Descent theory for open varieties

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Abstract

We extend the descent theory of Colliot-Thélène and Sansuc to arbitrary smooth algebraic varieties by removing the condition that every invertible regular function is constant. This links the Brauer–Manin obstruction for integral points on arithmetic schemes to the obstructions defined by torsors under groups of multiplicative type.

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Let X be a smooth and geometrically integral variety over a number field k with points everywhere locally. Descent theory of Colliot-Thélène and Sansuc [7], [28] describes arithmetic properties of X in terms of X-torsors under k-groups of multiplicative type. It interprets the *Brauer–Manin obstruction* to the existence of a rational point (or to weak approximation) on X in terms of the obstructions defined by torsors.

Let $\bar{k}$ be an algebraic closure of k. Because of its first applications the descent theory was stated in [7] for proper varieties that become rational over $\bar{k}$; in this case it is enough to consider torsors under tori. It was pointed out in [27] that the theory works more generally under the sole assumption that the group $\bar{k}[X]^*$ of invertible regular functions on $\overline{X} := X \times_k \bar{k}$ is the group of constants $\bar{k}^*$. This assumption is satisfied when X is proper, but it often fails for complements to reducible divisors in smooth projective varieties; it also fails for many homogeneous spaces of algebraic groups. In the general case of an arbitrary smooth and geometrically integral variety Colliot-Thélène and Xu Fei have recently introduced a Brauer–Manin obstruction to the existence of integral points [6, Sect. 1]. Descent obstructions to the existence of integral points were briefly considered by Kresch and Tschinkel in [20], Remark 3, see also Section 5.3 of [8]. In the particular case of an open subset of $\mathbf{P}^1_k$ a variant of the main theorem of descent linking the two kinds of obstructions has recently turned up in connection with an old conjecture of Skolem, see [18, Thm. 1].

The goal of this paper is to extend the theory of descent to the general case of a smooth and geometrically integral variety. It turns out that the main results are almost entirely the same. The methods, however, must be completely overhauled. As it frequently happens, one needs to systematically consider Galois hypercohomology of complexes instead of Galois cohomology of individual Galois modules. For principal homogeneous spaces of algebraic groups this approach has already been used in [1], [17] and [11]. But even in the 'classical' case $\bar{k}[X]^* = \bar{k}^*$ working with derived categories and hypercohomology of complexes streamlines the proof of a key result of descent theory ([7], Prop. 3.3.2 and Lemme 3.3.3, [28], Thm. 6.1.2 (a)) by avoiding delicate explicit computations with cocycles (see our Theorem 3.5 and its proof).

Let us now describe the contents of the paper. Let S be a k-group of multiplicative type, that is, a commutative algebraic group whose connected component of the identity is an algebraic torus. In Section 1 we define the *extended type* of an X-torsor under S, an invariant that classifies X-torsors up to twists by a k-torsor. When $\bar{k}[X]^* \neq \bar{k}^*$ the extended type defines a stronger equivalence relation on $H^1(X, S)$ than the classical type introduced by Colliot-Thélène and Sansuc in [7].

Let $\mathcal{T}$ be an X-torsor under S. In Section 2 we show that if $U \subset X$ is an open set such that the classical type of the torsor $\mathcal{T}_U \to U$ is zero, then $\mathcal{T}_U$ is canonically isomorphic to the fibred product $Z \times_Y U$, where Z and $Y = Z/S$ are k-torsors under groups of multiplicative type, and $U \to Y$ is a certain canonical morphism (Theorem 2.6). This description follows the ideas of Colliot-Thélène and Sansuc [7] who used similar constructions to describe $\mathcal{T}_U$ by explicit equations. Our goal was to obtain a functorial description, so our results are not immediately related to theirs. Corollary 2.7 describes the restriction of torsors of given extended type to sufficiently small open subsets.

In Section 3 we prove the main results of our generalised descent theory. The proof of Theorem 3.5 relies on the previous work of T. Szamuely and the first named author [16, 17], in particular, on their version of the Poitou–Tate duality for tori, which was later extended by C. Demarche to the groups of multiplicative type [10].

In Section 4 we prove statements about the existence of integral points and strong approximation. As an application we give a short proof of a result by Colliot-Thélène and Xu Fei, generalised by C. Demarche, see Theorem 4.3.

1 The extended type of a torsor

Let Z be an integral regular Noetherian scheme, and let $p : X \to Z$ be a faithfully flat morphism of finite type. Let $\mathcal{D}(Z)$ be the derived category of bounded complexes of fppf or étale sheaves on Z. For an object $\mathcal{C}$ of $\mathcal{D}(Z)$, the hypercohomology groups $\mathbb{H}^i(Z, \mathcal{C})$ will be denoted simply by $H^i(Z, \mathcal{C})$. Notation such as $\mathrm{Hom}_Z(A, B)$ or $\mathrm{Ext}^i_Z(A, B)$ will be understood in the category of sheaves on Z, or in $\mathcal{D}(Z)$. The same conventions apply when Z is replaced by X.

Consider the truncated object $\tau_{\leq 1}\mathbf{R}p_*\mathbf{G}_{m,X}$ in $\mathcal{D}(Z)$. Its shift by 1, which has trivial cohomology outside the degrees -1 and 0, is denoted by

$$KD(X) = (\tau_{\leq 1}\mathbf{R}p_*\mathbf{G}_{m,X})[1].$$

There is a canonical morphism $i : \mathbf{G}_{m,Z} \to \tau_{\leq 1}\mathbf{R}p_*\mathbf{G}_{m,X}$, and we define

$$KD'(X) = \mathrm{Coker}\,(i)[1],$$

so that we have an exact triangle

$$\mathbf{G}_{m,Z}[1] \longrightarrow KD(X) \xrightarrow{v} KD'(X) \xrightarrow{w} \mathbf{G}_{m,Z}[2]. \qquad (1)$$

A group scheme of finite type over Z is called a Z-group of multiplicative type if locally on Z it is isomorphic to a group subscheme of $\mathbf{G}^n_{m,Z}$. By [13, IX, Prop. 2.1] such a group is affine and faithfully flat over Z. If S is a group of multiplicative type or a finite flat group scheme over Z, we denote by $\widehat{S}$ the *Cartier dual* of S. This is the group scheme over Z which represents the fppf sheaf $\mathcal{H}om_Z(S, \mathbf{G}_{m,Z})$, see [13, X, Cor. 5.9] when S is of multiplicative type, and [24, Ch. 14] when S is finite flat.

The following proposition is a generalisation of the fundamental exact sequence of Colliot-Thélène and Sansuc (see [28], Thm. 2.3.6 and Cor. 2.3.9).

Proposition 1.1 *Let S be a Z-group scheme. Assume that one of the two following properties is satisfied:*

(a) *S is of multiplicative type;*

(b) *S is finite and flat, and if 2 is a residual characteristic of Z, then the 2-primary torsion subgroup $S\{2\}$ is of multiplicative type (equivalently, the Cartier dual $\widehat{S\{2\}}$ is smooth over Z).*

Then there is an exact sequence

$$H^1(Z, S) \to H^1(X, S) \xrightarrow{\chi} \mathrm{Hom}_Z(\widehat{S}, KD'(X)) \xrightarrow{\partial} H^2(Z, S) \to H^2(X, S). \quad (2)$$

To simplify notation, here and elsewhere we write $H^n(X, S)$ for the fppf cohomology group $H^n(X, p^*S)$. If S is smooth, then the fppf topology can be replaced by étale topology.

Proof of Proposition 1.1 We apply the functor $\mathrm{Hom}_k(\widehat{S},.)$ to the exact triangle (1). To identify the terms of the resulting long exact sequence we use the following well known fact: for any scheme X/Z, any Z-group S of multiplicative type and any $n \geq 0$ we have

$$H^n(X,S) = \mathrm{Ext}^n_X(p^*\widehat{S}, \mathbf{G}_{m,X}),$$

see [7], Prop. 1.4.1, or [28], Lemma 2.3.7. Let us recall the argument for the convenience of the reader. One proves first that $\mathcal{E}xt^n_X(p^*\widehat{S}, \mathbf{G}_{m,X}) = 0$ for any $n \geq 1$, and then the local-to-global spectral sequence

$$H^m(X, \mathcal{E}xt^n_X(p^*\widehat{S}, \mathbf{G}_{m,X})) \Rightarrow \mathrm{Ext}^{m+n}_X(p^*\widehat{S}, \mathbf{G}_{m,X})$$

completely degenerates, giving the desired isomorphism.

In case (b) the same argument works for $1 \leq n \leq 3$: indeed, for $\ell \neq 2$ we have $\mathcal{E}xt^n_X(p^*\widehat{S}\{\ell\}, \mathbf{G}_{m,X}) = 0$ by the main result of [5]. (Note that the case $\ell = 2$ is exceptional: for example, $\mathrm{Ext}^2_K(\alpha_2, \mathbf{G}_{m,K}) \neq 0$ if K is a separably closed field of characteristic 2, see [4].)

The functor $\mathbf{R}\mathrm{Hom}_X(p^*\widehat{S},.)$ from $\mathcal{D}(X)$ to the derived category of abelian groups $\mathcal{D}(\mathrm{Ab})$ is the composition of the functors $\mathbf{R}p_*(.) : \mathcal{D}(X) \to \mathcal{D}(Z)$ and $\mathbf{R}\mathrm{Hom}_Z(\widehat{S},.) : \mathcal{D}(Z) \to \mathcal{D}(\mathrm{Ab})$. This formally entails a canonical isomorphism

$$\mathrm{Ext}^n_X(p^*\widehat{S}, \mathbf{G}_{m,X}) = R^n\mathrm{Hom}_Z(\widehat{S}, \mathbf{R}p_*\mathbf{G}_{m,X}).$$

In particular, we have

$$H^n(Z,S) = \mathrm{Ext}^n_Z(\widehat{S}, \mathbf{G}_{m,Z}) = \mathrm{Hom}_Z(\widehat{S}, \mathbf{G}_{m,Z}[n]).$$

Truncation produces an exact triangle

$$\tau_{\leq 1}\mathbf{R}p_*\mathbf{G}_{m,X} \to \mathbf{R}p_*\mathbf{G}_{m,X} \to \tau_{\geq 2}\mathbf{R}p_*\mathbf{G}_{m,X} \to (\tau_{\leq 1}\mathbf{R}p_*\mathbf{G}_{m,X})[1],$$

and here $\tau_{\geq 2}\mathbf{R}p_*\mathbf{G}_{m,X}$ is acyclic in degrees 0 and 1. We deduce canonical isomorphisms

$$R^1\mathrm{Hom}_Z(\widehat{S}, \tau_{\leq 1}\mathbf{R}p_*\mathbf{G}_{m,X}) = R^1\mathrm{Hom}_Z(\widehat{S}, \mathbf{R}p_*\mathbf{G}_{m,X}) = H^1(X,S),$$

and an injection of $R^2\mathrm{Hom}_Z(\widehat{S}, \tau_{\leq 1}\mathbf{R}p_*\mathbf{G}_{m,X})$ into $R^2\mathrm{Hom}_Z(\widehat{S}, \mathbf{R}p_*\mathbf{G}_{m,X}) = H^2(X,S)$. Now (2) is obtained by applying $\mathrm{Hom}_Z(\widehat{S},.)$ to (1). □

Remarks 1. Let k be a field of characteristic zero with algebraic closure $\bar{k}$ and Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$. In the case when X is smooth over k, $KD(X)$ was introduced in [17] as the following complex of Γ-modules in degrees -1 and 0:

$$[\bar{k}(X)^* \to \mathrm{Div}(\overline{X})].$$

Here $\bar{k}(X)$ is the function field of $\overline{X} = X \times_k \bar{k}$, and $\mathrm{Div}(\overline{X})$ is the group of divisors on $\overline{X}$ (see [2], Lemma 2.3 and Remark 2.6). In this case $KD'(X)$ is quasi-isomorphic to the complex of Γ-modules

$$[\bar{k}(X)^*/\bar{k}^* \to \mathrm{Div}(\overline{X})].$$

Up to shift, $KD'(X)$ was independently introduced by Borovoi and van Hamel in [2]: in their notation we have $KD'(X) = U\mathrm{Pic}(\overline{X})[1]$. Furthermore, if X^c is a smooth compactification of X, and $\mathrm{Div}_\infty(\overline{X}^c)$ is the group of divisors of $\overline{X}^c$ supported on $\overline{X} - \overline{X}^c$, then, by [17], Lemma 2.2, $KD'(X)$ is quasi-isomorphic to the complex

$$[\mathrm{Div}_\infty(\overline{X}^c) \to \mathrm{Pic}(\overline{X}^c)].$$

2. In the relative case, when X is smooth over Z, our $KD(X)$ and $KD'(X)$ coincide with analogous objects defined in [17], Remark 2.4 (2). See Appendix A to the present paper for the proof of this fact.

3. In the relative case, when X is proper over Z with geometrically integral fibres, $KD'(X)$ identifies with the sheaf $R^1p_*\mathbf{G}_{m,X}$, the relative Picard functor. When X is also assumed projective over Z, the relative Picard functor is representable by a Z-scheme, separated and locally of finite type, see [3], Ch. 8, Thm. 1 on p. 210.

Let $\mathcal{D}(k)$ be the bounded derived category of the category of continuous discrete Γ-modules.

Definition 1.2 Let X be a smooth and geometrically integral variety over k. Let Y be an X-torsor under a k-group of multiplicative type S, and let $[Y]$ be its class in $H^1(X,S)$. We shall say that the morphism $\chi([Y]) : \widehat{S} \to KD'(X)$ in the derived category $\mathcal{D}(k)$ is the *extended type* of the torsor $Y \to X$.

Remarks 1. There is a canonical morphism from (2) to the sequence of Colliot-Thélène and Sansuc ([28], Thm. 2.3.6):

$$\mathrm{Ext}^1_k(\widehat{S}, \bar{k}[X]^*) \to H^1(X,S) \to \mathrm{Hom}_k(\widehat{S}, \mathrm{Pic}(\overline{X})) \to \mathrm{Ext}^2_k(\widehat{S}, \bar{k}[X]^*) \to H^2(X,S) \tag{3}$$

Indeed, (3) is obtained by applying the functor $\mathrm{Hom}_k(\widehat{S}, .)$ to the exact triangle

$$\bar{k}[X]^*[1] \to KD(X) \to \mathrm{Pic}(\overline{X}) \to \bar{k}[X]^*[2] \tag{4}$$

(cf. [28], p. 26), and there is an obvious canonical morphism from (1) to (4). Recall that if $Y \to X$ is a torsor under S, then the image of the class $[Y] \in H^1(X, S)$ in $\mathrm{Hom}_k(\widehat{S}, \mathrm{Pic}(\overline{X}))$ is called the *type* of $Y \to X$. We see that the notion of extended type defines a stronger equivalence relation on $H^1(X, S)$ than the notion of type. For example two torsors have the same extended type if and only if their classes in $H^1(X, S)$ coincide up to a 'constant element'.

2. If we assume further that $\bar{k}[X]^* = \bar{k}^*$ (e.g. X proper), then $KD'(X)$ is quasi-isomorphic to $[0 \to \mathrm{Pic}(\overline{X})]$, and the exact sequence (2) is just the fundamental exact sequence of Colliot-Thélène and Sansuc ([28], Cor. 2.3.9):

$$H^1(k, S) \to H^1(X, S) \to \mathrm{Hom}_k(\widehat{S}, \mathrm{Pic}(\overline{X})) \to H^2(k, S) \to H^2(X, S).$$

3. The other 'extreme' case is when $\mathrm{Pic}(\overline{X}) = 0$. Then $KD'(X)$ is quasi-isomorphic to $(\bar{k}[X]^*/\bar{k}^*)[1]$, and the extended type is an element of $\mathrm{Ext}^1_k(\widehat{S}, \bar{k}[X]^*/\bar{k}^*)$. One case of interest is when X is a principal homogeneous space of a k-torus T, so that $\mathrm{Pic}(\overline{X}) = \mathrm{Pic}(\overline{T}) = 0$, then the extended type is an element of $\mathrm{Ext}^1_k(\widehat{S}, \widehat{T})$. Suppose that $X = T$, and let $T' \to T$ be a surjective homomorphism of k-tori with kernel S. This is of course a T-torsor under S. We shall show in Remark 2 after Proposition 2.5 below that the extended type of this torsor is given by the natural extension

$$0 \to \widehat{T} \to \widehat{T'} \to \widehat{S} \to 0.$$

This fact was implicitly used in [18, Lemma 2.2].

4. Unlike the classical type, the extended type of a torsor $Y \to X$ is in general not determined by the $\overline{X}$-torsor $\overline{Y}$. For example, if $\mathrm{Pic}(\overline{X}) = 0$ and S is a torus, then $\mathrm{Ext}^1_k(\widehat{S}, \bar{k}[X]^*/\bar{k}^*) = 0$ because $\widehat{S}$ is a free abelian group.

Proposition 1.3 *Let X be a smooth and geometrically integral variety over k, and let S be a k-group of multiplicative type. If $X(k) \neq \emptyset$, then the map $\chi : H^1(X, S) \to \mathrm{Hom}_k(\widehat{S}, KD'(X))$ is onto. In other words, if $X(k) \neq \emptyset$, then there exist X-torsors of every extended type.*

Proof Since $X(k) \neq \emptyset$, the map $H^2(k, S) \to H^2(X, S)$ has a retraction, hence is injective. Therefore the map ∂ is zero and χ is surjective. □

Let $\mathrm{Br}(X) = H^2(X, \mathbf{G}_{m,X})$ be the cohomological Brauer–Grothendieck group of X. As usual, $\mathrm{Br}_0(X)$ will denote the image of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(X)$, and $\mathrm{Br}_1(X)$ the kernel of the natural map $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$.

It is easy to check that $\mathrm{Br}_1(X)$ is canonically isomorphic to $H^1(k, KD(X))$, see [2], Prop. 2.18, or [17], Lemma 2.1. Thus the exact triangle (1) induces an exact sequence in Galois hypercohomology

$$\mathrm{Br}(k) \to \mathrm{Br}_1(X) \xrightarrow{r} H^1(k, KD'(X)) \to H^3(k, \bar{k}^*). \tag{5}$$

The cup-product in étale cohomology defines the pairing

$$\cup \ : \ H^1(k, \widehat{S}) \times H^1(X, S) \to H^1(X, \widehat{S}) \times H^1(X, S) \to \mathrm{Br}(X),$$

whose image visibly belongs to $\mathrm{Br}_1(X)$. The following statement generalises [28, Thm. 4.1.1].

Theorem 1.4 *Let X be a smooth and geometrically integral variety over k, and let $f : Y \to X$ be a torsor under a k-group of multiplicative type S. Let $\lambda : \widehat{S} \to KD'(X)$ be the extended type of this torsor. Then for any $a \in H^1(k, \widehat{S})$ we have*

$$r(a \cup [Y]) = \lambda_*(a),$$

where λ_* is the induced map $H^1(k, \widehat{S}) \to H^1(k, KD'(X))$.

Proof We have canonical isomorphisms

$$H^1(X, S) = \mathrm{Hom}_k(\widehat{S}, \mathbf{R}p_*\mathbf{G}_{m,X}[1]) = \mathrm{Hom}_k(\widehat{S}, KD(X)),$$

cf. the proof of Proposition 1.1. By [22], Prop. V.1.20, these isomorphisms fit into the following commutative diagram of pairings

$$\begin{array}{ccccc}
H^1(k, \widehat{S}) & \times & H^1(X, S) & \to & \mathrm{Br}(X) \\
|| & & || & & || \\
H^1(k, \widehat{S}) & \times & \mathrm{Hom}_k(\widehat{S}, \mathbf{R}p_*\mathbf{G}_{m,X}[1]) & \to & H^1(k, \mathbf{R}p_*\mathbf{G}_{m,X}[1]) \\
|| & & || & & \uparrow \\
H^1(k, \widehat{S}) & \times & \mathrm{Hom}_k(\widehat{S}, KD(X)) & \to & H^1(k, KD(X))
\end{array}$$

Here the vertical arrow is induced by the canonical map

$$KD(X) = (\tau_{\leq 1}\mathbf{R}p_*\mathbf{G}_{m,X})[1] \to \mathbf{R}p_*\mathbf{G}_{m,X}[1].$$

Let $u : \widehat{S} \to KD(X)$ be the morphism corresponding to the class $[Y]$. By the commutativity of the diagram we have

$$a \cup [Y] = u_*(a) \in H^1(k, KD(X)) = \mathrm{Br}_1(X).$$

By definition, λ is the composed map

$$\widehat{S} \xrightarrow{u} KD(X) \xrightarrow{v} KD'(X).$$

By construction, the map r from the exact sequence (5) is the induced map $v_* : H^1(k, KD(X)) \to H^1(k, KD'(X))$, hence $r(a \cup [Y]) = v_*(u_*(a)) = \lambda_*(a)$. $\square$

2 Localisation of torsors

Let U be a smooth and geometrically integral variety over k. The abelian group $\bar{k}[U]^*/\bar{k}^*$ is torsion free, so we can define a k-torus R as the torus whose module of characters $\widehat{R}$ is the Γ-module $\bar{k}[U]^*/\bar{k}^*$. The natural exact sequence of Γ-modules

$$1 \to \bar{k}^* \to \bar{k}[U]^* \to \bar{k}[U]^*/\bar{k}^* \to 1 \qquad (*_U)$$

defines a class

$$[*_U] \in \mathrm{Ext}^1_k(\bar{k}[U]^*/\bar{k}^*, \bar{k}^*) = H^1(k, R).$$

Let Y be the k-torsor under R whose class in $H^1(k, R)$ is $-[*_U]$.

Lemma 2.1 *There exists a morphism $q_U : U \to Y$ such that q_U^* identifies $(*_Y)$ with $(*_U)$. Any morphism from U to a k-torsor under a torus factors through q_U.*

Proof This is Lemma 2.4.4 of [28]. $\square$

To deal with the case of torsors under arbitrary groups of multiplicative type we need to extend this constrution to certain geometrically reducible varieties. However, Lemma 2.1 does not readily generalise, because its essential ingredient is Rosenlicht's lemma which is valid only for connected groups. If S is a torus, it says that the natural map of Γ-modules $\widehat{S} \to \bar{k}[S]^*$ induces an isomorphism $\widehat{S} \cong \bar{k}[S]^*/\bar{k}^*$ (every invertible regular function that takes value 1 at the neutral element of S is a character). This is no longer true if $\widehat{S}$ has non-zero torsion subgroup. For general groups of multiplicative type we propose the following substitute.

Definition 2.2 For a (not necessarily integral) k-variety V with an action of S we define $\bar{k}[V]^*_S$ as the subgroup of $\bar{k}[V]^*$ consisting of the functions $f(x)$ for which there exists a character $\chi \in \widehat{S}$ such that $f(sx) = \chi(s)f(x)$ for any $s \in S(\bar{k})$.

It is easy to see that $\bar{k}[V]^*_S$ is a Γ-submodule of $\bar{k}[V]^*$.

Remark If S is a torus and V is geometrically connected, then $\bar{k}[V]^*_S = \bar{k}[V]^*$, so we are not getting anything new. Indeed, if $x \in V(\bar{k})$ and $f \in \bar{k}[V]^*$, then $f(sx)/f(x)$ is a regular invertible function on $\overline{S}$ with value 1 at the neutral element $e \in S(\bar{k})$. By Rosenlicht's lemma such a function is a character in $\widehat{S}$. We obtain a morphism from a connected variety $\overline{V}$ to a discrete group $\widehat{S}$, which must be a constant map. Hence there exists $\chi \in \widehat{S}$ such that $f(sx) = \chi(s)f(x)$ for any $x \in V(\bar{k})$ and any $s \in S(\bar{k})$.

Proposition 2.3 *Let S be a k-group of multiplicative type. Then the natural map $\widehat{S} \to \bar{k}[S]^*_S$ induces an isomorphism of Γ-modules $\widehat{S} \tilde{\longrightarrow} \bar{k}[S]^*_S/\bar{k}^*$.*

Proof The image of the natural inclusion $\widehat{S} \to \bar{k}[S]^*$ is contained in $\bar{k}[S]^*_S$, so it remains to show that any function from $f(x) \in \bar{k}[S]^*_S$ that takes value 1 at the neutral element e of S is a character. Indeed, for any $s \in S(\bar{k})$ we have $f(sx) = \chi(s)f(x)$, and taking $x = e$ we obtain $f(s) = \chi(s)$. □

Corollary 2.4 *Let V be a k-torsor of S. Then we have an exact sequence of Γ-modules*

$$0 \to \bar{k}^* \to \bar{k}[V]^*_S \to \widehat{S} \to 0. \tag{6}$$

The class of extension (6) *in* $\mathrm{Ext}^1_k(\widehat{S}, \bar{k}^*) = H^1(k, S)$ *is* $-[V]$.

Proof The action of $S(\bar{k})$ on $\widehat{S} = \bar{k}[S]^*_S/\bar{k}^*$ is trivial, hence the first statement follows from Proposition 2.3 by Galois descent. In the case when S is a torus the last statement is a well known lemma of Sansuc [26], (6.7.3), (6.7.4), see also Lemma 5.4 of [2]. The same calculation works in the general case. □

We shall need a relative version of Corollary 2.4. Recall that $\pi_*\mathbf{G}_{m,Y}$ is the sheaf on X such that for an étale morphism $U \to X$ we have

$$\pi_*\mathbf{G}_{m,Y}(U) = \mathrm{Mor}_U(Y_U, \mathbf{G}_{m,U}) = \mathrm{Mor}_k(Y_U, \mathbf{G}_{m,k}).$$

Define $(\pi_*\mathbf{G}_{m,Y})_S$ as the subsheaf of $\pi_*\mathbf{G}_{m,Y}$ such that for an étale morphism $U \to X$ the group of sections $(\pi_*\mathbf{G}_{m,Y})_S(U)$ consists of the functions $f(x) \in \mathrm{Mor}_U(Y_U, \mathbf{G}_{m,U})$ for which there exists a group scheme homomorphism $\chi : S_U \to \mathbf{G}_{m,U}$ such that $f(sx) = \chi(s)f(x)$ for any $s \in S_U(\bar{k})$ and any $x \in Y_U(\bar{k})$. If $m : S_U \times_U Y_U = S \times_k Y_U \to Y_U$ is the action of S on Y_U, then the last condition is $m^*f = \chi \cdot f$.

Proposition 2.5 *Let $p : X \to \mathrm{Spec}(k)$ be a smooth and geometrically integral variety, and let $\pi : Y \to X$ be a torsor under S.*

(i) *We have an exact sequence of étale sheaves on X:*

$$0 \to \mathbf{G}_{m,X} \to (\pi_* \mathbf{G}_{m,Y})_S \to p^* \widehat{S} \to 0. \tag{7}$$

Applying p_* to (7) *we obtain an exact sequence of* Γ*-modules*

$$0 \to \bar{k}[X]^* \to \bar{k}[Y]^*_S \to \widehat{S} \to \mathrm{Pic}\overline{X}. \tag{8}$$

(ii) *The class of extension* (7) *in* $\mathrm{Ext}^1_X(p^*\widehat{S}, \mathbf{G}_{m,X}) = H^1(X, S)$ *is the class* $[Y/X]$ *of the X-torsor Y (up to sign).*

(iii) *The last arrow in* (8) *is the type of the torsor* $\pi : Y \to X$.

(iv) *When the type of $\pi : Y \to X$ is zero, the extension given by the first three non-zero terms of* (8) *maps to the class of* (7) *by the canonical injective map*

$$0 \to \mathrm{Ext}^1_k(\widehat{S}, \bar{k}[X]^*) \to \mathrm{Ext}^1_X(p^*\widehat{S}, \mathbf{G}_{m,X}) = H^1(X, S).$$

Proof (i) The maps in this sequence are obvious maps. The exactness can be checked locally, so we can assume that $Y = X \times_k S$, but in this case the exactness is clear. The exact sequence (8) follows from (7) once we note that the canonical morphism $\widehat{S} \to p_* p^* \widehat{S}$ is an isomorphism since $\overline{X}$ is connected.

(ii) The proof of [7, Prop. 1.4.3] applies as is.

(iii-iv) More generally, let A be a Γ-module, and $\mathcal{F}$ be a sheaf on X. Recall that we have the spectral sequence of the composition of functors $\mathbf{R}p_*$ and $\mathbf{R}\mathrm{Hom}_k(A, \cdot)$:

$$\mathrm{Ext}^m_k(A, H^n(\overline{X}, \mathcal{F})) \Rightarrow \mathrm{Ext}^{m+n}_X(p^*A, \mathcal{F}).$$

It gives rise to the exact sequence

$$0 \to \mathrm{Ext}^1_k(A, p_*\mathcal{F}) \to \mathrm{Ext}^1_X(p^*A, \mathcal{F}) \to \mathrm{Hom}_k(A, R^1 p_* \mathcal{F}). \tag{9}$$

The arrows in (9) have explicit description. The canonical map $\mathrm{E}^1 \to \mathrm{E}^{0,1}$ sends the class of the extension of sheaves on X

$$0 \to \mathcal{F} \to \mathcal{E} \to p^*A \to 0$$

to the last arrow in

$$0 \to p_*\mathcal{F} \to p_*\mathcal{E} \to p_*p^*A \to R^1 p_* \mathcal{F},$$

composed with the canonical map $A \to p_*p^*A$. If the class of the extension $\mathcal{E}$ goes to $0 \in \mathrm{E}^{0,1}$, then this class comes from the extension of Γ-modules

$$0 \to p_*\mathcal{F} \to p_*\mathcal{E} \to \mathrm{Ker}[p_*p^*A \to R^1 p_*\mathcal{F}] \to 0$$

pulled back by the same canonical map. See Appendix B to this paper for a proof of these facts. In our case take $A = \widehat{S}$ and $\mathcal{F} = \mathbf{G}_{m,X}$. □

Remarks 1. The type of the torsor $\pi : Y \to X$, at least up to sign, can also be described explicitly as follows. Let $K = \bar{k}(X)$. The fibre of $p : Y \to X$ over $\mathrm{Spec}(K)$ is a K-torsor Y_K under S. By Corollary 2.4 we can lift any character $\chi \in \widehat{S}$ to a rational function $f \in K[Y_K]^*_S \subset \bar{k}(Y)^*$. By construction f is an invertible regular function on Y_K, hence $\mathrm{div}_{\overline{Y}}(f) = \pi^*(D)$ where D is a divisor on $\overline{X}$. Note that D is uniquely determined by χ up to a principal divisor on $\overline{X}$. It is not hard to check that the class of this divisor in $\mathrm{Pic}\overline{X}$ is the image of χ (up to sign). Indeed, by [28], Lemma 2.3.1 (ii), the type associates to χ the subsheaf $\mathcal{O}_\chi$ of χ-semiinvariants of $p_*(\mathcal{O}_Y)$. The function f is a rational section of $\mathcal{O}_\chi$, hence the class $[D]$ represents $\mathcal{O}_\chi \in \mathrm{Pic}\overline{X}$. If this description was used as a definition of type, then the exactness of (8) is easily checked directly.

2. From (8) we obtain the following exact sequence:

$$0 \to \bar{k}[X]^*/\bar{k}^* \to \bar{k}[Y]^*_S/\bar{k}^* \to \widehat{S} \to \mathrm{Pic}\overline{X}. \tag{10}$$

In the same way as in Proposition 2.5 (iii) one shows that when the type of the torsor $\pi : Y \to X$ is zero, the extension given by the first three non-zero terms of (10) maps to the extended type of $\pi : Y \to X$ by the canonical injective map

$$0 \to \mathrm{Ext}^1_k(\widehat{S}, \bar{k}[X]^*/\bar{k}^*) \to \mathrm{Hom}_k(\widehat{S}, KD'(X)).$$

In particular, a surjective homomorphism of k-tori $T_1 \to T_2$ with kernel S is a T_2-torsor under S. The extended type of this torsor comes from the extension

$$0 \to \bar{k}[T_2]^*/\bar{k}^* \to \bar{k}[T_1]^*_S/\bar{k}^* \to \widehat{S} \to 0,$$

which is precisely the dual exact sequence

$$0 \to \widehat{T}_2 \to \widehat{T}_1 \to \widehat{S} \to 0.$$

Theorem 2.6 *Let U be a smooth and geometrically integral variety over k, let S be a k-group of multiplicative type, and let $\pi : \mathcal{T} \to U$ be a torsor under S of type zero. Then we have the following statements.*

(i) *There is a natural exact sequence of Γ-modules*

$$0 \to \bar{k}^* \to \bar{k}[\mathcal{T}]^*_S \to \widehat{M} \to 0,$$

which is the definition of the k-group of multiplicative type M.

(ii) *There is a natural exact sequence of Γ-modules*

$$0 \to \bar{k}[U]^* \to \bar{k}[\mathcal{T}]^*_S \to \widehat{S} \to 0. \tag{11}$$

Let

$$1 \to S \to M \to R \to 1$$

be the dual exact sequence of k-groups of multiplicative type.

(iii) *We have $\mathcal{T} = Z \times_Y U$, where Z is a k-torsor under M that represents the negative of the class of the extension* (i)*, $Z \to Y = Z/S$ is the natural quotient, and $U \to Y$ is the morphism q_U from Lemma* 2.1*.*

Proof We note that the abelian group $\bar{k}[\mathcal{T}]^*_S/\bar{k}^*$ is finitely generated since the same is true for $\bar{k}[U]^*/\bar{k}^*$ and $\widehat{S}$. Thus we can define M as in (i). The extension (11) gives rise to the following commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \bar{k}^* & = & \bar{k}^* & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \bar{k}[U]^* & \to & \bar{k}[\mathcal{T}]^*_S & \to & \widehat{S} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \widehat{R} & \to & \widehat{M} & \to & \widehat{S} & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{12}$$

Similarly to Lemma 2.1 the extension (i) defines a k-torsor Z under M and a morphism $q : \mathcal{T} \to Z$ which identifies (i) with the extension

$$0 \to \bar{k}^* \to \bar{k}[Z]^*_S \to \widehat{M} \to 0.$$

The functoriality of this construction and the commutativity of (12) imply that there is an isomorphism $Z/S \cong Y$ of torsors under R which makes the diagram commute

$$\begin{array}{ccc} \mathcal{T} & \to & U \\ \downarrow & & \downarrow \\ Z & \to & Y \end{array}$$

This gives a morphism $\mathcal{T} \to Z \times_Y U$ of U-torsors under S, which, as any such morphism, is an isomorphism. □

Corollary 2.7 *Let X be a smooth geometrically integral variety over k, let S be a k-group of multiplicative type, and let $\lambda \in \mathrm{Hom}_k(\widehat{S}, KD'(X))$. Let U be a dense open set of X such that the induced element $\lambda_U \in \mathrm{Hom}_k(\widehat{S}, KD'(U))$ has trivial image in $\mathrm{Hom}_k(\widehat{S}, \mathrm{Pic}\overline{U})$, so that*

$$\lambda_U \in \mathrm{Ext}^1_k(\widehat{S}, \bar{k}[U]^*/\bar{k}^*) = \mathrm{Ext}^1_k(\widehat{S}, \widehat{R}) = \mathrm{Ext}^1_{k-\mathrm{groups}}(R, S),$$

where $\widehat{R} = \bar{k}[U]^*/\bar{k}^*$. *Let*

$$1 \to S \to M \to R \to 1$$

be an extension representing this class. Then we have the following statements.

(i) *The restriction of an* X*-torsor of extended type* λ *to* U *is isomorphic to* $Z \times_Y U$*, where* Y *is a* k*-torsor under* R*,* $U \to Y$ *is the morphism* q_U *defined in Lemma* 2.1*, and* Z *is a* k*-torsor under* M *such that* $Y = Z/S$.

(ii) *Conversely, any* U*-torsor* $Z \times_Y U \to U$ *extends to an* X*-torsor under* S *of extended type* λ.

Proof By Remark 2 after Proposition 2.5 we know that the extension (11) represents the class λ_U. Now part (i) follows from Theorem 2.6.

Recall that the embedding $j : U \to X$ gives a natural injective map $\mathbf{G}_{m,X} \to j_*\mathbf{G}_{m,U}$ of étale sheaves on X. On applying $\mathbf{R}p_*$ and the truncation $\tau_{\leq 1}$ we obtain a natural morphism $\tau_{\leq 1}\mathbf{R}p_*\mathbf{G}_{m,X} \to \tau_{\leq 1}\mathbf{R}(pj)_*\mathbf{G}_{m,U}$ in $\mathcal{D}(k)$. It is clear that we have a commutative diagram of exact triangles in $\mathcal{D}(k)$

$$\begin{array}{ccccc}
\bar{k}^* & \to & \tau_{\leq 1}\mathbf{R}p_*\mathbf{G}_{m,X} & \to & KD'(X)[-1] \\
|| & & \downarrow & & \downarrow \\
\bar{k}^* & \to & \tau_{\leq 1}\mathbf{R}(pj)_*\mathbf{G}_{m,U} & \to & KD'(U)[-1]
\end{array}$$

It gives rise to the following commutative diagrams of abelian groups:

$$\begin{array}{ccccccc}
H^1(k,S) & \to & H^1(X,S) & \to & \mathrm{Hom}_k(\widehat{S}, KD'(X)) & \to & H^2(k,S) \\
|| & & \downarrow & & \downarrow & & || \\
H^1(k,S) & \to & H^1(U,S) & \to & \mathrm{Hom}_k(\widehat{S}, KD'(U)) & \to & H^2(k,S)
\end{array} \qquad (13)$$

Now it is easy to complete the proof of the corollary. From Corollary 2.4 and Remark 2 after Proposition 2.5 we see that the extended type of $Z \to Y$ is λ_U. This implies that the extended type of $Z \times_Y U \to U$ is λ_U. Now (ii) is an immediate consequence of (13). □

3 Descent theory

In this and the next chapters k is a number field with the ring of integers $\mathcal{O}_k$. Let Ω_k be the set of places of k, and let Ω_∞ (resp. Ω_f) be the set of archimedean (resp. finite) places of k. For $v \in \Omega_k$ we write k_v for the completion of k at v.

For a variety X over k we denote by $X(\mathbf{A}_k)$ the topological space of adelic points of X; it coincides with $\prod_{v\in\Omega_k} X(k_v)$ when X is proper. Recall (cf. [28], Ch. 5) that the Brauer–Manin pairing

$$X(\mathbf{A}_k) \times \mathrm{Br}(X) \to \mathbf{Q}/\mathbf{Z}$$

is defined by the formula

$$((P_v), \alpha) \mapsto \sum_{v\in\Omega_k} j_v(\alpha(P_v)),$$

where $j_v : \mathrm{Br}(k_v) \to \mathbf{Q}/\mathbf{Z}$ is the local invariant in class field theory. By global class field theory we have $((P_v), \alpha) = 0$ for every $\alpha \in \mathrm{Br}_0(X)$. For a subgroup $B \subset \mathrm{Br}(X)$ (or $B \subset \mathrm{Br}(X)/\mathrm{Br}_0(X)$) we denote by $X(\mathbf{A}_k)^B$ the set of those adelic points that are orthogonal to B, and we write $X(\mathbf{A}_k)^{\mathrm{Br}}$ for $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$. By the reciprocity law in global class field theory, we have $X(k) \subset X(\mathbf{A}_k)^{\mathrm{Br}}$.

If $f : Y \to X$ is a torsor under a k-group of multiplicative type S, the *descent set* $X(\mathbf{A}_k)^f$ is defined as the set of adelic points $(P_v) \in X(\mathbf{A}_k)$ such that the family $([Y](P_v))$ is in the image of the diagonal map $H^1(k, G) \to \prod_{v\in\Omega_k} H^1(k_v, G)$, see [28], Section 5.3.

Proposition 3.1 *Let X be a smooth and geometrically integral variety over a number field k, and let S be a k-group of multiplicative type. An adelic point $(P_v) \in X(\mathbf{A}_k)$ belongs to the descent set $X(\mathbf{A}_k)^f$ associated to the torsor $f : Y \to X$ under S if and only if (P_v) is orthogonal to the subgroup*

$$\mathrm{Br}_\lambda(X) := r^{-1}(\lambda_*(H^1(k, \widehat{S}))) \subset \mathrm{Br}_1(X)$$

with respect to the Brauer–Manin pairing.

Proof The property $(P_v) \in X(\mathbf{A}_k)^f$ means that the family $([Y](P_v))$ is in the image of the diagonal map $H^1(k, S) \to \mathbf{P}^1(S)$, where $\mathbf{P}^1(S)$ is the restricted product of the groups $H^1(k_v, S)$. By the Poitou–Tate exact sequence (see, for example, [10], Thm. 6.3) this is equivalent to the condition

$$\sum_{v\in\Omega_k} j_v((a \cup [Y])(P_v)) = 0$$

for every $a \in H^1(k, \widehat{S})$. On the other hand, by Theorem 1.4 and exact sequence (5) every element of $\mathrm{Br}_\lambda(X)$ can be written as $a \cup [Y] + \alpha_0$, where $\alpha_0 \in \mathrm{Br}_0(X)$. The proposition follows. □

What we want now is an 'integral version' of Proposition 3.1. If Σ is a finite set of places of k, we denote by $\mathcal{O}_\Sigma$ the subring of k consisting of the elements integral at the non-archimedean places outside Σ. Then $U = \mathrm{Spec}\,(\mathcal{O}_\Sigma)$ is an open subset of $\mathrm{Spec}\,(\mathcal{O}_k)$. Let us assume that there are

- a faithfully flat and separated U-scheme of finite type $\mathcal{X}$,
- a flat commutative group U-scheme $\mathcal{S}$ of finite type, and
- an fppf $\mathcal{X}$-torsor $\mathcal{Y}$ under $\mathcal{S}$,

such that $X = \mathcal{X} \times_U k$, $S = \mathcal{S} \times_U k$, and $Y = \mathcal{Y} \times_U k$. This assumption can always be satisfied if Σ is large enough.

Let $[\mathcal{Y}]$ be the class of $\mathcal{Y}$ in the fppf cohomology group $H^1(\mathcal{O}_\Sigma, \mathcal{S}) = H^1(U, \mathcal{S})$. For every $\mathcal{O}_\Sigma$-torsor c under $\mathcal{S}$, one defines the *twisted torsor* $\mathcal{Y}^c = (\mathcal{Y} \times_U c)/\mathcal{S}$. This is a U-torsor under $\mathcal{S}$ such that $[\mathcal{Y}^c] = [\mathcal{Y}] - c$ (see [28], Lemma 2.2.3).

Corollary 3.2 *Let $(P_v) \in \prod_{v\in\Sigma} X(k_v) \times \prod_{v\notin\Sigma} \mathcal{X}(\mathcal{O}_v)$. Then the following conditions are equivalent:*

(a) *The adelic point (P_v) is orthogonal to* $\mathrm{Br}_\lambda(X)$.

(b) *There exists a class $[c] \in H^1(\mathcal{O}_\Sigma, \mathcal{S})$ that goes to $([\mathcal{Y}](P_v))$ under the diagonal map*

$$H^1(\mathcal{O}_\Sigma, \mathcal{S}) \;\to\; \prod_{v\in\Sigma} H^1(k_v, S) \times \prod_{v\notin\Sigma} H^1(\mathcal{O}_v, \mathcal{S}).$$

(c) *There exists an $\mathcal{O}_\Sigma$-torsor c under $\mathcal{S}$ such that the adelic point (P_v) lifts to an adelic point in $\prod_{v\in\Sigma} Y^c(k_v) \times \prod_{v\notin\Sigma} \mathcal{Y}^c(\mathcal{O}_v)$, where $Y^c = \mathcal{Y}^c \times_{\mathcal{O}_\Sigma} k$ is the generic fibre of the twisted torsor $\mathcal{Y}^c$.*

Proof Let c be an $\mathcal{O}_\Sigma$-torsor under $\mathcal{S}$ with cohomology class $[c] \in H^1(\mathcal{O}_\Sigma, \mathcal{S})$. Then $[c]$ goes to $([\mathcal{Y}(P_v)])$ if and only if $[\mathcal{Y}^c(P_v)] = 0$ for every place v. But this is equivalent to the fact that P_v lifts to a point in $Y^c(k_v)$ if $v \in \Sigma$, and to a point in $\mathcal{Y}^c(\mathcal{O}_v)$ if $v \notin \Sigma$. This proves the equivalence of (b) and (c).

Condition (b) implies that the adelic point (P_v) is in the descent set $X(\mathbf{A}_k)^f$. Hence (a) follows from (b) by Proposition 3.1.

Assume condition (a). By Proposition 3.1, the element

$$([\mathcal{Y}(P_v)]) \;\in\; \prod_{v\in\Sigma} H^1(k_v, S) \times \prod_{v\notin\Sigma} H^1(\mathcal{O}_v, \mathcal{S})$$

is in the diagonal image of some $\sigma \in H^1(k,S)$. Since σ is unramified outside Σ, Harder's lemma ([19], Lemma 4.1.3 or [12], Corollary A.8) implies that σ is in the image of the restriction map $H^1(\mathcal{O}_\Sigma,\mathcal{S}) \to H^1(k,S)$. Thus (a) implies (b). □

Remarks 1. If we assume further that $\mathcal{S}$ and $\mathcal{X}$ are smooth over U, then everywhere in the previous corollary we can replace fppf cohomology by étale cohomology. This can be arranged by choosing a sufficiently large set Σ.

2. We refer the reader to [20] and [8] for examples of descent on the torsor $\mathcal{Y} \to \mathcal{X}$ under μ_d, where $\mathcal{X} \subset \mathbf{P}^2_{\mathbf{Z}}$ is the complement to the closed subscheme given by a homogeneous polynomial $f(x,y,z)$ of degree d with integral coefficients, and $\mathcal{Y}$ is given by the equation $u^d = f(x,y,z)$.

Below is a "truncated" variant of Proposition 3.1 where we consider all places of k except finitely many. Keep the notation as above and let Σ_0 be a finite set of places of k. Let $X(\mathbf{A}_k^{\Sigma_0})$ be the topological space of "truncated" adelic points, defined as the restricted product of the spaces $X(k_v)$ for $v \notin \Sigma_0$ with respect to the subsets $\mathcal{X}(\mathcal{O}_v)$, $v \notin \Sigma \cup \Sigma_0$. We define $\mathbf{P}^1_{\Sigma_0}(S)$ as the restricted product of the groups $H^1(k_v,S)$ for $v \notin \Sigma_0$ with respect to the subgroups $H^1(\mathcal{O}_v,\mathcal{S})$, $v \notin \Sigma \cup \Sigma_0$. As in the classical case $\Sigma_0 = \emptyset$, the sets $\mathbf{P}^1_{\Sigma_0}(S)$ and $X(\mathbf{A}_k^{\Sigma_0})$ are independent of the choices of models $\mathcal{S}$ and $\mathcal{X}$. Let $H^1_{\Sigma_0}(k,\widehat{S})$ be the kernel of the restriction map $H^1(k,\widehat{S}) \to \prod_{v\in\Sigma_0} H^1(k_v,\widehat{S})$.

Proposition 3.3 *Let $(P_v)_{v\notin\Sigma_0} \in X(\mathbf{A}_k^{\Sigma_0})$. Then $([Y](P_v))_{v\notin\Sigma_0}$ is in the image of the diagonal map $H^1(k,S) \to \mathbf{P}^1_{\Sigma_0}(S)$ if and only if the "truncated" adelic point $(P_v)_{v\notin\Sigma_0}$ is orthogonal to the subgroup*

$$\mathrm{Br}_{\lambda,\Sigma_0}(X) := r^{-1}(\lambda_*(H^1_{\Sigma_0}(k,\widehat{S}))) \subset \mathrm{Br}_\lambda(X).$$

Proof Using the local Tate duality, we see that the Poitou–Tate exact sequence for S (see [10], Thm. 6.3.) gives rise to the Σ_0-truncated exact sequence

$$H^1(k,S) \to \mathbf{P}^1_{\Sigma_0}(S) \to H^1_{\Sigma_0}(k,\widehat{S})^{\mathrm{D}},$$

where the superscript D denotes the Pontryagin dual $\mathrm{Hom}(\cdot,\mathbf{Q}/\mathbf{Z})$. By this sequence, $(s_v)_{v\notin\Sigma_0} \in \mathbf{P}^1_{\Sigma_0}(S)$ is in the image of $H^1(k,S)$ if and only if

$$\sum_{v\notin\Sigma_0} j_v(a \cup s_v) = 0$$

for every $a \in H^1_{\Sigma_0}(k,\widehat{S})$. The proof finishes in the same way as the proof of Proposition 3.1. □

Taking $\Sigma = \Sigma_0$ we obtain a "truncated" analogue of Corollary 3.2.

Corollary 3.4 *Let $(P_v) \in \prod_{v \notin \Sigma} \mathcal{X}(\mathcal{O}_v)$. Then the following conditions are equivalent.*

(a) *(P_v) is orthogonal to* $\mathrm{Br}_{\lambda,\Sigma}(X)$.

(b) *There exists a class $[c] \in H^1(\mathcal{O}_\Sigma, \mathcal{S})$ that goes to $([\mathcal{Y}](P_v))$ under the diagonal map*

$$H^1(\mathcal{O}_\Sigma, \mathcal{S}) \;\to\; \prod_{v \notin \Sigma} H^1(\mathcal{O}_v, \mathcal{S}).$$

(c) *There exists an $\mathcal{O}_\Sigma$-torsor c under $\mathcal{S}$ such that P_v lifts to a point in $\mathcal{Y}^c(\mathcal{O}_v)$ for every $v \notin \Sigma$.*

Thm. 1 of [18] is a particular case of this result.

Let X be a smooth and geometrically integral k-variety. Define

$$\text{Б}(X) := \ker[\mathrm{Br}_1(X)/\mathrm{Br}(k) \to \prod_{v \in \Omega_k} \mathrm{Br}_1(X_v)/\mathrm{Br}(k_v)],$$

where $X_v := X \times_k k_v$. For $\alpha \in \text{Б}(X)$ and $(P_v) \in X(\mathbf{A}_k)$ the image α_v of α in $\mathrm{Br}_1(X_v)$ is constant for every place v, hence

$$i(\alpha) = \sum_{v \in \Omega_k} j_v(\alpha(P_v)) \;\in\; \mathbf{Q}/\mathbf{Z}$$

is well defined and does not depend on the choice of (P_v). Let us assume that $X(\mathbf{A}_k) \neq \emptyset$. Then we obtain a map $i : \text{Б}(X) \to \mathbf{Q}/\mathbf{Z}$. Note also that this assumption, by global class field theory, implies that the natural map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is injective. For a number field k we have $H^3(k, \bar{k}^*) = 0$, so we see from (5) that the map $r : \mathrm{Br}_1(X) \to H^1(k, KD'(X))$ induces an isomorphism

$$\mathrm{Br}_1(X)/\mathrm{Br}(k) \tilde{\longrightarrow} H^1(k, KD'(X)).$$

If $\mathcal{C}$ is an object of $\mathcal{D}(k)$, and $i > 0$ we define

$$\text{Ш}^i(\mathcal{C}) = \ker[H^i(k, \mathcal{C}) \to \prod_{v \in \Omega_k} H^i(k_v, \mathcal{C})].$$

Thus we get an isomorphism $\text{Б}(X) \tilde{\longrightarrow} \text{Ш}^1(KD'(X))$, using which we obtain a map $i : \text{Ш}^1(KD'(X)) \to \mathbf{Q}/\mathbf{Z}$.

Let S be a k-group of multiplicative type. There is a perfect Poitou–Tate pairing of finite groups (cf. [10], Thm. 5.7)

$$\langle , \rangle_{PT} : \text{Ш}^2(S) \times \text{Ш}^1(\widehat{S}) \to \mathbf{Q}/\mathbf{Z}$$

defined as follows. Let $a \in \text{Ш}^1(\widehat{S})$ and $b \in \text{Ш}^2(S)$. By [22], Lemma III.1.16, the group $H^2(k, S)$ is the direct limit of the groups $H^2(U, \mathcal{S})$ where U runs over non-empty open subsets of Spec $(\mathcal{O}_k)$. Note that by taking a smaller U we can assume that S and $\widehat{S}$ extend to smooth U-group schemes $\mathcal{S}$ and $\widehat{\mathcal{S}}$, respectively. For U sufficiently small we can lift b to some $b_U \in H^2(U, \mathcal{S})$, and lift a to some $\tilde{a}_U \in H^1(U, \widehat{\mathcal{S}})$. For any object $\mathcal{C}$ of $\mathcal{D}(U)$ we have the hypercohomology groups with *compact support* $H^i_c(U, \mathcal{C})$, see Section 3 of [16] for definitions. By [21], Prop. II.2.3 (a) (see also [16], Sect. 3), since a is locally trivial everywhere, $\tilde{a}_U$ comes from some $a_U \in H^1_c(U, \widehat{\mathcal{S}})$ under the natural map

$$H^1_c(U, \widehat{\mathcal{S}}) \to H^1(U, \widehat{\mathcal{S}}).$$

Define $\langle b, a \rangle_{PT}$ as the cup-product $b_U \cup a_U \in H^3_c(U, \mathbf{G}_{m,U}) \simeq \mathbf{Q}/\mathbf{Z}$ (the last isomorphism comes from the trace map, see [21], Prop. II.2.6). It is not clear to us whether this definition of the Poitou–Tate pairing coincides with the classical definition in terms of cocycles, but we shall only use the fact that it leads to a perfect pairing.

Recall that the map $\partial : \text{Hom}_k(\widehat{S}, KD'(X)) \to H^2(k, S)$ was defined in the exact sequence (2).

Theorem 3.5 *Let X be a smooth and geometrically integral variety over a number field k such that $X(\mathbf{A}_k) \neq \emptyset$. Let S be a k-group of multiplicative type, $\lambda \in \text{Hom}_k(\widehat{S}, KD'(X))$ and $a \in \text{Ш}^1(\widehat{S})$. Then $\partial(\lambda) \in \text{Ш}^2(S)$, and we have*

$$\langle \partial(\lambda), a \rangle_{PT} = i(\lambda_*(a)).$$

Proof The image of $\partial(\lambda)$ in $H^2(X, S)$ is zero because (2) is a complex. The assumption $X(\mathbf{A}_k) \neq \emptyset$ implies that the map $H^2(k_v, S) \to H^2(X_v, S)$ is injective for every place v (cf. also Proposition 1.3). Therefore, we have $\partial(\lambda) \in \text{Ш}^2(S)$.

Recall that $w : KD'(X) \to \mathbf{G}_{m,k}[2]$ is the natural map defined in (1) for X/k. Since (2) is obtained by applying the functor $\text{Hom}_k(\widehat{S}, .)$ to (1), under the canonical isomorphism $\text{Hom}_k(\widehat{S}, \mathbf{G}_{m,k}[2]) = H^2(k, S)$ we have the equality $w \circ \lambda = \partial(\lambda)$. Let us write $\alpha = \lambda_*(a) \in \text{Ш}^1(KD'(X))$.

Let $U \subset$ Spec $(\mathcal{O}_k)$ be a sufficiently small non-empty open subset such that there exists a smooth U-scheme $\mathcal{X}$ with geometrically integral fibres and the generic fibre $X = \mathcal{X} \times_U k$, and a smooth U-group of multiplicative type $\mathcal{S}$ with the generic fibre $S = \mathcal{S} \times_U k$.

Write $w_U \in \text{Hom}_U(KD'(\mathcal{X}), \mathbf{G}_{m,U}[2])$ for the map in the exact triangle (1) for $\mathcal{X}/U$. The passage to the generic point Spec (k) of U defines the restriction map

$$\mathrm{Hom}_U(\widehat{\mathcal{S}}, KD'(\mathcal{X})) \to \mathrm{Hom}_k(\widehat{S}, KD'(X)),$$

Consider the exact sequence (2) for $\mathcal{X}_V = \mathcal{X} \times_U V$ and $\mathcal{S}_V = \mathcal{S} \times_U V$, where $V \subset U$ is a non-empty open set, and also for X and S. We obtain a commutative diagram

$$\begin{array}{ccccccc} H^1(\mathcal{X}_V, \mathcal{S}_V) & \to & \mathrm{Hom}_V(\widehat{\mathcal{S}}, KD'(\mathcal{X}_V)) & \to & H^2(V, \mathcal{S}_V) & \to & H^2(\mathcal{X}_V, \mathcal{S}_V) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(X, S) & \to & \mathrm{Hom}_k(\widehat{S}, KD'(X)) & \to & H^2(k, S) & \to & H^2(X, S) \end{array}$$

Passing to the inductive limit over V and using [22], Lemma III.1.16, we deduce from this diagram a canonical surjective homomorphism

$$\varinjlim_V \mathrm{Hom}_V(\widehat{\mathcal{S}}_V, KD'(\mathcal{X}_V)) \to \mathrm{Hom}_k(\widehat{S}, KD'(X)).$$

Thus, by shrinking U, if necessary, we can lift λ to some $\lambda_U \in \mathrm{Hom}_U(\widehat{\mathcal{S}}, KD'(\mathcal{X}))$. Then

$$w_U \circ \lambda_U \in \mathrm{Hom}_U(\widehat{\mathcal{S}}, \mathbf{G}_{m,U}[2]) = H^2(U, \mathcal{S})$$

(see the proof of Proposition 1.1 for the equality here) goes to $\partial(\lambda)$ under the restriction map to $H^2(k, S)$.

As was explained above, we can lift $a \in Ш^1(\widehat{S})$ to some $a_U \in H^1_c(U, \widehat{\mathcal{S}})$. Write $\alpha_U = \lambda_{U*}(a_U)$. Then α_U is sent to α by the natural map

$$H^1_c(U, KD'(\mathcal{X})) \to H^1(k, KD'(X)).$$

By the remark before Proposition 1.1 we can use [17], Prop. 3.3, which gives

$$\begin{array}{rcl} i(\lambda_*(a)) = i(\alpha) & = & w_U \cup \alpha_U = w_{U*}(\alpha_U) = w_{U*}(\lambda_{U*}(a_U)) \\ & = & (w_U \circ \lambda_U)_*(a_U) = (w_U \circ \lambda_U) \cup a_U. \end{array}$$

The above definition of the Poitou–Tate pairing shows that this equals $\langle \partial(\lambda), a \rangle_{PT}$. □

Remark This proof avoids delicate computations with cocycles as in [28], the proof of Thm. 6.1.2, which follows [7], Prop. 3.3.2.

Corollary 3.6 *Let X be a smooth and geometrically integral variety over a number field k such that $X(\mathbf{A}_k)^{\text{Б}(X)} \neq \emptyset$. Then the map*

$$\chi : H^1(X, S) \to \mathrm{Hom}_k(\widehat{S}, KD'(X))$$

is surjective (there exist X-torsors of every extended type). The converse is true when $\mathrm{Pic}(\overline{X})$ is a finitely generated abelian group.

Proof Let $\lambda \in \mathrm{Hom}_k(\widehat{S}, KD'(X))$. Since $X(\mathbf{A}_k)^{\text{Б}(X)} \neq \emptyset$, Theorem 3.5 ensures that $\langle \partial(\lambda), a\rangle_{PT} = 0$ for every $a \in \text{Ш}^1(\widehat{S})$. The non-degeneracy of the Poitou–Tate pairing implies that $\partial(\lambda) = 0$. By Proposition 1.1 this is equivalent to $\lambda \in \mathrm{Im}(\chi)$.

To prove the converse it is enough to show that $i : \text{Б}(X) \to \mathbf{Q}/\mathbf{Z}$ is the zero map. The formation of $\text{Б}(X)$ is functorial in X, so there is a natural restriction map $\text{Б}(X^c) \to \text{Б}(X)$. By [26], formula (6.1.4), this is an isomorphism. The map $i^c : \text{Б}(X^c) \to \mathbf{Q}/\mathbf{Z}$ is the composition

$$\text{Б}(X^c) \tilde{\longrightarrow} \text{Б}(X) \xrightarrow{i} \mathbf{Q}/\mathbf{Z},$$

so it is enough to show that i^c is identically zero.

By functoriality of the exact sequence (2) we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^1(X^c, S) & \xrightarrow{\chi^c} & \mathrm{Hom}_k(\widehat{S}, \mathrm{Pic}(\overline{X}^c)) & \xrightarrow{\partial^c} & H^2(k, S) & \to & H^2(X^c, S) \\
\downarrow & & \downarrow & & || & & \downarrow \\
H^1(X, S) & \xrightarrow{\chi} & \mathrm{Hom}_k(\widehat{S}, KD'(X)) & \xrightarrow{\partial} & H^2(k, S) & \to & H^2(X, S)
\end{array}$$

The commutativity of this diagram implies that if χ is surjective, so that ∂ is zero, then ∂^c is also zero, hence χ^c is surjective. The assumption that $\mathrm{Pic}(\overline{X})$ is finitely generated implies that $\mathrm{Pic}(\overline{X}^c)$ is also finitely generated. Using [28], Prop. 6.1.4, we see that $X^c(\mathbf{A}_k)^{\text{Б}(X^c)}$ is not empty, thus i^c is identically zero. $\square$

See [29], Thm. 3.3.1, for miscellaneous characterisations of the property $X(\mathbf{A}_k)^{\text{Б}(X)} \neq \emptyset$ in terms of the so called elementary obstruction and the generic period.

4 Application: existence of integral points, obstructions to strong approximation

Recall that for a finite set of places $\Sigma_0 \subset \Omega_k$ we denote by $\mathbf{A}_k^{\Sigma_0}$ the ring of k-adèles without v-components for $v \in \Sigma_0$. Let X be a smooth and geometrically integral k-variety such that $X(\mathbf{A}_k) \neq \emptyset$. There exists a finite set of places Σ containing $\Sigma_0 \cup \Omega_\infty$, and a faithfully flat morphism $\mathcal{X} \to \mathrm{Spec}\,(\mathcal{O}_\Sigma)$ such that $X = \mathcal{X} \times_{\mathcal{O}_\Sigma} k$. We shall say that X satisfies *strong approximation*[1] *outside* Σ_0 if $X(k)$ is dense in the restricted product $X(\mathbf{A}_k^{\Sigma_0})$

[1]We adopt the convention that a variety X such that $X(\mathbf{A}_k) = \emptyset$ satisfies strong approximation outside Σ_0 for every Σ_0.

of the sets $X(k_v)$ for $v \notin \Sigma_0$ with respect to the subsets $\mathcal{X}(\mathcal{O}_v)$ (defined for $v \notin \Sigma$). The restricted product topology is called the *strong* topology. Explicitly, the base of open subsets of this topology consists of the sets

$$\prod_{v \in T} U_v \times \prod_{v \notin T} \mathcal{X}(\mathcal{O}_v),$$

where T is a finite subset of $\Omega_k \setminus \Sigma_0$ such that $\Sigma \subset T$, and U_v is an open subset of $X(k_v)$ for $v \in T$.

The following theorem gives sufficient conditions for "the Brauer–Manin obstruction to strong approximation outside Σ_0" to be the only obstruction on X.

Theorem 4.1 *Let X be a smooth and geometrically integral k-variety such that $X(\mathbf{A}_k) \neq \emptyset$, and let S be a k-group of multiplicative type. Let Σ_0 be a finite set of places of k. Assume that there exists an X-torsor Y under S with the following property:* For all k-torsors c under S, the twisted torsor Y^c has the strong approximation property outside Σ_0. *If $(P_v) \in X(\mathbf{A}_k)$ is orthogonal to* $\mathrm{Br}_\lambda(X)$*, then $(P_v)_{v \notin \Sigma_0}$ belongs to the closure of $X(k)$ in $X(\mathbf{A}_k^{\Sigma_0})$ for the strong topology.*

Proof Choose a finite set of places Σ containing $\Sigma_0 \cup \Omega_\infty$ such that X is the generic fibre of a flat smooth $\mathcal{O}_\Sigma$-scheme of finite type $\mathcal{X}$. We can also assume that the torsor $Y \to X$ extends to a torsor $\mathcal{Y} \to \mathcal{X}$ under a smooth $\mathcal{O}_\Sigma$-group scheme of multiplicative type $\mathcal{S}$ such that $S = \mathcal{S} \times_{\mathcal{O}_\Sigma} k$. Furthermore, we can assume that the adelic point $(P_v) \in X(\mathbf{A}_k)$ belongs to $\prod_{v \in \Sigma} X(k_v) \times \prod_{v \notin \Sigma} \mathcal{X}(\mathcal{O}_v)$. We want to find a rational point on X very close to P_v for $v \in (\Sigma - \Sigma_0)$ and integral outside Σ.

By Corollary 3.2, the property that (P_v) is orthogonal to $\mathrm{Br}_\lambda(X)$ implies that it can be lifted to an adelic point $(Q_v) \in \prod_{v \in \Sigma} Y^c(k_v) \times \prod_{v \notin \Sigma} \mathcal{Y}^c(\mathcal{O}_v)$ on some twisted torsor Y^c. In particular, $Y^c(\mathbf{A}_k) \neq \emptyset$. Since Y^c satisfies strong approximation outside Σ_0, we can find a rational point $m \in Y^c(k)$ very close to Q_v for $v \in (\Sigma - \Sigma_0)$ and integral outside Σ. Sending m to X produces a rational point $m' \in X(k)$ very close to P_v for $v \in (\Sigma - \Sigma_0)$ and integral outside Σ. □

The following corollary gives sufficient conditions for "the Brauer–Manin obstruction to the integral Hasse principle" to be the only obstruction.

Corollary 4.2 *Let $\mathcal{X}$ be a faithfully flat and separated scheme of finite type over $\mathcal{O}_k$ such that $X = \mathcal{X} \times_{\mathcal{O}_k} k$. Assume that Y^c has the strong approximation property outside Ω_∞ for every k-torsor c under S. If there exists an adelic point $(P_v) \in \prod_{v \in \Omega_k} \mathcal{X}(\mathcal{O}_v)$ orthogonal to* $\mathrm{Br}_\lambda(X)$*, then $\mathcal{X}(\mathcal{O}_k) \neq \emptyset$.*

Proof Theorem 4.1 says that (P_v) can be approximated by a rational point $m \in X(k)$ for the strong topology on $X(\mathbf{A}_k^{\Omega_\infty})$. Since $P_v \in \mathcal{X}(\mathcal{O}_v)$ for $v \in \Omega_f$, this implies that $m \in \mathcal{X}(\mathcal{O}_k)$. $\square$

As an application of Theorem 4.1 we get a short proof of a result that already appeared in C. Demarche's thesis [9, Remark 4.8.2] (see also [6, Thm. 4.5], where there is an additional assumption that the geometric stabiliser $\overline{H}$ is finite).

Theorem 4.3 *Let G be a semi-simple, simply connected linear group over a number field k. Let Σ_0 be a finite set of places of k such that for every almost k-simple factor G_1 of G there exists a place $v \in \Sigma_0$ such that $G_1(k_v)$ is not compact (for example, if k is not totally real we can take $\Sigma_0 = \{v_0\}$, where v_0 is a complex place of k). Let X be a homogeneous space of G such that the geometric stabiliser $\overline{H}$ is a $\bar{k}$-group of multiplicative type. Then for every adelic point $(P_v)_{v\in\Omega_k}$ of X orthogonal to $\mathrm{Br}_1(X)$, the point $(P_v)_{v\notin\Sigma_0}$ is in the closure of $X(k)$ in $X(\mathbf{A}_k^{\Sigma_0})$ for the strong topology.*

In other words: the Brauer–Manin obstruction to strong approximation outside Σ_0 is the only one on X.

Proof Let us assume that G acts on X on the left. Then $\overline{X}$ with the left action of $\overline{G}$ is isomorphic to $\overline{G}/\overline{H}$. Since $\mathrm{Pic}(\overline{G}) = 0$ and $\bar{k}[G]^* = \bar{k}^*$, the abelian group $\mathrm{Pic}(\overline{X})$ is finitely generated, and $\bar{k}[X]^* = \bar{k}^*$. Now the existence of a point $(P_v)_{v\in\Omega_k}$ orthogonal to $\mathrm{Br}_1(X)$ implies that $X(k) \neq \emptyset$ by [28], Prop. 6.1.4, and [15], Prop 3.7 (3) and Example 3.4. Therefore X with the left action of G is isomorphic to $X = G/H$, where H is a k-group of multiplicative type. Taking $Y = G$, we obtain a right torsor $Y \to X$ under H such that for any k-torsor c under H the twist Y^c is a left k-torsor under G.

By the Hasse principle for semi-simple simply connected groups (a theorem of Kneser–Harder–Chernousov), $Y^c(\mathbf{A}_k) \neq \emptyset$ implies $Y^c(k) \neq \emptyset$, hence $Y^c \simeq G$. By the strong approximation theorem (see, for example, [25], Thm. 7.12), G satisfies strong approximation outside Σ_0. It remains to apply Theorem 4.1. $\square$

Remarks 1. It is not clear to us whether Corollary 4.2 still holds if we only assume that all the twists Y^c satisfy the integral Hasse principle: indeed, we do not know in general whether the torsor $Y \to X$ can be extended to an fppf torsor $\mathcal{Y} \to \mathcal{X}$ over $\mathrm{Spec}\,(\mathcal{O}_k)$.

2. The assumptions of Theorem 4.1 and Corollary 4.2 imply that $\bar{k}[X]^* = \bar{k}^*$, that is, we are still in "the classical case" of descent theory. Indeed, if Y satisfies strong approximation outside a finite set of places, then $\overline{Y}$ is simply connected (this was first observed in [23], Thm. 1, see also [14], Cor. 2.4). This implies $\bar{k}[Y]^* = \bar{k}^*$, and hence $\bar{k}[X]^* = \bar{k}^*$. (Otherwise pick up a function $f \in \bar{k}[Y]^*$ such that the image of f in the free abelian group $\bar{k}[Y]^*/\bar{k}^*$ is not divisible by a prime ℓ. Then the normalisation of $\overline{Y}$ in $\bar{k}(Y)(f^{1/\ell})$ is a connected étale covering of $\overline{Y}$ of degree ℓ.)

Appendix A

Let Z be an integral regular Noetherian scheme, and let $p : X \to Z$ be a smooth faithfully flat morphism of finite type with geometrically integral fibres. The goal of this appendix is to show that the object $\tau_{\leq 1}\mathbf{R}p_*\mathbf{G}_{m,X}$ of the derived category $\mathcal{D}(Z)$ of étale sheaves on Z can be represented by an explicit two-term complex. This links our $KD(X)$ and $KD'(X)$ with their analogues introduced in [17], Remark 2.4 (2).

Let $j : \eta = \mathrm{Spec}(k(X)) \hookrightarrow X$ be the inclusion of the generic point. Since X is regular, there is no difference between Weil and Cartier divisors, so we have the following exact sequence of sheaves on X, see [22], Examples II.3.9 and III.2.22:

$$0 \to \mathbf{G}_{m,X} \to j_*\mathbf{G}_{m,\eta} \to \mathrm{Div}_X \to 0,$$

where Div_X is the sheaf of divisors on X, that is, the sheaf associated to the presheaf such that the group of sections over an étale U/X is the group of divisors on U.

We call an irreducible effective divisor D on X *horizontal* if it is the Zariski closure of a divisor on the generic fibre of $p : X \to Z$. If $D = p^{-1}(D')$ for a divisor D' on Z, we call D *vertical.* The sheaf Div_X is the direct sum of sheaves

$$\mathrm{Div}_X = \mathrm{Div}_{X/Z} \oplus \mathrm{Div}_X^v,$$

where $\mathrm{Div}_{X/Z}$ is the subsheaf of horizontal divisors, and Div_X^v is the subsheaf of vertical divisors.

Define a subsheaf $K_{X/Z}^{\times} \subset j_*\mathbf{G}_{m,\eta}$ by the condition that the following diagram is commutative and has exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{G}_{m,X} & \to & K^{\times}_{X/Z} & \to & \mathrm{Div}_{X/Z} & \to & 0 \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{G}_{m,X} & \to & j_*\mathbf{G}_{m,\eta} & \to & \mathrm{Div}_X & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Div}^v_X & = & \mathrm{Div}^v_X & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The complex of étale sheaves on Z

$$p_*K^{\times}_{X/Z} \to p_*\mathrm{Div}_{X/Z},$$

after the shift by 1 to the left, is the complex $\mathcal{KD}(\mathcal{X})$ defined in [17], Remark 2.4 (2), see also the formulae on the bottom of page 538. There is a natural injective morphism $\mathbf{G}_{m,Z} \to p_*K^{\times}_{X/Z}$; the complex

$$p_*K^{\times}_{X/Z}/\mathbf{G}_{m,Z} \to p_*\mathrm{Div}_{X/Z}$$

was introduced in [17] and denoted there by $\mathcal{KD}'(\mathcal{X})$.

Proposition *The object $\tau_{\leq 1}\mathbf{R}p_*\mathbf{G}_{m,X}$ of the derived category of étale sheaves on Z is represented by the complex $p_*K^{\times}_{X/Z} \to p_*\mathrm{Div}_{X/Z}$.*

Proof The proof of Lemma 2.3 of [2] works in our situation. To complete the proof we only need to show that $R^1p_*K^{\times}_{X/Z} = 0$. Note that the canonical morphism $\mathrm{Div}_Z \to p_*\mathrm{Div}^v_X$ is an isomorphism because p is surjective with geometrically integral fibres. Now the exact sequence of sheaves on X

$$0 \to K^{\times}_{X/Z} \to j_*\mathbf{G}_{m,\eta} \to \mathrm{Div}^v_X \to 0$$

gives rise to the following exact sequence of sheaves on Z:

$$p_*j_*\mathbf{G}_{m,\eta} \to \mathrm{Div}_Z \to R^1p_*(K^{\times}_{X/Z}) \to R^1p_*(j_*\mathbf{G}_{m,\eta}).$$

Using the spectral sequence of the composition of functors $\mathbf{R}p_*$ and $\mathbf{R}j_*$ we see that the sheaf $R^1p_*(j_*\mathbf{G}_{m,\eta})$ has a canonical embedding into $R^1(pj)_*\mathbf{G}_{m,\eta}$. The latter sheaf is zero by Grothendieck's version of Hilbert's theorem 90.

It remains to prove the surjectivity of $(pj)_*\mathbf{G}_{m,\eta} \to \mathrm{Div}_Z$, which is enough to check at the stalk at any geometric point of Z. But locally every divisor on Z is the divisor of a function, since Z is regular. This completes the proof. $\square$

Remark In this appendix we worked over the small étale site of Z. Applying our arguments to an arbitrary smooth scheme of finite type S/Z one shows that the same results remain true for the smooth site Sm/Z used in [17].

Appendix B

The functor $\mathbf{R}\mathrm{Hom}_X(p^*A, \cdot) : \mathcal{D}(X) \to \mathcal{D}(\mathrm{Ab})$ is the composition of functors $\mathbf{R}p_* : \mathcal{D}(X) \to \mathcal{D}(k)$ and $\mathbf{R}\mathrm{Hom}_k(A, \cdot) : \mathcal{D}(k) \to \mathcal{D}(\mathrm{Ab})$, hence we have

$$\mathbf{R}\mathrm{Hom}_X(p^*A, \mathcal{F}) = \mathbf{R}\mathrm{Hom}_k(A, \mathbf{R}p_*\mathcal{F}).$$

Explicitly, this isomorphism associates to $p^*A \to \mathcal{F}$ the composition

$$A \to \mathbf{R}p_*(p^*A) \to \mathbf{R}p_*\mathcal{F},$$

where the first map is the canonical adjunction morphism. The inverse associate to $A \to \mathbf{R}p_*\mathcal{F}$ the composition

$$p^*A \to p^*(\mathbf{R}p_*\mathcal{F}) \to \mathcal{F},$$

where the last map is the second canonical adjunction morphism.

Let us now complete the proof of Proposition 2.5 (iii). To give an equivalence class of the extension of sheaves on X

$$0 \to \mathcal{F} \to \mathcal{E} \to p^*A \to 0 \tag{14}$$

is the same as to give a morphism $p^*A \to \mathcal{F}[1]$ in the derived category $\mathcal{D}(X)$. By the above, to this morphism we associate the composition

$$A \to \mathbf{R}p_*p^*A \to \mathbf{R}p_*\mathcal{F}[1].$$

Since A is a one-term complex concentrated in degree 0 this composition comes from a morphism $\alpha : A \to (\tau_{\leq 1}\mathbf{R}p_*\mathcal{F})[1]$ in $\mathcal{D}(k)$. By taking the 0-th cohomology we obtain a homomorphism $\beta : A \to R^1p_*\mathcal{F}$ of discrete Galois modules. Clearly, β is the composition of the canonical map $A \to p_*p^*A$ with the differential in the long exact sequence of cohomology attached to (14):

$$0 \to p_*\mathcal{F} \to p_*\mathcal{E} \to p_*p^*A \to R^1p_*\mathcal{F}.$$

To finish the proof of (iii) we need to show that β can also be obtained through the spectral sequence, that is, as the image of the class of (14) under the right arrow in (9). But (9) is obtained by applying $\mathbf{R}\mathrm{Hom}_k(A,\cdot)$ to the exact triangle

$$(p_*\mathcal{F})[1] \to (\tau_{\leq 1}\mathbf{R}p_*\mathcal{F})[1] \to R^1p_*\mathcal{F}. \tag{15}$$

By definition, β is the composition of α with the right map in (15), so the proof of (iii) is now complete.

Let us complete the proof of Proposition 2.5 (iv). The exact triangle (15) gives rise to the exact sequence of abelian groups

$$0 \to \mathrm{Hom}_k(A,(p_*\mathcal{F})[1]) \to \mathrm{Hom}_k(A,(\tau_{\leq 1}\mathbf{R}p_*\mathcal{F})[1]) \to \mathrm{Hom}_k(A,R^1p_*\mathcal{F}),$$

which is the same as (9). Since the right arrow here sends α to β, we see that if $\beta = 0$, then α comes from a morphism $A \to (p_*\mathcal{F})[1]$. Hence the class of (14) comes from the class of an extension of A by $p_*\mathcal{F}$, say

$$0 \to p_*\mathcal{F} \to B \to A \to 0, \tag{16}$$

in the sense that (14) is the push-out of

$$0 \to p^*p_*\mathcal{F} \to p^*B \to p^*A \to 0$$

by the adjunction map $p^*p_*\mathcal{F} \to \mathcal{F}$. Therefore, by the description of the adjunction isomorphism and its inverse given above, applying p_* to (14), and pulling back the resulting short exact sequence via the adjunction map $A \to p_*p^*A$ (this makes sense when $\beta = 0$) gives back the extension (16). □

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