}(\overline{G})$.

A k-form G of $\overline{G}$ defines a trivial k-lien which we denote by lien(G). If G and G' are k-forms of $\overline{G}$, then $\mathrm{lien}(G) = \mathrm{lien}(G')$ if and only if G' is an inner form of G.

The second Galois cohomology set $H^2(k, \overline{G}, \kappa)$ is defined in terms of cocycles; it contains a distinguished subset of *neutral elements* ([FSS], (1.17) and (1.25)).

*In this paper the homomorphisms between topological groups are not necessarily continuous.
**Or k-band, or k-kernel.
‡*Representable* in the terminology of [Gi].

EXAMPLE 1.10. When k is either a non-Archimedean local field or a totally imaginary number field, and $\overline{G}$ is semi-simple, Douai [D] proved that all elements of $H^2(k, \overline{G}, \kappa)$ are neutral. If G is a k-form of $\overline{G}$, we set $H^2(k, G) := H^2(k, \overline{G}, \mathrm{lien}(G))$.

DEFINITION 1.11. An extension of topological groups

$$1 \to \overline{G}(\bar{k}) \xrightarrow{i} E \xrightarrow{q} \Gamma \to 1 \tag{3}$$

is called *compatible* with a lien κ if the maps are open onto their images (i.e. i is continuous and q is open), and the induced homomorphism $\Gamma \to \mathrm{Out}(\overline{G}(\bar{k}))$ is $\kappa\colon \Gamma \to \mathtt{SOut}(\overline{G})$ followed by the canonical map $\mathtt{SOut}(\overline{G}) \to \mathrm{Out}(\overline{G}(\bar{k}))$.

DEFINITION 1.12. We shall say that an exact sequence of topological groups (3) is *locally split* (being understood as locally in the étale topology) if there exists a finite field extension K/k such that the induced map $q_K\colon E_K \to \Gamma_K$ admits a continuous homomorphic section, where $\Gamma_K := \mathrm{Gal}(\bar{k}/K)$ and $E_K := E \cap q^{-1}(\Gamma_K)$.

The condition 'locally split' is usually quite easy to check, and will be satisfied in the examples we are going to consider. See Appendix A for more details about extensions of topological groups.

PROPOSITION 1.13 ([FSS], (1.19)). *The set $H^2(k, \overline{G}, \kappa)$ is in natural bijection with the equivalence classes of extensions of topological groups which are compatible with the lien κ. The neutral elements correspond to the extensions which admit a continuous homomorphic section $\Gamma \to E$. The set $H^2(k, \overline{G}, \kappa)$ contains neutral elements precisely when κ is trivial.*

Remark 1.14. Let κ be a k-lien on $\overline{G}$ and E be an extension which is compatible with κ. In the language of [Gi], the fibred category $\mathcal{G}$ such that for any finite extension K/k the fibre of $\mathcal{G}$ at $\mathrm{Spec}\, K$ consists of the sections of $q_K\colon E_K \to \Gamma_K$, is a k-gerb. By [Gi], VIII.6.2.5 and VIII.7.2.5, the cohomology class of such an extension in the sense of Proposition 1.13 is the cohomology class of the gerb of sections $\mathcal{G}$ of this extension in the sense of [Gi].

Remark 1.15. Let $\overline{Z}$ be the center of $\overline{G}$. Then a lien κ defines a k-form Z of $\overline{Z}$. Either $H^2(k, \overline{G}, \kappa)$ is empty or $H^2(k, Z)$ acts simply transitively on it ([Sp], 1.17). If $\overline{G}$ is abelian and $\kappa = \mathrm{lien}(G)$ is a k-lien on $\overline{G}$, then it is easy to check that $H^2(k, \overline{G}, \kappa)$ is just the usual Galois cohomology group $H^2(k, G)$ which has zero for a unique neutral element (see also [Gi], IV.3.4).

It is actually possible to use locally split sequences to define $H^2(k, \overline{G}, \kappa)$ in terms of extensions of groups (see Appendix A).

2. Obstruction Given by a Class in H^2

Until the end of this section, we fix a triple $(X, \overline{Y}, \overline{G})$, where $\overline{G}$ is an algebraic $\bar{k}$-group, X is a reduced and geometrically connected k-variety, and $\overline{Y}$ is a quasi-projective $\bar{k}$-variety equipped with a map $\bar{f}: \overline{Y} \to \overline{X}$ which makes $\overline{Y}$ a right torsor under $\overline{G}$.

2.1. DESCENT DATUM

The following sequence of topological groups is exact:

$$1 \to \mathtt{Aut}\,(\overline{Y}/\overline{X}) \to \mathtt{SAut}\,(\overline{Y}/X) \xrightarrow{q} \Gamma \tag{4}$$

Note that $\overline{G}(\bar{k})$ is a subgroup of $\mathtt{Aut}\,(\overline{Y}/\overline{X})$ via its right action on $\overline{Y}$. The weak topology on $\mathtt{Aut}\,(\overline{Y}/\overline{X})$ induces the discrete topology on $\overline{G}(\bar{k})$ because the stabilizer in $\overline{G}(\bar{k})$ of an arbitrary $\bar{k}$-point of $\overline{Y}$ is just $\{\bar{e}\}$.

DEFINITION 2.1. Let E be a topological subgroup of $\mathtt{SAut}\,(\overline{Y}/X)$. We shall say that E *satisfies the condition* (*) (with respect to $(X, \overline{Y}, \overline{G})$) if there exists a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{G}(\bar{k}) & \longrightarrow & E & \xrightarrow{q} & \Gamma & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathtt{Aut}\,(\overline{Y}/\overline{X}) & \longrightarrow & \mathtt{SAut}\,(\overline{Y}/X) & \xrightarrow{q} & \Gamma & & \end{array} \tag{$*$}$$

where the top row is an exact and locally split sequence of topological groups, and the map $\overline{G}(\bar{k}) \to \mathtt{Aut}\,(\overline{Y}/\overline{X})$ is the natural inclusion.

In Section 3, we shall review three situations where condition $(*)$ holds for a well chosen E, namely, the case of a connected torsor $\overline{Y}$ under a finite $\bar{k}$-group scheme $\overline{G}$, the case when X is a homogeneous space of a connected k-group, and the case of an abelian group $\overline{G}$.

We are now ready to define a k-lien attached to E satisfying $(*)$.

PROPOSITION 2.2. *Assume that* $E \subset \mathtt{SAut}\,(\overline{Y}/X)$ *satisfies* (*).

(1) *If* $g \mapsto \varphi_g$ *is a set-theoretic section of* $q: E \to \Gamma$, *then there exists a unique map* $\theta: \Gamma \to \mathtt{SAut}^{\mathrm{gr}}(\overline{G})$ *such that*

$$\varphi_g(\bar{m}.\bar{s}) = \varphi_g(\bar{m}).\theta_g(\bar{s}), \quad \bar{m} \in \overline{Y}(\bar{k}), \quad \bar{s} \in \overline{G}(\bar{k}), \quad g \in \Gamma. \tag{5}$$

(2) *The map* $\kappa_E\colon \Gamma \to \mathtt{SOut}\,(\overline{G})$ *induced by* θ *is independent on the set-theoretic section* φ. *It is a* k*-lien on* $\overline{G}$, *such that the exact sequence*

$$1 \to \overline{G}(\bar{k}) \to E \xrightarrow{q} \Gamma \to 1 \tag{6}$$

is compatible with it.

(3) *If the class* $\mathrm{Cl}(E)$ *of the sequence (6) in* $H^2(k, \overline{G}, \kappa_E)$ *is neutral, then there exists a* k*-form* G *of* $\overline{G}$ *with* $\mathrm{lien}(G) = \kappa_E$, *and a* G*-torsor* Y *over* X *such that the* $\overline{G}$*-torsor* $\overline{Y}$ *is obtained from* Y *by extension of scalars from* k *to* $\bar{k}$.

Proof. (1) Fix for each $g \in \Gamma$ an element φ_g of E such that $q(\varphi_g) = g$. Then for any $\bar{s} \in \overline{G}(\bar{k})$ the element $\varphi_g \circ \bar{s} \circ \varphi_g^{-1} \in E$ is in the kernel of q. Let us call it $\theta_g(\bar{s})$. It is clearly the only element of $\overline{G}(\bar{k})$ satisfying (5). By definition of a torsor the map $\Phi\colon (\bar{m}, \bar{s}) \mapsto (\bar{m}, \bar{m}.\bar{s})$ is an isomorphism from $\overline{Y} \times_{\bar{k}} \overline{G}$ to $\overline{Y} \times_{\overline{X}} \overline{Y}$. Fixing $\bar{m} \in \overline{Y}(\bar{k})$ we have $\theta_g(\bar{s}) = p_2[\Phi^{-1}(\varphi_g(\bar{m}), \varphi_g(\bar{m}.\bar{s}))]$, where $p_2\colon \overline{Y} \times_{\bar{k}} \overline{G} \to \overline{G}$ is the second projection. Therefore the map $\bar{s} \mapsto \theta_g(\bar{s})$ is indeed an element of $\mathtt{SAut}\,(\overline{G})$. Formula (5) now shows that it belongs to $\mathtt{SAut}^{\,\mathrm{gr}}(\overline{G})$.

(2) If we change the set-theoretic section φ, then φ_g is replaced by $\bar{t}_g \circ \varphi_g$ for some $\bar{t}_g \in \overline{G}(\bar{k})$. Then $\theta_g(\bar{s})$ is replaced by $\bar{t}_g\theta_g(\bar{s})\bar{t}_g^{-1}$. Therefore θ induces a well defined section $\kappa_E\colon \Gamma \to \mathtt{SOut}\,(\overline{G})$. For any $g, h \in \Gamma$ we have $\varphi_{gh} = \bar{u}_{g,h} \circ (\varphi_g \circ \varphi_h)$ for some $\bar{u}_{g,h} \in \overline{G}(\bar{k})$. Thus κ_E is a homomorphism. To show that κ_E is a lien, it remains to check the continuity condition (cf. [FSS], (1.7)); in particular, it is sufficient to show that one can choose the maps φ_g such that for each $\bar{m} \in \overline{Y}(\bar{k})$ the map $g \mapsto \varphi_g(\bar{m})$ is locally constant. But this follows from Appendix A since (6) is locally split by assumption, hence $q\colon E \to \Gamma$ admits a continuous set-theoretic section. The compatibility of (6) with κ_E is obvious.

(3) If a section $\varphi\colon \Gamma \to E$ of q is a continuous homomorphism, then the corresponding $\theta\colon \Gamma \to \mathtt{SAut}\,(\overline{G})$ is also a homomorphism. Then the lien κ_E is trivial, and θ defines a k-form G of $\overline{G}$ as the quotient of $\overline{G}$ by the continuous action of $\theta(\Gamma)$. Since $\overline{Y}$ is quasi-projective, we can also define Y as the quotient of $\overline{Y}$ by $\varphi(\Gamma)$. Now formula (5) shows that the right G-torsor Y is a k-form of the $\overline{G}$-torsor $\overline{Y}$. □

Remark 2.3. The condition that there exists $E \subset \mathtt{SAut}\,(\overline{Y}/X)$ which satisfies $(*)$ means that for any $g \in \Gamma$ the associated conjugate $\overline{X}^g$-torsor $\overline{Y}^g$ under $\overline{G}^g$ is isomorphic to the $\overline{X}$-torsor $\overline{Y}$ under $\overline{G}$. In the language of [DM] this is a descent datum on the $\overline{X}$-torsor $\overline{Y}$.

DEFINITION 2.4. Assume that E satisfies $(*)$, and that there exists a k-form (X, Y, G) of the right torsor $(\overline{X}, \overline{Y}, \overline{G})$. We shall say that this k-form is *compatible* with E if the map $\Gamma \to \mathtt{SAut}\,(\overline{Y}/X)$ (defined using the universal property of the fibred product $\overline{Y} = Y \times_X \overline{X}$) takes values in E.

Therefore the class $\mathrm{Cl}(E)$ can be regarded as the obstruction to descend the torsor $(\overline{X}, \overline{Y}, \overline{G})$ to a k-form compatible with E.

2.2. THE ELEMENTARY OBSTRUCTION

To a subgroup $E \subset \mathtt{SAut}(\overline{Y}/X)$ satisfying $(*)$ we associate an obstruction to the existence of a k-rational point on X.

THEOREM 2.5. *Let E be a subgroup of $\mathtt{SAut}(\overline{Y}/X)$ satisfying $(*)$. Assume that X contains a k-rational point x. Then for any $\bar{k}$-point $\bar{y}$ of $\overline{Y}$ above x, there exists a canonical continuous homomorphic section $j_{\bar{y}}$ of $q: E \to \Gamma$ uniquely defined by the property that the image of the induced section of $q: \mathtt{SAut}(\overline{Y}/X) \to \Gamma$ leaves $\bar{y}$ invariant. In particular, the class $\mathrm{Cl}(E)$ in $H^2(k, \overline{G}, \kappa_E)$ is neutral.*

Proof. Taking the push-out of (6) with respect to $i_x: x = \operatorname{Spec} k \hookrightarrow X$ we get a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{G}(\bar{k}) & \longrightarrow & E & \xrightarrow{q} & \Gamma & \longrightarrow & 1 \\ & & \Big\downarrow & & \Big\downarrow{\scriptstyle\alpha} & & \Big\| & & \\ 1 & \longrightarrow & \mathtt{Aut}(\overline{Y}_x/\bar{k}) & \longrightarrow & \mathtt{SAut}(\overline{Y}_x/k) & \xrightarrow{q} & \Gamma & \longrightarrow & 1 \end{array}$$

The fibre $\overline{Y}_x$ of $\overline{Y}$ at x is a $\bar{k}$-torsor under $\overline{G}$. Choosing a point $\bar{y} \in \overline{Y}_x(\bar{k})$ defines an isomorphism of right $\bar{k}$-torsors $\overline{G} \to \overline{Y}_x$. Since (6) is locally split, it admits a continuous set-theoretic section $\varphi: \Gamma \to E$ (cf. Appendix A). Then for every $g \in \Gamma$, there exists a unique $s_g \in \overline{G}(\bar{k})$ such that $\alpha(\varphi(g))(\bar{y}) = \bar{y}.s_g$. The map $g \mapsto s_g$ is locally constant. Define $j_{\bar{y}}: \Gamma \to E$ by $j_{\bar{y}}(g) = s_g^{-1} \circ \varphi(g)$. It is clear that $j_{\bar{y}}(g)$ is the only lifting of g which leaves $\bar{y}$ invariant. From this it follows that $j_{\bar{y}}$ is a homomorphic section of $E \to \Gamma$. It is continuous because φ and $g \mapsto s_g$ are continuous. $\square$

DEFINITION 2.6. Suppose that E satisfies $(*)$. By Theorem 2.5, the condition that the class $\mathrm{Cl}(E)$ is not neutral is an obstruction to the existence of a k-rational point on X. By analogy with the abelian case (see Subsection 3.4 below and [CS], 2.2.8) we call this the *elementary obstruction given by the quadruple* $c := (X, \overline{Y}, \overline{G}, E)$ (or by E if no confusion is possible).

Remark 2.7. Let $Y \to X$ be a G-torsor (defined over k), and let $m \in X(k)$ be such that the fibre Y_m of Y at m has no k-rational point. Then the section $j_{\bar{y}}$ associated to a geometric point $\bar{y} \in Y(\bar{k})$ lying over m defines a k-form $Y' = \overline{Y}/j_{\bar{y}}(\Gamma)$ which is *not* isomorphic to Y. (Y' contains a k-rational point lying over m.) Actually, Y' is isomorphic to the twist Y^{σ} of the torsor Y by a cocycle σ such that Y_m is a k-torsor under G defined by σ.

2.3. THE ELEMENTARY OBSTRUCTION OVER A NUMBER FIELD

Recall the following classical definitions:

DEFINITION 2.8. Let X be a smooth variety defined over a number field k. We shall say that X is a counterexample to the *Hasse principle* if $X(\mathbf{A}_k) \neq \emptyset$ and $X(k) = \emptyset$. If X is proper, it is said to satisfy *weak approximation* if $X(k)$ is dense in $X(\mathbf{A}_k) = \prod_{v \in \Omega} X(k_v)$ (equipped with the product of the v-adic topologies). If Σ is a finite subset of Ω and $X(k)$ is dense in $\prod_{v \notin \Sigma} X(k_v)$, then we shall say that *weak approximation outside* Σ holds for X.

Now let k be a number field and $(X, \overline{Y}, \overline{G})$ as in Subsection 2.1. Assume that $E \subset \mathtt{SAut}(\overline{Y}/X)$ satisfies $(*)$ and that $X(\mathbf{A}_k) \neq \emptyset$. Then by Theorem 2.5, the obstruction $\mathrm{Cl}(E)$ lies in the set of elements of $H^2(k, \overline{G}, \kappa_E)$ which become neutral for all completions of k. If $\mathrm{Cl}(E)$ is not neutral, we get an *elementary obstruction to the Hasse principle*.

Remark 2.9. A result by Borovoi ([Bo93], Proposition 6.5) shows that in the case when $\overline{G}$ is a connected linear algebraic group and κ is a k-lien on $\overline{G}$, an element of $H^2(k, \overline{G}, \kappa)$ which is locally neutral is neutral if and only if its image in $H^2(k, G^{\mathrm{tor}})$ is trivial, where G^{tor} is the k-form of the toric part of $\overline{G}$ defined by κ. Thus, the most interesting case of elementary obstruction to the Hasse principle for a non-abelian torsor is the case $\overline{G}$ finite.

3. Applications

In this section we review three realizations of the elementary obstruction treated in the literature. We also establish a link with the elementary obstruction of Colliot-Thélène and Sansuc.

3.1. TORSORS UNDER A FINITE GROUP SCHEME; GEOMETRIC FUNDAMENTAL GROUP

Let X be a reduced and geometrically connected k-variety equipped with a geometric point $\bar{x}$. In [Gr], IX.6.1, Grothendieck has proved that there is an exact sequence of profinite groups

$$1 \to \pi_1(\overline{X}, \bar{x}) \to \pi_1(X, \bar{x}) \to \Gamma \to 1 \qquad (\pi)$$

which is split if $X(k) \neq \emptyset$.

Let $\overline{Y}$ be a connected and Galois étale covering of $\overline{X}$ and take for $\overline{G}$ the constant $\bar{k}$-group scheme $\mathtt{Aut}(\overline{Y}/\overline{X})$ (as a scheme, it is a finite disjoint union of copies of $\mathrm{Spec}\,\bar{k}$). Chosing a point $\bar{y} \in \overline{Y}(\bar{k})$ above $\bar{x}$, we realize $\mathtt{Aut}(\overline{Y}/\overline{X})$ as a quotient of $\pi_1(\overline{X}, \bar{x})$. Take for E the whole of $\mathtt{Aut}(\overline{Y}/X)$ and assume that $q\colon \mathtt{Aut}(\overline{Y}/X) \to \Gamma$ is surjective (this is equivalent to saying that the kernel of the corresponding map $\pi_1(\overline{X},\bar{x}) \to \overline{G}(\bar{k})$ is stable under the outer action of Γ); then, one comes to consider the obstruction related to the splitting of the push-out of (π) by the map $\pi_1(\overline{X},\bar{x}) \to \mathtt{Aut}\,(\overline{Y}/\overline{X}) = \overline{G}(\bar{k})$, namely the elementary obstruction associated to

$$1 \to \mathtt{Aut}\,(\overline{Y}/\overline{X}) \to \mathtt{Aut}\,(\overline{Y}/X) \to \Gamma \to 1 \tag{7}$$

This sequence is locally split because the étale covering $\overline{Y} \to \overline{X}$ admits a K-form for some finite field extension K/k. Thus we obtain a k-lien κ on $\overline{G}$ and a cohomology class in $H^2(k,\overline{G},\kappa)$. By Grothendieck's result (or by Theorem 2.5), this class is neutral if $X(k) \neq \emptyset$.

The general problem of determining whether a (possibly ramified) covering $\overline{Y}$ is defined over k has been studied by Dèbes and Douai in [DD].

DEFINITION 3.1. We shall refer to the fact that the sequence (π) does not split as to the *fundamental obstruction* to the existence of a k-rational point on X. Considering the push-forward of (π) by the map $\pi_1(\overline{X},\bar{x}) \to \pi_1^{\mathrm{ab}}(\overline{X})$

$$1 \to \pi_1^{\mathrm{ab}}(\overline{X}) \to P \to \Gamma \to 1 \tag{π^{ab}}$$

we shall speak of the *abelianized fundamental obstruction* if the sequence (π^{ab}) does not split.

If $\bar{f}\colon \overline{Y} \to \overline{X}$ is a connected torsor under a finite $\bar{k}$-group scheme $\overline{G}$, then $\bar{f}$ is an étale Galois covering ([Mi80], I.5.4). Thus if $\overline{G}$ is finite, our general framework essentially reduces to considering torsors arising from the geometric fundamental group of X.

Remark 3.2. It can happen that for an étale connected Galois covering $\overline{Y} \to \overline{X}$ the map $\mathtt{Aut}\,(\overline{Y}/X) \to \Gamma$ is not surjective, in other words, the normal subgroup $\pi_1(\overline{Y},\bar{y}) \subset \pi_1(\overline{X},\bar{x})$ is not stable under the action $\Gamma \to \mathrm{Out}\,(\pi_1(\overline{X}))$. However, there always exists an étale connected Galois covering $\overline{Z} \to \overline{X}$ which factors through $\overline{Y} \to \overline{X}$ and such that $\mathtt{Aut}\,(\overline{Z}/X) \to \Gamma$ is surjective (take the intersection of the images of $\pi_1(\overline{Y},\bar{y})$ by Γ, this is an open normal subgroup of $\pi_1(\overline{X},\bar{x})$ which is stable under the outer action of Γ).

An example of this situation is an elliptic curve X containing two points of order 2 which are conjugate by an involution in Γ. The corresponding double étale coverings of $\overline{X}$ are conjugate varieties but they may not be isomorphic as $\bar{k}$-varieties.

3.2. THE CASE WHEN X IS A HOMOGENEOUS SPACE OF AN ALGEBRAIC k-GROUP

Let G be an algebraic k-group. Let X be a *left* * homogeneous space under G, that is, a k-variety equipped with a left action of G which is transitive on $X(\bar{k})$. Springer

*We have to take this convention to be consistent with Section 2.

([Sp], 1.20) has constructed a k-lien L_X canonically associated to X and a class $\alpha_X \in H^2(k, L_X)$ which is neutral if and only if X is dominated by a principal homogeneous space under G (in particular, $X(k) \neq \emptyset$ implies the neutrality of α_X). This construction was taken up by Borovoi for the study of the Hasse principle on homogeneous spaces ([Bo93], 7.7), and also by Flicker, Scheiderer and Sujatha ([FSS], (5.2)). In this subsection, we want to reinterpret Springer's construction as a special case of our general set-up.

Fix a point $\bar{x}_0 \in X(\bar{k})$ and let $\overline{H}$ be the stabilizer of $\bar{x}_0$. Let us call Y the k-variety G (without the group structure) and put $\overline{Y} := Y \times_k \bar{k}$. Then the map $\bar{f}\colon \overline{Y} \to \overline{X}$ which sends $\bar{m}$ to $\bar{m}.\bar{x}_0$ makes $\overline{Y}$ a right $\overline{X}$-torsor under $\overline{H}$. There is also the natural left action of $\overline{G}$ on $\overline{Y}$ which makes $\overline{Y}$ a left $\bar{k}$-torsor under $\overline{G}$.

We define $\mathtt{SAut}_G(\overline{Y}/k)$ (resp. $\mathtt{SAut}_G(\overline{Y}/X)$) as the subgroup of $\mathtt{SAut}(\overline{Y}/k)$ (resp. of $\mathtt{SAut}(\overline{Y}/X)$) consisting of the elements φ which are compatible with the left action of G in the following sense:

$$\varphi(\bar{s}.\bar{m}) = (q(\varphi)(\bar{s})).\varphi(\bar{m}),$$

for any $\bar{s} \in \overline{G}(\bar{k}), \bar{m} \in \overline{Y}(\bar{k})$ (recall that $q\colon \mathtt{SAut}(\overline{Y}/k) \to \Gamma$ is the map defined in Subsection 1.2).

PROPOSITION 3.3. (1) *The subgroup* $E := \mathtt{SAut}_G(\overline{Y}/X)$ *of* $\mathtt{SAut}(\overline{Y}/X)$ *satisfies* $(*)$ (cf. 2.1) *with respect to* $(X, \overline{Y}, \overline{H})$.

(2) *The* k*-lien* κ_E *(resp. the class* $\mathrm{Cl}(E) \in H^2(k, \overline{H}, \kappa_E)$*) coincides with the lien* L_X *(resp. the class* α_X*) constructed by Springer.*

Proof. (1) The action of Γ on $\overline{Y}$ obviously defines a continuous and homomorphic section of the map $\mathtt{SAut}_G(\overline{Y}/k) \to \Gamma$. The kernel of this map consists of the automorphisms φ of the $\bar{k}$-variety $\overline{Y}$ satisfying $\varphi(\bar{s}.\bar{m}) = \bar{s}.\varphi(\bar{m})$ for any $(\bar{s}, \bar{m}) \in \overline{G} \times \overline{Y}$. Taking for $\bar{m}$ the unit element of $\overline{G}$, we see that this kernel is $\overline{G}(\bar{k})$ acting on the right on $\overline{Y}$. Thus the following sequence is exact and splits:

$$1 \to \overline{G}(\bar{k}) \to \mathtt{SAut}_G(\overline{Y}/k) \xrightarrow{q} \Gamma \to 1$$

By definition, an element $\varphi_g \in \mathtt{SAut}(\overline{Y}/k)$ where $g = q(\varphi_g)$, belongs to $\mathtt{SAut}(\overline{Y}/X)$ if and only if it satisfies $\bar{f}(\varphi_g(\bar{m})) = g(\bar{f}(\bar{m}))$ which is the same as $\varphi_g(\bar{m}).\bar{x}_0 = g(\bar{m}.\bar{x}_0)$, for any $\bar{m} \in \overline{Y}(\bar{k})$. Taking g to be the unit element of Γ we see that the set of elements of $\overline{G}(\bar{k}) \subset \mathtt{SAut}_G(\overline{Y}/k)$ which belong to $\mathtt{SAut}_G(\overline{Y}/X)$ is just $\overline{H}(\bar{k})$. It remains to show that the map $\mathtt{SAut}_G(\overline{Y}/X) \to \Gamma$ is surjective. For any $g \in \Gamma$ there exists $\bar{s}_g \in \overline{G}(\bar{k})$ such that $g(\bar{x}_0) = \bar{s}_g.\bar{x}_0$ (because X is a homogeneous space under G). Put $\varphi_g(\bar{m}) = g(\bar{m})\bar{s}_g$, then $\varphi_g \in \mathtt{SAut}_G(\overline{Y}/k)$ and $\varphi_g(\bar{m}).\bar{x}_0 = g(\bar{m}).(\bar{s}_g.\bar{x}_0) = g(\bar{m}).g(\bar{x}_0) = g(\bar{m}.\bar{x}_0)$, hence φ_g is a lifting of g to $\mathtt{SAut}_G(\overline{Y}/X)$. Therefore, the following sequence is exact:

$$1 \to \overline{H}(\bar{k}) \to \mathtt{SAut}_G(\overline{Y}/X) \xrightarrow{q} \Gamma \to 1 \tag{11}$$

It is locally split because it is split over K as soon as $\bar{x}_0$ is K-rational. Thus E satisfies $(*)$ with respect to $(X, \overline{Y}, \overline{H})$.

(2) By Proposition 2.2, the lien κ_E corresponds to the map $\theta\colon \Gamma \to \mathtt{SAut}\,(\overline{H})$ defined by $\varphi_g(\bar{m}.\bar{h}) = \varphi_g(\bar{m}).\theta_g(\bar{h})$, for any $\bar{h} \in \overline{H}(\bar{k})$ and $\bar{m} \in \overline{Y}(\bar{k})$, where φ_g is a lifting of g to $\mathtt{SAut}_G(\overline{Y}/X)$.

By definition* ([Sp], 1.20; [FSS], (5.1)), the map $f_g\colon \Gamma \to \mathtt{SAut}^{\mathrm{gr}}(\overline{H})$ defining L_X is given by $f_g(\bar{h}) = \bar{s}_g^{-1}.g(\bar{h}).\bar{s}_g$, with $\bar{s}_g.\bar{x}_0 = g.\bar{x}_0$. Choose the lifting φ_g defined by $\varphi_g(\bar{m}) = g(\bar{m})\bar{s}_g$. Then $\varphi_g(\bar{m}.\bar{h}) = g(\bar{m})g(\bar{h})\bar{s}_g = \varphi_g(\bar{m}).f_g(\bar{h})$, so the lien L_X coincides with κ_E. According to ([FSS], (5.1)) the class α_X is given by the extension

$$1 \to \overline{H}(\bar{k}) \to E' \to \Gamma \to 1,$$

where E' is the subgroup of the semi-direct product $\overline{G}(\bar{k}) \rtimes \Gamma$ consisting of the products $\bar{s}g$ such that $\bar{s}.\bar{x}_0 = g.\bar{x}_0$. The elements of $E = \mathtt{SAut}_G(\overline{Y}/X) \subset \mathtt{SAut}_G(\overline{Y}/k) = \overline{G}(\bar{k}) \rtimes \Gamma$ are precisely the products $\varphi_g = \bar{s}g$ satisfying $\varphi_g(\bar{m}).\bar{x}_0 = g(\bar{m}.\bar{x}_0)$. Taking $\bar{m}$ to be the unit element we obtain that $E = E'$ and $\mathrm{Cl}(E) = \alpha_X$. □

Here are two useful special cases:

EXAMPLE 3.4. Let X be a homogeneous space under G and assume that $H^1(k, G)$ is trivial (e.g. $G = \mathrm{GL}_n$, $G = \mathrm{SL}_n$, or G is a semi-simple simply connected group over a non-archimedean local field or a totally imaginary number field). Then $X(k) \neq \emptyset$ if and only if the class $\alpha_X = \mathrm{Cl}(E)$ is neutral in $H^2(k, \overline{H}, L_X) = H^2(k, \overline{H}, \kappa_E)$.

PROPOSITION 3.5. *Let N and Z be two algebraic k-groups with Z central in N. Let X be a k-torsor under the algebraic k-group N/Z given by a cocycle $\xi \in Z^1(k, N/Z)$. Let us consider X as a homogeneous space of N. Then $\alpha_X = \delta([\xi])$, where $[\xi]$ is the class of ξ in $H^1(k, N/Z)$, and $\delta\colon H^1(k, N/Z) \to H^2(k, Z)$ is the connecting map.*

Proof. Let $c\colon \Gamma \to N$ be a continuous cochain which lifts ξ. There is a $\bar{k}$-point $\bar{x}_0$ on X such that $g.\bar{x}_0 = c_g.\bar{x}_0$. By ([FSS], (5.1)) the class α_X is given by the 2-cocycle $h_{g,t} = c_g g(c_t)c_{gt}^{-1}$. By [Se94], I.5.6 and I.5.7, the class of h in $H^2(k, Z)$ is just the image of ξ by the connecting map. □

3.3. THE ABELIAN CASE

Let X be a reduced and geometrically connected k-variety and $\overline{Y}$ an $\overline{X}$-torsor under a linear *commutative* algebraic $\bar{k}$-group $\overline{S}$. Any k-lien on $\overline{S}$ defines a unique k-form S of $\overline{S}$.

DEFINITION 3.6. Let S be a k-form of $\overline{S}$. We denote by $\mathtt{SAut}_S(\overline{Y}/X)$ the subgroup of $\mathtt{SAut}\,(\overline{Y}/X)$ consisting of the elements φ satisfying $\varphi(\bar{m}.\bar{s}) = \varphi(\bar{m}).(q(\varphi)(\bar{s}))$ for any $\bar{m} \in \overline{Y}(\bar{k})$, $\bar{s} \in \overline{S}(\bar{k})$, where the Galois group acts on $\overline{S} = S \times_k \bar{k}$ via the second factor.

*For a right homogeneous space the formula would have been $f_g = \mathrm{int}(\bar{s}_g) \circ g$.

The kernel of the natural map $q : \mathtt{SAut}_S(\overline{Y}/X) \to \Gamma$ is the group of $\overline{X}$-automorphisms of $\overline{Y}$ which commute with the action of $\overline{S}$. Since $\overline{S}$ is abelian, this is just $\overline{S}(\overline{X})$ (cf. [Gi], III.1.4.8, III.1.5.7). Thus the following sequence is exact:

$$1 \to \overline{S}(\overline{X}) \to \mathtt{SAut}_S(\overline{Y}/X) \to \Gamma \tag{9}$$

Since any unipotent abelian group is cohomologically trivial ([Se94], II.1.2, Proposition 1), let us further assume that S is of multiplicative type. Denote by $\hat{S}$ the Γ-module dual to S. Under the hypothesis $\bar{k}[X]^* = \bar{k}^*$, which is used in the abelian descent theory ([CS], [Sk]), the two following sequences are exact and canonically isomorphic:

$$H^1(k, S) \to H^1(X, S) \xrightarrow{\chi} \mathrm{Hom}_\Gamma(\hat{S}, \mathrm{Pic}\,\overline{X}) \xrightarrow{\partial} H^2(k, S) \to H^2(X, S), \tag{10}$$

$$H^1(k, S) \to H^1(X, S) \to H^0(k, H^1(\overline{X}, \overline{S})) \xrightarrow{\delta} H^2(k, S) \to H^2(X, S). \tag{11}$$

Both sequences are functorial in k, X, and S. The first one was introduced by Colliot-Thélène and Sansuc ([CS], 1.5.1) as the exact sequence of low degree terms of the spectral sequence $Ext^p_\Gamma(\hat{S}, H^q(\overline{X}, \mathbf{G}_m)) \Rightarrow H^{p+q}(X, S)$. The second one is obtained from the Leray spectral sequence $H^p(k, H^q(\overline{X}, \overline{S})) \Rightarrow H^{p+q}(X, S)$ using the assumption $\overline{S}(\overline{X}) = \overline{S}(\bar{k})$ (which follows from $H^0(\overline{X}, \mathbf{G}_m) = \bar{k}^*$ because S is of multiplicative type). The comparison of the two spectral sequences is carried out in Appendix B (if S is not a torus, then these two spectral sequences do not necessarily coincide).

Recall that the *type* of an X-torsor Y under S is the image $\chi([Y])$ of $[Y]$ in $\mathrm{Hom}_\Gamma(\hat{S}, \mathrm{Pic}\,\overline{X})$. The following result shows that the obstruction O'_λ (by definition this is the class $\partial(\lambda)$) of Colliot-Thélène and Sansuc ([CS], 2.2.8) is a particular case of the elementary obstruction defined as in Section 2.

PROPOSITION 3.7. *Let X be a reduced and geometrically connected k-variety such that $H^0(\overline{X}, \mathbf{G}_m) = \bar{k}^*$, and let S be a k-group of multiplicative type.*

(1) *For any $\overline{X}$-torsor $\overline{Y}$ under $\overline{S}$, the map q: $\mathtt{SAut}_S(\overline{Y}/X) \to \Gamma$ is surjective if and only if the image $\chi([\overline{Y}])$ of $[\overline{Y}] \in H^1(\overline{X}, \overline{S})$ in $\mathrm{Hom}(\hat{S}, \mathrm{Pic}\,\overline{X})$ is Galois equivariant.*
(2) *The set of $\overline{X}$-torsors $\overline{Y}$ under $\overline{S}$ (up to isomorphism) such that $E := \mathtt{SAut}_S(\overline{Y}/X)$ satisfies (*) is naturally identified with the group $\mathrm{Hom}_\Gamma(\hat{S}, \mathrm{Pic}\,\overline{X})$.*
(3) *Let $\lambda \in \mathrm{Hom}_\Gamma(\hat{S}, \mathrm{Pic}\,\overline{X})$, and let $\overline{Y} \to \overline{X}$ be the corresponding torsor under $\overline{S}$. Set $E := \mathtt{SAut}_S(\overline{Y}/X)$. Then the class $\mathrm{Cl}(E) \in H^2(k, S)$ coincides (up to a sign) with $\partial(\lambda)$. In particular, $\mathrm{Cl}(E) = 0$ if and only if $\partial(\lambda) = 0$. It is the unique obstruction for the existence of X-torsors under S of type λ.*

Proof. (1) The map q: $\mathtt{SAut}_S(\overline{Y}/X) \to \Gamma$ is surjective if and only if for any $g \in \Gamma$ the $\overline{X}^g$-torsor $\overline{Y}^g$ under $\overline{S}^g$ is isomorphic to the $\overline{X}$-torsor $\overline{Y}$ under $\overline{S}$, the induced isomorphism $\overline{S}^g \to \overline{S}$ being defined by the form S. This is equivalent to saying that the class $[\overline{Y}] \in H^1(\overline{X}, \overline{S})$ is Galois equivariant. Now the result follows from the fact that the exact sequences (10) and (11) coincide.

(2) Since sequence (10) is functorial in k, on considering it over $\bar{k}$ we see that the Γ-modules $H^1(\overline{X}, \overline{S})$ and $\mathrm{Hom}(\hat{S}, \mathrm{Pic}\,\overline{X})$ are isomorphic. Now the result follows from (1) and from the exact sequence (11) because $\overline{S}(\overline{X}) = \overline{S}(\bar{k})$.

(3) The element $\partial(\lambda) \in H^2(k, S)$ is also the element $\delta(\lambda)$ obtained via the exact sequence (11). Let $D(\lambda)$ be the k-gerb associated to $\lambda \in H^0(k, H^1(\overline{X}, \overline{S}))$ as in ([Gi], V.3.1.6). By definition, for any finite extension $k \subset K \subset \bar{k}$, the fibre of $D(\lambda)$ at $\mathrm{Spec}\,K$ is the category of S_K-torsors over X_K (where $S_K := S \times_k K$, $X_K := X \times_k K$) which become isomorphic to the $\overline{S}$-torsor $\overline{Y}$ after the extension of scalars. By Proposition 2.2, the K-forms of the $\overline{S}$-torsor $\overline{Y}$ over $\overline{X}$ correspond to the sections of the exact sequence

$$1 \to \overline{S}(\bar{k}) \to E = \mathtt{SAut}_S(\overline{Y}/X) \to \Gamma \to 1. \tag{12}$$

Therefore $D(\lambda)$ is the gerb of sections of (12). By ([Gi], V.3.2.1) the cohomology class of the gerb $D(\lambda)$ coincides with $\delta(\lambda)$. On the other hand, the class $\mathrm{Cl}(E)$ of (12) is also the class of the gerb of sections of (12) (cf. Remark 1.14). □

3.4. THE ELEMENTARY OBSTRUCTION AND THE ABELIANIZED FUNDAMENTAL OBSTRUCTION

The aim of this subsection is to show that after replacing a variety by some dense open subset, the vanishing of the abelianized fundamental obstruction implies the vanishing of the elementary obstruction of Colliot-Thélène and Sansuc. When the geometric Picard group is of finite type, the two obstructions are essentially equivalent.

Let U be a geometrically connected and reduced k-variety, $\overline{U} := U \times_k \bar{k}$. We have an exact sequence of Γ-modules:

$$1 \to \bar{k}^* \to \bar{k}[U]^* \to \bar{k}[U]^*/\bar{k}^* \to 1. \tag{13}$$

Let R be the k-torus dual to the torsion-free Γ-module $\bar{k}[U]^*/\bar{k}^*$. We rewrite (13) as an extension of the Γ-modules $\widehat{R}$ by $\bar{k}^*$:

$$1 \to \bar{k}^* \to \bar{k}[U]^* \to \widehat{R} \to 1. \tag{$*_U$}$$

It is clear that a k-point of U (actually, even a 0-cycle of degree 1 on U) defines a splitting of $(*_U)$. Hence the class of this extension in $\mathrm{Ext}^1_k(\hat{R}, \bar{k}^*) = H^1(k, R)$ is an obstruction to the existence of k-points on U. The multiplication by n sequence $1 \to R[n] \to R \to R \to 1$ defines the boundary maps $\partial_n\colon H^1(k, R) \to H^2(k, R[n])$. We thus get a family of elements $\partial_n([*_U]) \in H^2(k, R[n])$.

On the other hand, we have the abelianized fundamental exact sequence

$$1 \to \pi_1^{\mathrm{ab}}(\overline{U}) \to P \to \Gamma \to 1. \tag{π^{ab}}$$

Recall (cf. [KL]) that $\pi_1^{\mathrm{ab}}(\overline{U})$ with its Γ-module structure is

$$\pi_1^{\mathrm{ab}}(\overline{U}) = \varprojlim \pi_1^{\mathrm{ab}}(\overline{U})/n = \varprojlim \mathrm{Hom}(H^1(\overline{U}, \mu_n), \bar{k}^*).$$

The Kummer sequence gives an exact sequence of Γ-modules

$$0 \to \hat{R}/n \to H^1(\overline{U}, \mu_n) \to \mathrm{Pic}\,(\overline{U})[n] \to 0.$$

Therefore we get a surjective map of Γ-modules

$$\pi_1^{\mathrm{ab}}(\overline{U}) \to \mathrm{Hom}(\hat{R}/n, \bar{k}^*) = R[n]. \tag{14}$$

(It is surjective because $\mathrm{Ext}^1_{\mathbf{Z}}(\cdot, \bar{k}^*) = 0$ since $\bar{k}^*$ is divisible). Fix a geometric point $\bar{u}$ of U. We define $\overline{Z}_n \to \overline{U}$ as the connected étale covering corresponding to the group homomorphism $\pi_1(\overline{U}, \bar{u}) \to R(\bar{k})[n]$ with Galois invariant kernel. We get a commutative diagram of group extensions

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\overline{U}, \bar{u}) & \longrightarrow & \pi_1(U, \bar{u}) & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & R(\bar{k})[n] & \longrightarrow & \mathtt{Aut}(\overline{Z}_n/U) & \longrightarrow & \Gamma & \longrightarrow & 1.
\end{array} \tag{15}$$

Note that the Γ-module structure on $R(\bar{k})[n] = \mathtt{Aut}\,(\overline{Z}_n/\overline{U})$ given by the lower row of (15) is its usual Γ-module structure.

THEOREM 3.8. *The class of the lower extension in (15) in $H^2(k, R[n])$ coincides with $\partial_n([*_U])$. The class $[*_U]$ vanishes if and only if the push-out of the extension (π^{ab}) by the map (14) is split for every n.*

Proof. The second assertion follows from the first one since $H^1(k, R)$ contains no divisible elements : indeed, let K be a finite extension of k over which R is isomorphic to $\mathbf{G}_m^r$. Now $H^1(K, \mathbf{G}_m^r) = 0$ by Hilbert's Theorem 90, and we conclude by a restriction-corestriction argument. $\square$

We need the following general lemma:

LEMMA 3.9. *Let Y be a k-torsor under R such that $[Y] = -[*_U] \in H^1(k, R)$. Then there exists a morphism $v: U \to Y$ such that $v^*: \bar{k}[Y]^* \to \bar{k}[U]^*$ is an isomorphism. In particular, v^* induces an equivalence of $(*_Y)$ and $(*_U)$.*

Proof. By Rosenlicht's lemma $\bar{k}[Y]^*/\bar{k}^* = \hat{R}$ as abelian groups. Actually this is an isomorphism of Γ-modules (see [CS], 1.4). We get an extension of Γ-modules

$$1 \to \bar{k}^* \to \bar{k}[Y]^* \to \hat{R} \to 1 \tag{$*_Y$}$$

The class $[*_Y]$ in $\mathrm{Ext}^1_k(\hat{R}, \bar{k}^*) = H^1(k, R)$ is the opposite of $[Y]$ ([S], Section 6, [CS], 1.4.3), hence $[*_U] = [*_Y]$. Therefore there exists an isomorphism of Γ-modules

$\rho: \bar{k}[Y]^* \to \bar{k}[U]^*$, such that we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \bar{k}^* & \longrightarrow & \bar{k}[Y]^* & \longrightarrow & \widehat{R} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \| & & \\ 1 & \longrightarrow & \bar{k}^* & \longrightarrow & \bar{k}[U]^* & \longrightarrow & \widehat{R} & \longrightarrow & 1 \end{array}$$

It is a general fact that $\rho \in \mathrm{Hom}_k(\bar{k}[Y]^*, \bar{k}[U]^*)$ (homomorphisms of Γ-modules) uniquely defines $\tilde{\rho} \in \mathrm{Hom}_{\Gamma,\bar{k}-alg}(\bar{k}[Y], \bar{k}[U])$ (Γ-equivariant homomorphisms of $\bar{k}$-algebras) such that $\tilde{\rho}$ gives ρ on invertible elements. Indeed, as $\overline{Y} := Y \times_k \bar{k}$ is isomorphic to a finite number of copies of $\mathbf{G}_m$, $\bar{k}[Y]^*$ generates the $\bar{k}$-algebra $\bar{k}[Y]$, hence $\tilde{\rho}$ is uniquely determined by ρ. To show that for any ρ there exists some $\tilde{\rho}$ we reason as follows. A $\bar{k}$-point of Y realizes $\hat{R}$ inside $\bar{k}[Y]^*$ as functions χ which equal 1 at this point. As a $\bar{k}$-vector space, $\bar{k}[Y]$ is freely generated by the characters $\chi \in \hat{R}$. Thus ρ restricted to the subgroup $\hat{R} \subset \bar{k}[Y]^*$ is a homomorphism $\hat{R} \to \bar{k}[U]^*$. It uniquely extends to a morphism of $\bar{k}$-algebras $\tilde{\rho}: \bar{k}[Y] \to \bar{k}[U]$, then $\tilde{\rho}$ restricted to $\bar{k}[Y]^*$ is just ρ. It is clear that $\tilde{\rho}$ is Γ-equivariant since such is its restriction to $\bar{k}[Y]^*$. (We thank J.-L. Colliot-Thélène for his help with this argument.)

Now we define $v: U \to Y = \mathrm{Spec}\,(k[Y])$ as the morphism dual to the morphism of k-algebras $k[Y] = \bar{k}[Y]^\Gamma \to \bar{k}[U]^\Gamma = k[U]$ defined by $\tilde{\rho}$. □

We resume the proof of Theorem 3.8. To prove the theorem we can replace U by Y. Indeed, by Lemma 3.9 we have $[*_U] = [*_Y]$. On the other hand, the map (14) is the composition of $v_*: \pi_1^{\mathrm{ab}}(\overline{U}) \to \pi_1^{\mathrm{ab}}(\overline{Y})$ with $\pi_1^{\mathrm{ab}}(\overline{Y}) \to \pi_1^{\mathrm{ab}}(\overline{Y})/n = R(\bar{k})[n]$. The last equality is a canonical isomorphism by the Kummer sequence and the fact that $\mathrm{Pic}\,\overline{Y} = 0$. Let $\bar{y}$ be the image of $\bar{u}$ by v. It is clear that $\overline{Z}_n = U \times_Y \overline{Y}_n$, where $\overline{Y}_n$ is the unramified covering of $\overline{Y}$ corresponding to the surjection $\pi_1(\overline{Y}, \bar{y}) \to \pi_1^{\mathrm{ab}}(\overline{Y})/n$. We have an extension of abelian k-groups

$$1 \to R[n] \to R \to R \to 1.$$

The k-torsor Y under R can be viewed as a homogeneous space under R with stabilizer $R[n]$. Let α_Y be its class as defined by Springer (see Subsection 3.2). We observe that the exact sequence

$$1 \to \mathtt{Aut}\,(\overline{Y}_n/\overline{Y}) \to \mathtt{Aut}\,(\overline{Y}_n/Y) \to \Gamma \to 1 \tag{16}$$

is (8) of Subsection 3.2 with $\overline{H}(\bar{k}) = \mathtt{Aut}\,(\overline{Y}_n/\overline{Y}) = R[n](\bar{k})$. Indeed, since $\overline{R}$ is connected we have $\mathtt{SAut}\,(\overline{Y}_n/Y) = \mathtt{Aut}\,(\overline{Y}_n/Y)$, therefore $E = \mathtt{SAut}_R(\overline{Y}_n/Y)$ must coincide with this group. By Proposition 3.3 the class of (16) is the class α_Y. Now Proposition 3.5 tells us that $\partial_n([*_Y])$ coincides with α_Y. This finishes the proof. □

COROLLARY 3.10. *Let X be a smooth and geometrically integral variety over k such that $\bar{k}[X]^* = \bar{k}^*$. Assume that* $\mathrm{Pic}\,\overline{X}$ *is of finite type, and let S be the k-group which is dual to* $\mathrm{Pic}\,\overline{X}$. *Let $U \subset X$ be a dense open subset such that* $\mathrm{Pic}\,\overline{U} = 0$. *Consider* *the image* $\partial(\mathrm{Id}) \in H^2(k, S)$ *of* $\mathrm{Id} \in \mathrm{Hom}_\Gamma(\hat{S}, \mathrm{Pic}\,\overline{X})$. *Then* $\partial(\mathrm{Id}) = 0$ *if and only if the push-out of* (π^{ab}) *for* U *by the maps* $\pi^{\mathrm{ab}}(\overline{U}) \to \pi^{\mathrm{ab}}(\overline{U})/n$ *splits for all* n.

The class $\partial(\mathrm{Id})$ is the elementary obstruction for the existence of a k-point on X introduced by Colliot-Thélène and Sansuc ([CS], Proposition 2.2.8). Its vanishing is the necessary and sufficient condition for the existence of universal torsors on X (by definition, these are torsors of type Id).

Proof. Since $\mathrm{Pic}\,\overline{U} = 0$, we have an exact sequence of Γ-modules

$$1 \to \bar{k}[U]^*/\bar{k}^* \to \mathrm{Div}_{\overline{X}\setminus\overline{U}}(\overline{X}) \to \mathrm{Pic}\,\overline{X} \to 1$$

where $\mathrm{Div}_{\overline{X}\setminus\overline{U}}(\overline{X})$ is the group of divisors with support in $\overline{X} \setminus \overline{U}$. Consider the dual sequence of k-groups of multiplicative type

$$1 \to S \to Q \to R \to 1$$

Since $\mathrm{Div}_{\overline{X}\setminus\overline{U}}(\overline{X})$ is a permutation Γ-module, the connecting map $\delta\colon H^1(k, R) \to H^2(k, S)$ is injective by Shapiro's lemma. It is proved in [CS], pp. 417–419, that $\delta([*_U])$ coincides with $\partial(\mathrm{Id})$ up to a sign. It remains to apply Theorem 3.8. $\square$

4. Descent Obstructions

4.1. NON-ABELIAN DESCENT

Let $(X, \overline{Y}, \overline{G})$ be as in Section 2, and E a subgroup of $\mathtt{SAut}\,(\overline{Y}/X)$ satisfying $(*)$ (cf. 2.1). If $\mathrm{Cl}(E)$ is neutral, we can choose a continuous homomorphic section of $q\colon E \to \Gamma$. This defines k-forms G of $\overline{G}$ and Y of $\overline{Y}$ such that $f\colon Y \to X$ is a torsor under G which is a k-form of $\bar{f}\colon \overline{Y} \to \overline{X}$ (see Proposition 2.2 (3)).

We define a pairing $H^1(X, G) \times X(k) \to H^1(k, G)$ by taking the pull-back with respect to the inclusion of a k-point into X. If Y is an X-torsor under G and $P \in X(k)$ we denote it by $[Y].P$. This is the same thing as the class of the fibre of Y at P.

Let $\sigma \in Z^1(k, G)$. There is an obvious commutative diagram

$$\begin{array}{ccc} H^1(X, G) & \longrightarrow & H^1(X, G^\sigma) \\ {\scriptstyle .P}\downarrow & & \downarrow{\scriptstyle .P} \\ H^1(k, G) & \longrightarrow & H^1(k, G^\sigma) \end{array} \qquad (17)$$

where the horizontal maps are bijections which associate to a torsor its twist by σ. In particular, if $[Y].P = [\sigma]$, then $[Y^\sigma].P$ is the trivial element of $H^1(k, G^\sigma)$. In the abelian case, the inner form G^σ can be identified with G and the map $H^1(X, G) \to H^1(X, G^\sigma)$ is just the translation by $-[\sigma]$.

The following lemma is similar to classical descent statements:

LEMMA 4.1.

- *For any field extension K/k the subset $f^\sigma(Y^\sigma(K))$ of $X(K)$ depends only on the class $[\sigma] \in H^1(k, G)$.*
- *We have:* $X(k) = \bigcup_{[\sigma]\in H^1(k,G)} f^\sigma(Y^\sigma(k))$.

Proof. Let $P \in X(K)$. Then $P \in f^\sigma(Y^\sigma(K))$ if and only if $[Y^\sigma].P$ is trivial in $H^1(K, G^\sigma)$, that is if and only if $[\sigma] = [Y].P$ (by the commutativity of (17)). The last condition depends only on $[\sigma]$.

In particular, let $P \in X(k)$ and put $[\sigma] = [Y].P$; then $[Y^\sigma].P$ is trivial in $H^1(k, G^\sigma)$. Thus the fibre of Y^σ at P is a trivial torsor, and there exists a k-rational point Q of Y^σ such that $f^\sigma(Q) = P$. □

From now on let k be a number field. We suppose that $X(\mathbf{A}_k) \neq \emptyset$. 'Evaluating' $f: Y \to X$ at an adelic point of X gives a map

$$\theta_f: X(\mathbf{A}_k) \to \prod_{v\in\Omega} H^1(k_v, G).$$

Note that if G is linear, then the set $H^1(k_v, G)$ is finite ([Se94], III.4). For each $\sigma \in Z^1(k, G)$, we let σ_v denote its image in $Z^1(k_v, G)$. (This image is defined by first choosing a place w of $\overline{k}$ over v, and then restricting σ to the decomposition group D_w of w. The union of completions at w of finite subextensions of $\bar{k}$ is an algebraic closure of k_v, and D_w is its Galois group over k_v, cf. [Se94], II.6.1.).

DEFINITION 4.2. Let $f: Y \to X$ be a torsor under G. Define $X(\mathbf{A}_k)^f$ as the subset of $X(\mathbf{A}_k)$ consisting of adelic points whose image under θ_f comes from an element of $H^1(k, G)$

$$X(\mathbf{A}_k)^f = \bigcup_{[\sigma]\in H^1(k,G)} f^\sigma(Y^\sigma(\mathbf{A}_k)).$$

We have $X(k) \subset X(\mathbf{A}_k)^f \subset X(\mathbf{A}_k)$. The emptiness of $X(\mathbf{A}_k)^f$ is thus an obstruction to the existence of a k-point on X, that is, an obstruction to the Hasse principle. We shall call it *the descent obstruction* defined by $f: Y \to X$.

The motivation to introduce the descent obstruction in the non-abelian case is to refine the classical Brauer–Manin obstruction, as will become apparent in the explicit examples given in the last section of the paper. Note that if G is a k-group of multiplicative type, the diagonal image of $H^1(k, G)$ in the product $\prod_{v\in\Omega} H^1(k_v, G)$ is described by the Poitou–Tate exact sequence (cf. [Mi86], I.4.20 (b), I.4.13). There is a generalization, due to Kottwitz, of this sequence to the case when G is connected reductive (cf. [Bo98], Theorem 5.15):

$$H^1(k, G) \to \oplus_{v\in\Omega} H^1(k_v, G) \to (\pi_1(\overline{G})_\Gamma)_{\mathrm{tors}}$$

where $\pi_1(\overline{G})$ is the algebraic fundamental group of $\overline{G}$ (this is a Γ-module which is of finite type as an abelian group, see [Bo98], 1.4) and $\pi_1(\overline{G})_\Gamma$ is its coinvariant module. Here $\oplus_{v\in\Omega} H^1(k_v, G)$ is the subset of the product $\prod_{v\in\Omega} H^1(k_v, G)$ consisting of (ξ_v) such that $\xi_v = 1$ for almost all places v.

PROPOSITION 4.3. *Let $(X, \overline{Y}, \overline{G})$ be as in Section 2, and let E be a subgroup of* $\texttt{SAut}\,(\overline{Y}/X)$ *satisfying* (*) (cf. 2.1), *such that the class* $\mathrm{Cl}(E) \in H^2(k, \overline{G}, \kappa_E)$ *is neutral (that is, the elementary obstruction given by E is empty). Then for any k-form $f: Y \to X$ of the torsor $\overline{Y}$ compatible with E, the set $X(\mathbf{A}_k)^f$ depends only on $c := (X, \overline{Y}, \overline{G}, E)$ and not on f.*

Proof. If we change the homomorphic section $j: \Gamma \to E$, then the forms G and Y are respectively replaced by G^σ and Y^σ for some $\sigma \in Z^1(k, G)$. Indeed, the homomorphism $g \mapsto \theta_g$ defining the action of Γ on $\overline{G}$ is well defined up to conjugation, and $j_g(\bar{m}.\bar{s}) = j_g(\bar{m}).\theta_g(\bar{s})$ (with $Y = \overline{Y}/j(\Gamma)$ and $G = \overline{G}/\theta(\Gamma)$) by Proposition 2.2. □

Because of Proposition 4.3 we shall sometimes write $X(\mathbf{A}_k)^c$ for $X(\mathbf{A}_k)^f$ where $f : Y \to X$ is a k-form compatible with E, with the convention $X(\mathbf{A}_k)^c = \emptyset$ if $\mathrm{Cl}(E)$ is not neutral.

Now we are going to prove the following 'non-abelian' version of [Sk], Theorem 3(b).

PROPOSITION 4.4. *Let $f: Y \to X$ be a torsor under a linear algebraic group G and assume that X is a proper k-variety. Fix a finite set of places $\Sigma \subset \Omega$ and let $\mathbf{A}_k^\Sigma$ denote the image of $\mathbf{A}_k$ by the projection to $\prod_{v\notin\Sigma} k_v$. Then there are only finitely many classes $[\sigma] \in H^1(k, G)$ such that $Y^\sigma(\mathbf{A}_k^\Sigma) \neq \emptyset$.*

Proof. Let G^0 be the connected component (of unity) of G. Then $F = G/G^0$ is a finite k-group.

For a finite set of places $\Sigma' \supset \Sigma$ containing the Archimedean ones let $\mathcal{O}_{k,\Sigma'} \subset k$ be the ring of Σ'-integers of k (integers away from Σ'). Let us fix Σ' large enough so that the following properties hold:

G extends to a smooth group scheme $\mathcal{G}$ over $\mathrm{Spec}\,(\mathcal{O}_{k,\Sigma'})$,
X extends to a proper scheme $\mathcal{X}$ over $\mathrm{Spec}\,(\mathcal{O}_{k,\Sigma'})$,
Y extends to an $\mathcal{X}$-torsor $\mathcal{Y}$ under $\mathcal{G}$.

We denote by $\mathcal{G}^0$ and $\mathcal{F}$ the group schemes over $\mathrm{Spec}\,(\mathcal{O}_{k,\Sigma'})$ extending G^0 and F respectively. Up to enlarging Σ' we can assume that the these group schemes fit into an exact sequence

$$1 \to \mathcal{G}^0 \to \mathcal{G} \to \mathcal{F} \to 1$$

Let $[\sigma] \in H^1(k, G)$ be such that $Y^\sigma(\mathbf{A}_k^\Sigma) \neq \emptyset$. The condition $Y^\sigma(k_v) \neq \emptyset$ implies that there exists a k_v-point $M_v \in X(k_v)$ such that $[Y_{M_v}] = [\sigma_v]$. By the properness of $\mathcal{X}/\mathcal{O}_{k,\Sigma'}$ for all $v \notin \Sigma'$ we have $X(k_v) = \mathcal{X}(\mathcal{O}_v)$. By our choice of Σ', for all $v \notin \Sigma'$

the class $[\sigma_v]$ is the image of $[\mathcal{Y}_{M_v}]$ under the natural map $H^1(\mathcal{O}_v, \mathcal{G}) \to H^1(k_v, G)$. Thus the image of $[\sigma_v]$ in $H^1(k_v, F)$ comes from $H^1(\mathcal{O}_v, \mathcal{F})$ for all $v \notin \Sigma'$. The image of $[\sigma]$ in $H^1(k, F)$ can be represented by a k-torsor Z under F. This is a 0-dimensional k-scheme, hence $Z = \mathrm{Spec}\,(k[Z])$. The étale k-algebra $k[Z]$ is a product of field extensions of k. The fact that the image of $[\sigma_v]$ in $H^1(k_v, F)$ comes from $H^1(\mathcal{O}_v, \mathcal{F})$ implies that all of these fields are not ramified at all $v \notin \Sigma'$. The degrees of these extensions of k are bounded by $|F(\bar{k})|$. There are only finitely many extensions of k of bounded degree, which are unramified away from Σ' ([Lan86], V.4, Theorem 5). In particular, there exists a finite Galois field extension k'/k which contains all these extensions. Thus the image of $[\sigma]$ in $H^1(k, F)$ is contained in a finite subset (the image of $H^1(\mathrm{Gal}\,(k'/k), F)$ in $H^1(k, F)$), which we can take to be the image of a finite subset $\Phi \subset H^1(k, G)$ consisting of elements coming from $H^1(\mathcal{O}_v, \mathcal{G})$ for all $v \notin \Sigma'$. On replacing G with its twist by a cocycle representing a class in Φ, it is now enough to prove that the set of classes $[\sigma] \in H^1(k, G)$ going to zero in $H^1(k, F)$, and such that for all $v \notin \Sigma'$ we have $[\sigma_v] \in \mathrm{Im}\,[H^1(\mathcal{O}_v, \mathcal{G}) \to H^1(k_v, G)]$, is finite. Let $[\rho_v] \in H^1(\mathcal{O}_v, \mathcal{G})$ be a class mapping to $[\sigma_v]$. We claim that $[\rho_v]$ goes to zero in $H^1(\mathcal{O}_v, \mathcal{F})$. For this it is enough to show that the kernel of the map $H^1(\mathcal{O}_v, \mathcal{F}) \to H^1(k_v, F)$ is trivial. To prove this we observe that a $\mathrm{Spec}\,\mathcal{O}_v$-torsor under a finite (hence proper) group $\mathcal{F}$ is proper over $\mathrm{Spec}\,\mathcal{O}_v$, hence, by the valuative criterion of properness, a section over the generic point $\mathrm{Spec}\,k_v \subset \mathrm{Spec}\,\mathcal{O}_v$ extends to a section over the whole of $\mathrm{Spec}\,\mathcal{O}_v$. Therefore $[\rho_v]$ goes to zero in $H^1(\mathcal{O}_v, \mathcal{F})$, and hence comes from $H^1(\mathcal{O}_v, \mathcal{G}^0)$. However, every $\mathrm{Spec}\,\mathcal{O}_v$-torsor under the smooth and connected group $\mathcal{G}^0$ is trivial by Lang's theorem [Lan56] (which allows one to a find a rational point in the closed fibre) and Hensel's lemma (which allows one to lift it to a section over $\mathrm{Spec}\,\mathcal{O}_v$). Thus $H^1(\mathcal{O}_v, \mathcal{G}^0)$ is trivial, and $[\sigma_v] = 0$ for all $v \notin \Sigma'$. Since every set $H^1(k_v, G)$ is finite, $([\sigma_v])$ belongs to the finite subset of $\prod_{v \in \Omega \setminus \Sigma} H^1(k_v, G)$ consisting of (α_v) such that α_v is arbitrary for $v \in \Sigma' \setminus \Sigma$, and $\alpha_v = 0$ otherwise. By a theorem of Borel–Serre ([Se94], III.4.6; [BS], 7.1) the natural diagonal map $H^1(k, G) \to \prod_{v \in \Omega \setminus \Sigma} H^1(k_v, G)$ has finite fibres, hence the inverse image of our finite subset is also finite. Thus the set of classes $[\sigma] \in H^1(k, G)$ such that $Y^\sigma(k_v) \neq \emptyset$ for any $v \notin \Sigma$ is finite. □

4.2. OBSTRUCTIONS TO WEAK APPROXIMATION

Throughout this subsection we assume that X is a proper, smooth, and geometrically connected variety over a number field k, with $X(k) \neq \emptyset$. Let G be a linear algebraic k-group. Let $f : Y \to X$ be a torsor under G.

DEFINITION 4.5. Let Σ be a finite set of places of k. We let $X(\mathbf{A}_k^\Sigma)^f$ denote the subset of $X(\mathbf{A}_k^\Sigma)$ consisting of points whose image under the evaluation map $\theta_f^\Sigma \colon X(\mathbf{A}_k^\Sigma) \to \prod_{v \in \Omega \setminus \Sigma} H^1(k_v, G)$ comes from an element of $H^1(k, G)$. We have:

$$X(\mathbf{A}_k^\Sigma)^f = \bigcup_{[\sigma] \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbf{A}_k^\Sigma)).$$

Recall the following result (which is well known when G is abelian):

LEMMA 4.6. *Let k_v be a completion of k and $[Y] \in H^1(X, G)$. Then the evaluation map $\theta: X(k_v) \to H^1(k_v, G)$ induced by $[Y]$ is locally constant.*

Proof. (cf. [Du], II, (0.31)). Let $m \in X(k_v)$. We want to find a v-adic neighbourhood V of m such that θ is constant on V. Twisting G and Y by a cocycle representing $\theta(m)$ if necessary, we can assume that $\theta(m)$ is trivial. Let A be the Henselization of the local ring of X at m, then the restriction of $[Y]$ to $H^1(A, G)$ is trivial. Since A is the inductive limit of the étale ring extensions $B \supset O_{X,m}$ such that the fibre of $\operatorname{Spec} B \to \operatorname{Spec} O_{X,m}$ contains a point over $k(m) = k_v$, there exists an étale morphism $p: Z \to X$ such that $m \in p(Z(k_v))$ and the restriction $p^*[Y]$ of $[Y]$ to $H^1(Z, G)$ is trivial. By the implicit function theorem, the map $Z(k_v) \to X(k_v)$ induced by p is open. Therefore, its image contains a v-adic neighbourhood V of m. For any point $m' \in V$ there exists a k_v-point $n' \in Z(k_v)$ such that $p(n') = m'$. Then $[Y].m' = (p^*[Y]).n'$ is trivial. □

THEOREM 4.7. *Let $\overline{X(k)}^{\Sigma}$ be the closure of the image of $X(k)$ in $X(\mathbf{A}_k^{\Sigma})$, then $\overline{X(k)}^{\Sigma} \subset X(\mathbf{A}_k^{\Sigma})^f$.*

Proof. To begin with, let us show that $f(Y(\mathbf{A}_k^{\Sigma}))$ is a closed subset of $X(\mathbf{A}_k^{\Sigma})$. Let (M_v) be an adelic point of X which belongs to the closure of $f(Y(\mathbf{A}_k^{\Sigma}))$. Then for any place $v \notin \Sigma$, there exists a k_v-point P_v of Y such that $N_v := f(P_v)$ is arbitrary close to M_v. As $[Y].N_v$ is trivial, so is $[Y].M_v$ by Lemma 4.6. Similarly $f^{\sigma}(Y^{\sigma}(\mathbf{A}_k^{\Sigma}))$ is closed for any $\sigma \in Z^1(k, G)$ (this argument does not use the assumption X proper).

Since X is proper, Proposition 4.4 applies and $X(k)$ is a subset of $\bigcup_{[\sigma]\in\mathcal{S}} f^{\sigma}(Y^{\sigma}(\mathbf{A}_k^{\Sigma}))$, where $\mathcal{S}$ is a finite subset of $H^1(k, G)$. A finite union of closed subsets is closed, hence the closure of $X(k)$ in $X(\mathbf{A}_k^{\Sigma})$ is a subset of $\bigcup_{[\sigma]\in\mathcal{S}} f^{\sigma}(Y^{\sigma}(\mathbf{A}_k^{\Sigma}))$. □

COROLLARY 4.8. *The condition $X(\mathbf{A}_k)^f \neq X(\mathbf{A}_k)$ (resp. $X(\mathbf{A}_k^{\Sigma})^f \neq X(\mathbf{A}_k^{\Sigma})$) is an obstruction to weak approximation (resp. to weak approximation outside Σ) on X.*

We shall call this condition *the descent obstruction* to weak approximation (resp. to weak approximation outside Σ) associated to $f: Y \to X$.

4.3. RELATION WITH THE BRAUER–MANIN OBSTRUCTION

Recall that for any number field k, local class field theory gives an injective map $j_v: \operatorname{Br} k_v \to \mathbf{Q}/\mathbf{Z}$ which is an isomorphism for v finite. Let X be a smooth and geometrically integral k-variety, and $\operatorname{Br} X := H^2(X, \mathbf{G}_m)$ the cohomological Brauer group of X. Set

$$X(\mathbf{A}_k)^{\mathrm{Br}} = \left\{ (M_v) \in X(\mathbf{A}_k), \quad \forall A \in \mathrm{Br}\, X, \quad \sum_{v \in \Omega} j_v(A(P_v)) = 0 \right\}.$$

(The sum is well-defined, [CS], III.) The reciprocity law of global class field theory implies $X(k) \subset X(\mathbf{A}_k)^{\mathrm{Br}}$. In particular, the condition $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$ is an obstruction to the existence of a k-rational point on X, this is the *Manin or Brauer–Manin obstruction.*

If X is proper, we let $\overline{X(k)}$ denote the closure of $X(k)$ embedded diagonally in $X(\mathbf{A}_k) = \prod_{v \in \Omega} X(k_v)$. In this case we have $\overline{X(k)} \subset X(\mathbf{A}_k)^{\mathrm{Br}}$, and the condition $X(\mathbf{A}_k)^{\mathrm{Br}} \neq X(\mathbf{A}_k)$ is the *Brauer–Manin obstruction to weak approximation.*

If B is a subset of $\mathrm{Br}\, X$, we set

$$X(\mathbf{A}_k)^{B} = \left\{ (M_v) \in X(\mathbf{A}_k), \quad \forall A \in B, \quad \sum_{v \in \Omega} j_v(A(P_v)) = 0 \right\}.$$

We also set $\mathrm{Br}_1 X := \mathrm{Ker}\,[\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X}]$.

We now reformulate Theorem 3 (a) of [Sk] (which is an extension of one of the main results of [CS]) as the equivalence of the Brauer–Manin obstruction related to $\mathrm{Br}_1 X$ and the descent obstructions given by all torsors under groups of multiplicative type. More precisely:

THEOREM 4.9. *Let X be a smooth and geometrically integral variety over a number field k such that $\bar{k}[X]^* = \bar{k}^*$ (e.g. X proper). Denote by $\mathcal{T}_{\mathrm{ab}}$ the set of quadruples $c = (X, \overline{Y}, S, E)$ such that S is a k-group of multiplicative type, $\overline{Y}$ is an $\overline{X}$-torsor under $\overline{S}$, and $E = \mathtt{SAut}_S(\overline{Y}/X)$ satisfies* (*) (cf. 2.1). *Then:*

$$X(\mathbf{A}_k)^{\mathrm{Br}_1 X} = \bigcap_{c \in \mathcal{T}_{\mathrm{ab}}} X(\mathbf{A}_k)^c.$$

(Recall that $X(\mathbf{A}_k)^c$ has been defined after the proof of Proposition 4.3).

Proof. Let $r\colon \mathrm{Br}_1 X \to H^1(k, \mathrm{Pic}\, \overline{X})$ be the map given by the Hochschild–Serre spectral sequence ([CS], (1.5)):

$$0 \to \mathrm{Pic}\, X \to (\mathrm{Pic}\, \overline{X})^{\Gamma} \to \mathrm{Br}\, k \to \mathrm{Br}_1 X \xrightarrow{r} H^1(k, \mathrm{Pic}\, \overline{X}) \to 0$$

By ([Se94], I.2.2, Corollary 2) for any $\alpha \in \mathrm{Br}_1 X$ there exists a Γ-submodule $\lambda\colon M \hookrightarrow \mathrm{Pic}\, \overline{X}$ of finite type such that $r(\alpha) \in \lambda_*(H^1(k, M))$. Thus $X(\mathbf{A}_k)^{\mathrm{Br}_1 X} = \cap_\lambda X(\mathbf{A}_k)^{\mathrm{Br}_\lambda}$, where $\mathrm{Br}_\lambda = r^{-1}\lambda_*(H^1(k, M)) \subset \mathrm{Br}_1 X$. Let S be the k-group dual to M, and let $\overline{Y}$ be an $\overline{X}$-torsor of type λ. We have $\bar{k}[X]^* = \bar{k}^*$, then $E := \mathtt{SAut}_S(\overline{Y}/X)$ satisfies (*) by Proposition 3.7. Set $c = (X, \overline{Y}, \overline{S}, E)$, then $X(\mathbf{A}_k)^{\mathrm{Br}_\lambda} = X(\mathbf{A}_k)^c$ by [Sk], Theorem 3 (a) (note that $X(\mathbf{A}_k)^{\mathrm{Br}_\lambda}$ is empty if the elementary obstruction does not disappear, which is a nontrivial fact: see [Sk], proof of Theorem 3 (a)). This finishes the proof. $\square$

The set $X(\mathbf{A}_k)^{\mathrm{Br}}$ can be empty even if $X(\mathbf{A}_k)^{\mathrm{Br}_1 X}$ is not, because elements of $\mathrm{Br}\, X$ which are not killed in $\mathrm{Br}\, \overline{X}$ can give a Brauer–Manin obstruction; see [Ha96] for an example of this situation. Now we are going to show that such a 'transcendental' Manin obstruction, at least when it is realized by an Azumaya algebra on X, is still equivalent to a descent obstruction provided one uses non-abelian torsors.

We denote by $\mathrm{Br}_{\mathrm{Az}}\, X$ the Brauer group of X, defined as the group of similarity classes of Azumaya algebras on X. By a theorem of Grothendieck there is an injection $\mathrm{Br}_{\mathrm{Az}}\, X \to \mathrm{Br}\, X$. More precisely, the exact sequence of étale sheaves

$$1 \to \mathbf{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

gives rise to the exact sequence of pointed sets

$$H^1(X, \mathbf{G}_m) \to H^1(X, \mathrm{GL}_n) \to H^1(X, \mathrm{PGL}_n) \xrightarrow{d} \mathrm{Br}\, X,$$

and $\mathrm{Br}_{\mathrm{Az}}\, X$ is the union (for $n > 0$) of the images of $H^1(X, \mathrm{PGL}_n)$ by d (cf. [Mi80], IV.2.5). It is conjectured that in fact $\mathrm{Br}_{\mathrm{Az}}\, X = \mathrm{Br}\, X$. (There are results by O. Gabber and R. Hoobler in this direction.) For any field K it is well known that $\mathrm{Br}\, K = \mathrm{Br}_{\mathrm{Az}}\, K$; moreover the map $d: H^1(K, \mathrm{PGL}_n) \to {}_n\mathrm{Br}\, K$ is injective ([Se68], X.5). Note that for any field K of characteristic zero, we have ${}_n\mathrm{Br}\, K = H^2(K, \mu_n)$ by Kummer theory, and μ_n is the fundamental group of PGL_n. If K is a number field or the completion of a number field, then the map $d: H^1(K, \mathrm{PGL}_n) \to H^2(K, \mu_n) = {}_n\mathrm{Br}\, K$ is an isomorphism (this is also well-known, and can be viewed as a special case of [Bo98], 5).

We want to relate the Brauer–Manin obstruction associated to the subgroup $\mathrm{Br}_{\mathrm{Az}}\, X$ of $\mathrm{Br}\, X$ to the obstruction defined by X-torsors under the groups PGL_n, $n = 1, 2, \ldots$. Let $\mathcal{T}_{\mathrm{PGL}_n}$ denote the set of PGL_n-torsors $f: Y \to X$ considered up to isomorphism, and set $\mathcal{T}_{\mathrm{PGL}} = \bigcup_n \mathcal{T}_{\mathrm{PGL}_n}$.

THEOREM 4.10. *We have*

$$X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{Az}} X} = \bigcap_{f \in \mathcal{T}_{\mathrm{PGL}}} X(\mathbf{A}_k)^f.$$

Proof. Let $f: Y \to X$ be a torsor under PGL_n and $\alpha := d([Y])$ be the corresponding element of ${}_n\mathrm{Br}_{\mathrm{Az}}\, X$. Let $(M_v) \in X(\mathbf{A}_k)$. The following diagram is commutative:

$$\begin{array}{ccc} H^1(X, \mathrm{PGL}_n) & \xrightarrow{d} & \mathrm{Br}\ X \\ {\scriptstyle .M_v}\downarrow & & \downarrow{\scriptstyle .M_v} \\ H^1(k_v, \mathrm{PGL}_n) & \xrightarrow{d} & \mathrm{Br}\ k_v \end{array}$$

and the canonical maps $d: H^1(k, \mathrm{PGL}_n) \to {}_n\mathrm{Br}\, k$ and $d: H^1(k_v, \mathrm{PGL}_n) \to {}_n\mathrm{Br}\, k_v$ are isomorphisms. Now it follows from the commutativity of this diagram that

$$([Y].M_v) \in \mathrm{Im}\left[H^1(k, \mathrm{PGL}_n) \to \prod_{v\in\Omega} H^1(k_v, \mathrm{PGL}_n)\right]$$

if and only if $(\alpha(M_v)) \in \mathrm{Im}\,[\mathrm{Br}\,k \to \prod_{v\in\Omega} \mathrm{Br}\,k_v]$. This implies $X(\mathbf{A}_k)^{\alpha} = X(\mathbf{A}_k)^{f}$. Since $\mathrm{Br}_{\mathrm{Az}}\,X$ is the union of the images of $H^1(X, \mathrm{PGL}_n)$ in $\mathrm{Br}\,X$ for $n = 1, 2, \ldots$, we are done. □

Remark 4.11. The already mentioned example [Ha96] of a smooth and projective k-variety X with $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{Az}}X} = \emptyset$ and $X(\mathbf{A}_k)^{\mathrm{Br}_1 X} \neq \emptyset$, shows (by Theorems 4.9 and 4.10) that the descent obstruction to the Hasse principle for a connected linear algebraic group does not necessarily reduce to the corresponding obstruction for its toric part. From this point of view, the case of the descent obstruction differs from the case of the elementary obstruction (cf. [Bo93], Theorem 6.5).

5. Examples

5.1. A COUNTEREXAMPLE TO THE HASSE PRINCIPLE NOT ACCOUNTED FOR BY THE MANIN OBSTRUCTION

The first example of (proper, smooth, and geometrically integral) k-variety which does not satisfy the Hasse principle, but for which there is no Manin obstruction, was recently given by the second author in [Sk]. We are going to revisit this counterexample using the obstruction associated to a non-abelian torsor.

Let C and D be the curves of genus one over $k = \mathbf{Q}$, and $f: C \times D \to X$ be the quotient by a fixed point free involution, as defined by equations in ([Sk], Section 2). Let E be the Jacobian of C. We have $[C] \in Ш(E)$, $D(k) \neq \emptyset$, hence X has points everywhere locally. Let C' be the principal homogeneous space under E given by equations in ([Sk], App. A). We have $[C'] \in Ш(E)$ and $2[C'] = [C]$. Finally, let C'' be a principal homogeneous space under E whose class $[C''] \in H^1(k, E)$ is such that $2[C''] = [C']$ (any element in the Tate–Shafarevich group $Ш(E)$ is divisible by any prime number in $H^1(k, E)$, if E is an elliptic curve, [Mi86], I.6.18). Consider the tower of finite étale coverings

$$Y'' = C'' \times D \xrightarrow{\xi' \times Id} Y' = C' \times D \xrightarrow{\xi \times Id} Y = C \times D \xrightarrow{f} X,$$

where the morphisms ξ' and ξ are induced by multiplication by 2 on E. Let $f' = f \circ (\xi \times Id)$, $f'' = f \circ (\xi \times Id) \circ (\xi' \times Id)$. Let $E[4]$ be the group of points of E of order dividing 4. The map f'' makes Y'' an X-torsor with respect to a k-group scheme G whose underling group is $E[4] \rtimes \mathbf{Z}/2$, where the non-trivial element of $\mathbf{Z}/2$ acts on $E[4]$ as multiplication by -1. G contains $E[4]$ as a normal k-group sub-scheme. Suppose that $[C] \in Ш(E)[2]$ is not divisible by 4 in $Ш(E)$. (It follows from the Birch–Swinnerton-Dyer conjecture that $Ш(E) = (\mathbf{Z}/4)^2$, so it should be the case here.)

PROPOSITION 5.1. *Assume that the Birch–Swinnerton-Dyer conjecture is true for the elliptic curve E. Then no twisted form of the torsor* $f'': Y'' \to X$ *has points everywhere locally.*

Without assuming the Birch–Swinnerton-Dyer conjecture the same result remains true if one chooses $[C'] \in Ш(E)$ such that $[C] = 2^i[C']$ with i maximal with this property, and then defines C'' and Y'' in the same way as above. (The group $Ш(E)$ is finite by Rubin's theorem, see [Sk], the proof of Proposition 2.)

Proof. Let $\sigma \in Z^1(k, G)$, and let $(Y'')^\sigma \to X$ be the twist of $Y'' \to X$ by σ. Since $E[4]$ is normal in G, G^σ contains its twisted group $E[4]^\sigma$. The quotient of $(Y'')^\sigma$ by $E[4]^\sigma$ is an X-torsor under $\mathbf{Z}/2$. This is the same thing as the twist Y^τ of $Y \to X$ by the image $\tau \in Z^1(k, \mathbf{Z}/2)$ of σ. It is shown in ([Sk], the proof of Theorem 2 (a)) that if $Y^\tau(\mathbf{A_Q}) \neq \emptyset$, then $[\tau] = 1$ and Y^τ is isomorphic to Y. Hence the morphism $(Y'')^\sigma \to X$ factors as $(Y'')^\sigma \to Y \to X$. From the exact sequence of pointed sets

$$H^1(k, E[4]) \to H^1(k, G) \to H^1(k, \mathbf{Z}/2)$$

it follows that there exists $\rho \in Z^1(k, E[4])$ such that $[\sigma]$ is the image of $[\rho]$. Then $(Y'')^\sigma \to Y$ can be considered as the twist of $Y'' = C'' \times D \to Y$ by ρ. The action of $E[4]$ on $Y'' = C'' \times D$ is given by the natural action on C'' and the trivial action on D. Thus $(Y'')^\sigma = (C'')^\rho \times D$. It is clear that $(C'')^\rho$ is an 8-covering of E which is a lifting of the 2-covering $C \to E$. Since $Ш(E) = (\mathbf{Z}/4)^2$ by the Birch–Swinnerton-Dyer conjecture, we conclude that $(C'')^\rho$ cannot represent an element of $Ш(E)$ for any class $[\rho] \in H^1(k, E[4])$. Thus no twist of $Y'' \to X$ has an adelic point. □

We thus have $X(\mathbf{A}_k)^{f''} = \emptyset$. Note that it is shown in [Sk] that $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$: the point is that in this case $\mathrm{Br}\, X = \mathrm{Br}_1 X$, hence the Brauer–Manin obstruction is controlled by abelian torsors over X (cf. Theorem 4.9). But although the coverings $Y \to X$ and $Y'' \to Y$ are abelian, the covering $Y'' \to X$ is not; the condition that $Y^\tau(\mathbf{A}_k)^{\mathrm{Br}}$ is empty for any twisted form Y^τ of $Y \to X$ (this is the 'refined obstruction' defined in [Sk]) follows from the fact that $(Y'')^\sigma(\mathbf{A}_k)$ is empty for any twisted form $(Y'')^\sigma$ of Y'' (apply Theorem 4.9 to the abelian torsor $Y'' \to Y$).

This example shows that the descent obstruction associated to a non-abelian torsor can be finer than the Manin obstruction.

5.2. COUNTEREXAMPLES TO WEAK APPROXIMATION RELATED TO NON-ABELIAN FUNDAMENTAL GROUPS

The following result is proven in [Ha99]:

Let X be a proper, smooth, and geometrically integral variety over a number field k such that $X(k) \neq \emptyset$. Assume that $(\mathrm{Br}\,\overline{X})^{\Gamma}$ is finite (e.g. $H^2(X, \mathcal{O}_X) = 0$) and that the geometric fundamental group $\pi_1(\overline{X})$ is non-abelian. Assume further that $H^1(X, \mathcal{O}_X) = 0$ or that the tangent bundle T_X of X is nef. Then, $p^{\Sigma}(X(\mathbf{A}_k)^{\mathrm{Br}}) \not\subset \overline{X(k)}^{\Sigma}$ for any finite set $\Sigma \subset \Omega$, where p^{Σ} is the canonical projection $X(\mathbf{A}_k) \to X(\mathbf{A}_k^{\Sigma})$.

This statement can be reinterpreted using the obstruction to weak approximation defined in Subsection 4.2. Indeed, the assumption $\pi_1(\overline{X})$ non-abelian gives a torsor $f\colon Y \to X$ under a finite non-abelian group scheme G (cf. 3.1). Now, using the additional hypothesis and abelian descent theory as in Theorem 4.9 ([CS], [Sk]), one shows that infinitely many elements of $p^{\Sigma}(X(\mathbf{A}_k)^{\mathrm{Br}})$ do not belong to $X(\mathbf{A}_k^{\Sigma})^f$. Thus the fact that G is not abelian implies that the descent obstruction to weak approximation (or to weak approximation outside Σ) can be finer than the Brauer–Manin obstruction.

This result applies for example to étale quotients of abelian varieties, in particular, to bielliptic surfaces (this is not surprising in view of the counterexample in the previous subsection). It is worth noting that in this case, $X(K)$ is Zariski-dense in X for some finite field extension K/k. See ([Ha99], Section 6) for more details.

5.3. HOMOGENEOUS SPACES OF BOROVOI AND KUNYAVSKIĬ

In this subsection we apply the general machinery of the elementary obstruction to homogeneous spaces constructed by Borovoi and Kunyavskiĭ. In particular, we show how to find adelic points on such a space X, which satisfy the Manin conditions with respect to $\mathrm{Br}_1 X$.

Let k be a field of characteristic zero. Consider a finite k-group F with center Z. Let A be a finite abelian k-group containing Z and define $N := (F \times A)/Z$, where Z is embedded into $(F \times A)/Z$ by the map $z \mapsto (z, z^{-1})$. For some m, N can be realized as a subgroup of $\mathrm{SL}_{m,k}$. Note that $Z = F \cap A$, A is central in N, and F is normal in N. Let X be the homogeneous space of $\mathrm{SL}_{m,k}$ defined by twisting (cf. [Se94], I.5.3) $\mathrm{SL}_{m,k}/F$ by a cocycle $\xi \in Z^1(k, A/Z)$ (the natural right action of N on $\mathrm{SL}_{m,k}/F$ induces an action of N/Z on $\mathrm{SL}_{m,k}/F$, hence an action of A/Z on $\mathrm{SL}_{m,k}/F$). Let η be the image of $[\xi] \in H^1(k, A/Z)$ by the connecting homomorphism $H^1(k, A/Z) \to H^2(k, Z)$, and denote by Δ the connecting map of pointed sets $H^1(k, F/Z) \to H^2(k, Z)$. Since Z is the center of F, we have a simply transitive action $(b, \alpha) \mapsto b.\alpha$ of $H^2(k, Z)$ on $H^2(k, F)$ (Remark 1.15). Then we have:

PROPOSITION 5.2. *The variety X has a k-point if and only if $\eta \in \mathrm{Im}\,\Delta$.*

We use the following lemma:

LEMMA 5.3. *Let ν_F be the neutral class in $H^2(k, F)$ corresponding to the extension $E_0 := F(\bar{k})\rtimes\Gamma$, and let α_X be the Springer class of the homogeneous space X. Then $\alpha_X = \eta.\nu_F$.*

Proof. Fix a continuous cochain $g \mapsto c_g\colon \Gamma \to A(\bar{k})$ which is a lifting of the cocycle ξ. Let B be the subgroup of $N(\bar{k})\rtimes\Gamma$ consisting of the elements ng with $nZ = c_gZ$, and E the subgroup of $N(\bar{k})\rtimes\Gamma$ consisting of the elements ng with $nF = c_gF$. Then (cf. Proposition 3.5) η is the class in $H^2(k, Z)$ of the exact sequence

$$1 \to Z(\bar{k}) \to B \to \Gamma \to 1. \tag{18}$$

Applying the results of Subsection 3.2. with $\bar{x}_0 = \bar{e}F$, we see that α_X is the class in $H^2(k, F)$ of the exact sequence:

$$1 \to F(\bar{k}) \to E \to \Gamma \to 1 \tag{19}$$

(We have $L_X = \mathrm{lien}(F)$ because A is central in N). Now, by [FSS], (1.24), the class $\eta.\nu_F$ is represented by the exact sequence:

$$1 \to F(\bar{k}) \to E' \to \Gamma \to 1, \tag{20}$$

where $E' = B \times_\Gamma E_0/D$ and D is the subgroup (z, z^{-1}), $z \in Z(\bar{k})$, of the fibre product. An element of $B \times_\Gamma E_0$ corresponds to a triple (n_1, n_2, g), where $n_1 \in A(\bar{k})$, $n_2 \in F(\bar{k})$ and $n_1Z = c_gZ$. The map $\Phi\colon B \times_\Gamma E_0 \to E$ sending (n_1, n_2, g) to (n_1n_2, g) is well defined because $n_1n_2F = c_gn_2F = c_gF$ (recall that A is central in N). Since $F \cap A = Z$, the kernel of Φ is D. On the other hand Φ is clearly surjective so Φ identifies E' with E and $\alpha_X = \eta.\nu_F$. □

COROLLARY 5.4. *The class $\alpha_X \in H^2(k, F)$ is neutral if and only if $\eta \in \mathrm{Im}\,\Delta$.*

Proof. For any $\alpha \in H^2(k, F)$ there exists a unique $b \in H^2(k, Z)$ such that $\alpha = b.\nu_F$. By [Bo93], Proposition 2.3, $b.\nu_F$ is neutral if and only if b belongs to the image of Δ. Now the result follows from Lemma 5.3. □

Since $H^1(k, \mathrm{SL}_m)$ is trivial, Proposition 5.2 follows from Corollary 5.4 and Example 3.4. □

PROPOSITION 5.5. *Let k be a number field. Suppose that the restriction of η to $H^2(k_v, Z)$ is trivial for any $v \in \Omega$. Then the set $X(\mathbf{A}_k)^{\mathrm{Br}_1 X}$ is not empty.*

Proof. The map $\mathrm{SL}_{m,\bar{k}} \to \overline{X}$ is a finite étale Galois covering with group $F(\bar{k})$. We have $\mathrm{Pic}\,\mathrm{SL}_{m,\bar{k}} = 0$ ([S], 6.9), and $H^0(\mathrm{SL}_{m,\bar{k}}, \mathbf{G}_m) = \bar{k}^*$. Now the Hochschild–Serre spectral sequence ([Mi80], III.2.20) $H^p(F(\bar{k}), H^q(\mathrm{SL}_{m,\bar{k}}, \mathbf{G}_m)) \Rightarrow H^{p+q}(\overline{X}, \mathbf{G}_m)$ shows that $\mathrm{Pic}\,\overline{X}$ is the group of characters of $F(\bar{k})$, hence is $F^{\mathrm{ab}}(\bar{k}) = \widehat{F/Z}(\bar{k})$.

Let T be the twist of $\mathrm{SL}_{m,k}/Z$ by ξ, then $f\colon T \to X$ is an X-torsor under the k-group F/Z. Its type $\lambda\colon \widehat{F/Z} \to \mathrm{Pic}\,\overline{X}$ is injective because $\overline{T}$ is connected, hence it is an isomorphism ($\widehat{F/Z}$ and $\mathrm{Pic}\,\overline{X}$ have the same cardinality). In other words, T is a universal torsor over X. Using ([Sk], Theorem 3 (a)) we obtain $X(\mathbf{A}_k)^{\mathrm{Br}_1 X} = X(\mathbf{A}_k)^f$.

For each place v of k, set $T_v = T \times_k k_v$. Since the image of η in $H^2(k_v, Z)$ is trivial, by Lemma 5.3 there exists a T_v-torsor $Y_v \to T_v$ under $Z \times_k k_v$, which is a principal homogeneous space under SL_{m,k_v}. Since $H^1(K, \mathrm{SL}_{m,K}) = 0$ for any field K, we have $Y_v(k_v) \neq \emptyset$. Hence, $T(k_v) \neq \emptyset$ for any place v of k. Thus $X(\mathbf{A}_k)^f \neq \emptyset$ and we are done. $\square$

Remark 5.6. It is possible to choose the data above such that $\mathrm{Br}\,\overline{X^c} = 0$, where X^c is a smooth compactification of X (Borovoi–Kunyavskiĭ, personal communication). Then it follows from Proposition 5.5 that $X^c(\mathbf{A}_k)^{\mathrm{Br}}$ is not empty. It is an interesting question whether one can arrange that $X(k) = \emptyset$ or $X^c(k) = \emptyset$.

Appendix A: Extensions of Topological Groups

The following result is useful to deal with the various topological conditions used in the paper for exact sequences of groups.

PROPOSITION. *With the assumptions of Subsection 1.4 consider an exact sequence of topological groups, where the first inclusion is continuous (i.e. open onto its image):*

$$1 \to \overline{G}(\bar{k}) \to E \xrightarrow{q} \Gamma \to 1 \tag{3}$$

Then,

(1) *The map q is open onto its image if and only if it admits a continuous set-theoretic section. This is in particular the case if* (3) *is locally split.*
(2) *Let κ be a k-lien on the algebraic group $\overline{G}$. Assume that the induced homomorphism $\Gamma \to \mathrm{Out}(\overline{G}(\bar{k}))$ is $\kappa\colon \Gamma \to \mathtt{SOut}(\overline{G})$ followed by the canonical map $\mathtt{SOut}(\overline{G}) \to \mathrm{Out}(\overline{G}(\bar{k}))$. Then the sequence* (3) *is compatible with κ if and only if it is locally split.*

Proof. (1) Assume that q admits a continuous set-theoretic section s. Let U be an open subset of E. Take any $g \in q(U)$, say, $g = q(u)$ for some $u \in U$. We may assume $s(g) = u$ (translating s by an element of $\ker q$ if necessary). Now $s^{-1}(U)$ is an open subset of $q(U)$ which contains g. Since g is arbitrary in $q(U)$, the set $q(U)$ is open.

Conversely, assume that q is open. Since $\overline{G}(\bar{k})$ is discrete, one can find an open neighbourhood U of $\bar{e}$ such that $U \cap \overline{G}(\bar{k}) = \{\bar{e}\}$. Shrinking U if necessary, we may assume that the restriction of q to U is injective (consider the inverse image of U by the continuous map $(x, y) \mapsto x^{-1}y\colon E \times E \to E$; it is an open neighbourhood of $\{\bar{e}\} \times \{\bar{e}\}$ which contains a set of the form $V \times V$, where $V \subset E$ is an an open neighbourhood of $\{\bar{e}\}$. Then replace U by $U \cap V$). Since q is open, the image of $q(U)$ contains $\Gamma_K := \mathrm{Gal}(\bar{k}/K)$ for some finite Galois field extension K/k, and (on replacing U by $U \cap q^{-1}(\Gamma_K)$) we shall assume $q(U) = \Gamma_K$. Then the bijective map $q\colon U \to \Gamma_K$ (which is open) admits a continuous inverse map. In particular the restriction $q_K\colon E_K := E \cap q^{-1}(\Gamma_K) \to \Gamma_K$ admits a continuous set-theoretic section. Note that if we had supposed (3) locally split instead of q open, such a section would have existed by definition, so it just remains to prove that q admits a continuous set-theoretic section if q_K does.

Suppose that q_K admits a continuous set-theoretic section s_K. Then Γ is the finite disjoint union of cosets $g_i\Gamma_K$ ($g_i \in \Gamma$, $1 \leqslant i \leqslant n$). Take an arbitrary lifting $\varphi_i \in E$ of g_i; any $g \in \Gamma$ admits a unique decomposition $g = g_i h$ ($1 \leqslant i \leqslant n$, $h \in \Gamma_K$), put $s(g) = \varphi_i s_K(h)$. Let $g \in \Gamma$, for each finite field extension L/K, any element of the open neighbourhood $g\Gamma_L$ of g belongs to the same Γ_K-coset as g. Since $h \mapsto s_K(h)$ is continuous on Γ_K, so is $g \mapsto s(g)$ on Γ.

(2) Using (1), it remains to prove that if (3) is compatible with κ, then it is locally split, and by Proposition 1.13, this is equivalent to saying that any class in $H^2(k, \overline{G}, \kappa)$ becomes neutral after a finite field extension of k. To do that, we may assume that $\overline{G}$ admits a k-form G, which defines a continuous homomorphic splitting $\theta\colon \Gamma \to \mathtt{SAut}^{gr}(\overline{G})$ of (1). The lien κ lifts to a continuous set-theoretic section f of (1), which can be written $f_g = \theta_g \varphi_g$, where the map $\varphi\colon \Gamma \to \mathtt{Aut}^{\mathrm{gr}}(\overline{G})$ is locally constant by [FSS], (1.7). Let K be a finite field extension of k such that φ becomes constant on Γ_K, then this constant must be trivial in $\mathtt{SOut}(\overline{G}/K)$ because κ and θ are homomorphic. So we may assume that the k-lien κ is trivial. Now any element of $H^2(k, \overline{G}, \kappa)$ can be represented by a cocycle which is given by a pair of maps (f, s), where $f\colon \Gamma \to \mathtt{SAut}^{\mathrm{gr}}(\overline{G})$ is a homomorphic section of (1) and $s\colon \Gamma \times \Gamma \to \overline{G}(\bar{k})$ is locally constant ([FSS], (1.17)). If s is constant, say $s \equiv a$, then the cocycle (f, s) is equivalent to (f', s'), where $f'_g = \mathrm{int}(a^{-1}) \circ f_g$ and $s' \equiv 1$ (by the formula (6) in [FSS], (1.17)), so any cocycle is locally equivalent to a neutral cocycle (we are indebted to C. Scheiderer for this argument). □

Appendix B: Comparison of Two Spectral Sequences

In this appendix we relate the spectral sequence used by Colliot-Thélène and Sansuc to obtain the exact sequence (10) of Subsection 3.3 to the Leray spectral sequence which gives the sequence (11).

PROPOSITION. *Let k be a field of characteristic zero, and $p\colon X \to \operatorname{Spec} k$ a reduced and geometrically connected variety. Let S be a k-group of multiplicative type and $M := \hat{S}$ the Γ-module which is dual to S. Then there is a canonical morphism of the Leray spectral sequence $\mathrm{E}_2^{p,q} = H^p(k, H^q(\overline{X}, \overline{S}))$ to the spectral sequence $\mathrm{E}'^{p,q}_2 = \mathrm{Ext}^p_k(M, R^q p_* \mathbf{G}_m)$, both sequences converging to $H^i(X, S)$. When S is a torus, this morphism is an isomorphism. If S is an arbitrary k-group of multiplicative type, and X is a k-variety such that $\bar{k}[X]^*$ is divisible (for example, $\bar{k}[X]^* = \bar{k}^*$), then the exact sequences of the first five low degree terms of these two spectral sequences are canonically isomorphic.*

Proof. We write $\mathcal{H}om_X$ for *Hom* of sheaves on $X_{\text{ét}}$, and $\mathcal{H}om_k$ for *Hom* of sheaves on $\operatorname{Spec} k$. The sheaf of abelian groups on X given by S can be written as $\mathcal{H}om_X(p^*M, \mathbf{G}_m)$ (M is a G-module of finite type, and checking it locally one sees that here homomorphisms of sheaves over X are the same thing as morphisms of group X-schemes). We have

$$p_*\mathcal{H}om_X(p^*M, \mathcal{F}) = \mathcal{H}om_k(M, p_*\mathcal{F})$$

since p^* and p_* are adjoint (this is a sheaf version of $Hom_X(p^*M, \mathcal{F}) = Hom_k(M, p_*\mathcal{F})$). This implies that there is a natural isomorphism of functors from the derived category $\mathcal{D}^+(X)$ of sheaves on $X_{\text{ét}}$ to the derived category $\mathcal{D}^+(k)$ of sheaves on $\operatorname{Spec} k$ (complexes of G-modules)

$$\mathbf{R}p_*\mathbf{R}\mathcal{H}om_X(p^*M, \cdot) = \mathbf{R}\mathcal{H}om_k(M, \mathbf{R}p_*(\cdot)).$$

Let us apply this to the sheaf $\mathbf{G}_m$. By Lemma 1.3.3(ii) of [CS], we have $\mathcal{E}xt^i_X(p^*M, \mathbf{G}_m) = 0$ for any $i > 0$, hence the complex $\mathbf{R}\mathcal{H}om_X(p^*M, \mathbf{G}_m)$ consists of the sheaf $S = \mathcal{H}om_X(p^*M, \mathbf{G}_m)$ in degree 0. We obtain

$$\mathbf{R}p_*(S) = \mathbf{R}\mathcal{H}om_k(M, \mathbf{R}p_*\mathbf{G}_m).$$

Now we apply the derived functor $\mathbf{H}(k, \cdot)$ of the functor $A \mapsto A^G$ to both sides. On the left-hand side we get $\mathbf{H}(k, \mathbf{R}p_*S) = \mathbf{H}(X, S)$ ($\mathbf{H}(X, \cdot)$ is the derived functor of $H^0(X, \cdot)$). The resulting spectral sequence of composed functors is

$$H^p(k, R^q p_*S) \Rightarrow H^{p+q}(X, S).$$

On the right-hand side we get $\mathbf{R}Hom_k(M, \mathbf{R}p_*\mathbf{G}_m)$ which gives rise to the spectral sequence

$$\operatorname{Ext}^p_k(M, R^q p_*\mathbf{G}_m) \Rightarrow H^{p+q}(X, S).$$

Let us relate these spectral sequences.

Let $\mathcal{G}$ be a complex of sheaves on $X_{\text{ét}}$ which is an injective resolution of $\mathbf{G}_m$. There is a natural map of complexes of G-modules (=sheaves over $\operatorname{Spec} k$)

$$p_*\mathcal{H}om_X(p^*M, \mathcal{G}) \to \mathcal{H}om_k(M, p_*\mathcal{G}). \tag{1}$$

Let $I = I^{**}$ be a bicomplex which is a Cartan–Eilenberg resolution of $p_*\mathcal{G}$, and let $\mathcal{T}$ be the total complex of $\mathcal{H}om_k(M, I)$. The complex $\mathcal{T}$ represents $\mathbf{R}\mathcal{H}om_k(M, p_*\mathbf{G}_m)$ in $\mathcal{D}^+(k)$ ([W], 10.5.6). The resolution $p_*\mathcal{G} \to I$ induces a natural map of complexes $\mathcal{H}om_k(M, p_*\mathcal{G}) \to \mathcal{T}$, and on combining with (1) a map

$$p_*\mathcal{H}om_X(p^*M, \mathcal{G}) \to \mathcal{T}. \tag{2}$$

This gives a map between the corresponding hypercohomology spectral sequences, which is a natural morphism we are looking for. (We have actually proved more: the image of this morphism is contained in the image of the natural map $H^n(k, \mathcal{H}om_k(M, R^k p_*\mathbf{G}_m)) \to \operatorname{Ext}^n_k(M, R^k p_*\mathbf{G}_m)$.)

If S is a torus, then M is locally in the étale topology isomorphic to a finite direct sum of copies of $\mathbf{Z}$. Then (1) is an identity map, and the second assertion of the proposition becomes obvious. To prove our last assertion it is enough to prove that (2) induces isomorphisms on H^0 and H^1 (in other words, (2) induces a quasi-isomorphism of complexes truncated at 1). We can check it locally and suppose that $M = \mathbf{Z}/n$. Then on the level of H^0 we have $H^0(\overline{X}, \mu_n) \to \mathrm{Hom}(\mathbf{Z}/n, H^0(\overline{X}, \mathbf{G}_m))$ which is always an isomorphism. On the level of H^1 we have $H^1(\overline{X}, \mu_n) \to \mathrm{Hom}(\mathbf{Z}/n, H^1(\overline{X}, \mathbf{G}_m))$. This map is always surjective, and its kernel is $H^0(\overline{X}, \mathbf{G}_m)/n$ which is trivial under our assumptions. This finishes the proof of the proposition. □

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