Del Pezzo surfaces of degree 4 and their relation to Kummer surfaces

Alexei Skorobogatov

Introduction

In this note, which has little pretence to originality, we clarify the relation between the geometry of del Pezzo surfaces of degree 4 and their realization as the zero set of two quadratic forms in five variables. We also review the classical description of the desingularized Kummer surface K constructed from the Jacobian J of a curve C of genus 2 as the zero set of three quadratic forms in six variables (Plücker, Kummer, Klein [7], [6], see [5] or [3] for a modern treatment). If C has a rational Weierstrass point, a partial diagonalization of this system gives rise to a natural projection onto a hyperplane, defining a finite morphism $\pi : K \to X$ of degree 2 onto a del Pezzo surface X of degree 4 (see [9, §6]). We show that X is the blow-up of $\mathbb{P}^2_k$ in the images of the five other Weierstrass points of C under the embedding of $\mathbb{P}^1_k$ as a conic in $\mathbb{P}^2_k$. The morphism π sends the 16 lines on K to the 16 lines on X, and is equivariant with respect to the action of the subgroup of 2-division points $J[2] \subset J$. Thus π gives rise to a morphism from the twisted Kummer surface to the twisted del Pezzo surface. In our presentation it is obvious that all del Pezzo surfaces of degree 4 can be obtained in this way, an observation made by Victor Flynn in [4]. The fact that any 2-covering of J maps to a del Pezzo surface of degree 4 was first observed in [2], and used in [2], [1] and [9] to construct and visualize elements of order 2 in the Tate–Shafarevich group of J over $\mathbf{Q}$ using the theory of the Brauer–Manin obstruction on del Pezzo surfaces of degree 4. It was the author's desire to understand the geometry behind these calculations that prompted him to write this note. I would like to thank Igor Dolgachev for useful discussions.

1 Preliminaries

Let k be a field of characteristic not equal to 2 with separable closure $\overline{k}$, and Galois group $\Gamma = \mathrm{Gal}\,(\overline{k}/k)$.

Let L be an étale k-algebra, that is, $L = \oplus_{j=1}^m k_j$ for some finite separable field extensions k_j/k. The trace map $\mathrm{Tr}_{L/k} : L \to k$ is defined as the sum of traces

$\mathrm{Tr}_{k_j/k} : k_j \to k$. Similarly, the norm map $\mathrm{N}_{L/k} : L^* \to k^*$ is the product of norms $\mathrm{N}_{k_j/k} : k_j^* \to k^*$. Let $n = \dim_k L$. For example, if $P(x)$ is a separable polynomial of degree n, then $L = k[x]/(P(x))$ is an étale k-algebra of dimension n. Let $\theta \in L$ be the image of x. Lagrange interpolation gives rise to the well known relations

$$\mathrm{Tr}_{L/k}(P'(\theta)^{-1}\theta^i) = 0, \quad i = 0, 1, \ldots, n-2, \tag{1}$$

where $P'(x)$ is the derivative of $P(x)$.

Assume that n is *odd*.

Consider the finite étale abelian group k-scheme $G = \mathrm{R}_{L/k}(\mu_2)/\mu_2$, where $\mathrm{R}_{L/k}$ is the Weil restriction of scalars. The abelian group $G(\overline{k}) \simeq (\mathbf{Z}/2)^{n-1}$ is generated by n elements of order 2 whose product is the identity. These generators are permuted by Γ in the same way as the components of $L \otimes_k \overline{k} \simeq \overline{k}^{\,n}$. There is an exact sequence of k-groups

$$1 \to \mu_2 \to \mathrm{R}_{L/k}(\mu_2) \to G \to 1.$$

Since n is odd, the usual restriction-corestriction argument shows that the map

$$\mathrm{H}^2(k, \mu_2) \to \mathrm{H}^2(k, \mathrm{R}_{L/k}(\mu_2)) = \mathrm{H}^2(L, \mu_2)$$

is injective. Thus we have

$$\mathrm{H}^1(k, G) = L^*/k^*L^{*2} = \mathrm{Coker}\,[\Delta \;:\; k^*/k^{*2} \to \prod_j k_j^*/k_j^{*2}], \tag{2}$$

where Δ is the diagonal map.

We shall have to deal with 5-tuples of points on the projective line, as well as with 5-tuples of points and 5-tuples of lines in the projective plane. Recall that all these data are equivalent up to projective transformation. Indeed, to give five distinct points in $\mathbb{P}^1_k$ is equivalent to giving five points in $\mathbb{P}^2_k$ in general position (this means that no three points are on the same line). In one direction, use the Veronese embedding $\mathbb{P}^1_k \to S^2(\mathbb{P}^1_k) = \mathbb{P}^2_k$, where S^2 denotes the symmetric square. In the other direction take the unique conic $C \simeq \mathbb{P}^1_k$ through five points in the plane. Five lines in general position in $\mathbb{P}^2_k$ correspond to five points in general position in the dual projective plane.

Similarly, to give six distinct points on a smooth projective curve of genus 0 is equivalent to giving six points in $\mathbb{P}^2_k$ lying on a conic. This is also equivalent to giving six lines in the dual plane $\mathbb{P}^2_k$ which are tangent to a common conic.

2 Del Pezzo surfaces of degree 4

2.1 Equations

We assume that k has at least 5 elements.

Let X be a del Pezzo surface of degree 4, i.e. a smooth intersection of two quadrics in $\mathbb{P}^4_k$. Let Q_1 and Q_2 be quadratic forms in five variables such that X is given by $Q_1 = Q_2 = 0$. By [10, Prop. 2.1] exactly five quadrics in the pencil of quadrics containing X are singular. Using the assumption about k we can assume without loss of generality that $\det Q_1 \neq 0$. By a linear change of variables and the multiplication of Q_1 by an element of k^* we can arrange that $\det Q_1 = 1$. Then the characteristic polynomial $P(x) = \det(Q_1 x - Q_2)$ is a separable monic polynomial of degree 5, so that $P(x) = \prod_{i=1}^5 (x - \theta_i)$ for some distinct $\theta_i \in \overline{k}$. Then $L = k[x]/(P(x))$ is an étale k-algebra of dimension 5. Let θ be the image of x in L; then $(\theta_i) \in \overline{k}^{\,5}$ is the image of θ under the map $L \to L \otimes_k \overline{k} = \overline{k}^{\,5}$.

Over $\overline{k}$ the quadrics of the pencil can be simultaneously diagonalized (*ibidem*). More precisely, we can write $\mathbb{P}^4_k = \mathbb{P}(\mathrm{R}_{L/k}\mathbb{A}^1_L)$, and let $u = \sum_{i=0}^4 u_i \theta^i$ be a variable in $\mathbb{A}^1_L$. For an arbitrary del Pezzo surface X of degree 4 with characteristic polynomial $P(x)$ there exists $\alpha \in L^*$ such that X is given by equations

$$\mathrm{Tr}_{L/k}(\alpha u^2) = \mathrm{Tr}_{L/k}(\alpha\theta u^2) = 0, \quad \text{or, equivalently,} \quad \sum_{i=1}^5 \alpha_i z_i^2 = \sum_{i=1}^5 \alpha_i \theta_i z_i^2 = 0, \tag{3}$$

where $(\alpha_i) \in \overline{k}^{\,5}$ is the image of α in $L \otimes_k \overline{k} = \overline{k}^{\,5}$.

Let $G = \mathrm{R}_{L/k}(\mu_2)/\mu_2$. The abelian group $G(\overline{k}) \simeq (\mathbf{Z}/2)^4$ is generated by five elements of order 2 whose product is the identity. These generators are permuted by Γ in the same way as the indices of the θ_i. The k-group G acts on $\mathbb{P}^4_k$ by changing the signs of the coordinates z_i, so G leaves invariant every quadric that contains X, and thus preserves X. From (3) it is clear that the natural morphism $X \to X/G$ sends u to u^2, so that X/G is the subset of $\mathbb{P}^4_k = \mathbb{P}(\mathrm{R}_{L/k}\mathbb{A}^1_L)$ with L-coordinate $w = u^2$, given by

$$\mathrm{Tr}_{L/k}(\alpha w) = \mathrm{Tr}_{L/k}(\alpha\theta w) = 0. \tag{4}$$

In particular, $X/G \simeq \mathbb{P}^2_k$. Set $\delta = \alpha P'(\theta)$. By relations (1) the 3-dimensional subspace of $\mathrm{R}_{L/k}\mathbb{A}^1_L$ given by (4) is spanned by δ^{-1}, $\delta^{-1}\theta$, $\delta^{-1}\theta^2$. Thus we can write $w = \delta^{-1}(t_0 + t_1\theta + t_2\theta^2)$, where t_0, t_1, t_2 are coordinates over k. Therefore, X is given by the vanishing of the θ^3 and θ^4-terms in

$$t_0 + t_1\theta + t_2\theta^2 = \delta u^2 = \delta(\sum_{i=0}^4 u_i\theta^i)^2. \tag{5}$$

Thus every del Pezzo surface of degree 4 is isomorphic to the surface given by (5) for some separable polynomial $P(x)$ of degree 5, and $\delta \in L^*$. This was pointed out by E.V. Flynn [4].

Remark. We note that if $\delta = 1$, then X contains the line $\mathbb{P}^1_k$ with coordinates $(r : s)$, given by $u = r + s\theta$, $t_0 = r^2$, $t_1 = 2rs$, $t_2 = s^2$.

2.2 Geometry

To a del Pezzo surface X of degree 4 we associate the reduced closed 5-element subscheme $S = S_X \subset \mathbb{P}^1_k$ parameterizing singular quadrics in the pencil of quadrics through X.

Definition 2.1 *A del Pezzo surface X of degree 4 over k is called* **split** *if all the 16 lines on X are defined over k. Let us call a del Pezzo surface X of degree 4* **quasi-split** *if it has at least one line defined over k. Equivalently, X is quasi-split if it is the blow-up of $\mathbb{P}^2_k$ in a Galois-stable set of five $\overline{k}$-points in general position.*

To see the equivalence of the two definitions note that the five lines on $\overline{X}$ meeting a fixed k-line are disjoint, and so can be simultaneously contracted, which gives a morphism $X \to \mathbb{P}^2_k$. Conversely, the blow-up of $\mathbb{P}^2_k$ in a Galois-stable set of five points in general position contains the k-line which is the strict transform of the unique conic through these five points.

Lemma 2.2 *Any quasi-split del Pezzo surface Y of degree 4 is isomorphic to the blow-up of $\mathbb{P}^2_k$ in the image of S_Y under the Veronese embedding $\mathbb{P}^1_k \hookrightarrow S^2(\mathbb{P}^1_k) = \mathbb{P}^2_k$.*

Proof Let Y be a quasi-split del Pezzo surface of degree 4 with a k-line ℓ. The contraction of the five $\overline{k}$-lines of Y that meet ℓ represents Y as the blow-up of $\mathbb{P}^2_k$ in a Galois-stable set of five $\overline{k}$-points, and identifies ℓ with the unique conic through them. It is enough to prove that the resulting 5-element subscheme $F \subset \ell \simeq \mathbb{P}^1_k$ is projectively equivalent to S_Y. Choose a k-point x_0 in $\ell \setminus F$, which is possible since $|k| \geq 5$. We identify ℓ with the pencil Π of quadrics through Y as follows. The tangent spaces $T_{x_0,Q}$, where Q is a quadric in Π, are precisely the hyperplanes in $\mathbb{P}^4_k$ containing the tangent plane $T_{x_0,Y}$. If x is a $\overline{k}$-point in $\ell \setminus F$, then the union of ℓ and the inverse image of the line $(x_0x) \subset \mathbb{P}^2_k$ in Y is the hyperplane section $T_{x_0,Q} \cap Y$ for a unique non-singular quadric Q in Π. This defines an isomorphism $\Pi \simeq \ell$ which identifies F and S_Y. QED

The scheme $S = S_X$ defines the étale k-algebra $L = k[S]$ and hence the k-group $G = \mathrm{R}_{L/k}(\mu_2)/\mu_2$. The singular quadrics containing X are cones over smooth quadric surfaces. The action of G on X has the following geometric description. The five generators of $G(\overline{k})$ correspond to the five singular quadrics containing X, so that each generator acts on $\overline{X}$ as the deck transformation of the double covering given by the projection of $\overline{X}$ from the vertex of the corresponding quadratic cone to its base.

As a projective variety with an action of G, X can be twisted by a 1-cocycle of the Galois group Γ with coefficients in $G(\overline{k})$ (see [11, Ch. 2] for details). The classes in $\mathrm{H}^1(k, G)$ bijectively correspond to the isomorphism classes of k-torsors under G. A k-torsor τ under G is a k-scheme with an action of G such that $\tau \times_k \overline{k}$ is isomorphic to $\overline{G}$ with its action on itself by translations. The twist ${}^\tau X$ of X by τ is the quotient of $\tau \times_k X$ by the diagonal action of G. This is a del Pezzo surface of degree 4 over k which is isomorphic to $\overline{X}$ over $\overline{k}$. The action of G on X comes from its action on $\mathbb{P}^4_k$ that leaves invariant every quadric through X. Thus the twisting has no effect on $S = S_X$. If $\lambda \in L^*$ represents a class in $\mathrm{H}^1(k, G)$ given by formula (2), and X is given by (3), then the twisted surface is given by

$$\mathrm{Tr}_{L/k}(\alpha\lambda u^2) = \mathrm{Tr}_{L/k}(\alpha\theta\lambda u^2) = 0.$$

It is easy to check that $G(\overline{k})$ acts simply transitively on the 16 lines of $\overline{X}$. This action defines a k-torsor τ_X under G, which we call the torsor of lines of X. A del Pezzo surface of degree 4 is quasi-split if and only if its torsor of lines is trivial, i.e. has a k-point.

Theorem 2.3 *Let X be a del Pezzo surface of degree* 4*, and let S_X be the attached reduced* 5*-element subscheme of $\mathbb{P}^1_k$. Let X_0 be the blow-up of $\mathbb{P}^2_k$ in the image of S_X under the Veronese embedding $\mathbb{P}^1_k \hookrightarrow S^2(\mathbb{P}^1_k) = \mathbb{P}^2_k$. Then X_0 is*

(a) *the unique (up to isomorphism) quasi-split twist of X by a k-torsor under G;*

(b) *the unique (up to isomorphism) quasi-split del Pezzo surface of degree* 4 *such that S_X and S_{X_0} are projectively equivalent as subschemes of $\mathbb{P}^1_k$.*

Proof The surface X_0 is clearly quasi-split, moreover, the subschemes S_X and S_{X_0} of $\mathbb{P}^1_k$ are projectively equivalent by Lemma 2.2. Let us show that X_0 is the unique quasi-split twist of X. If τ is a k-torsor under G, then the torsor of lines of the twist ${}^\tau X$ is $\tau \times_k \tau_X$. The class of this torsor is $[\tau_X] - [\tau] \in \mathrm{H}^1(k, G)$, hence ${}^\tau X$ is quasi-split if and only if $\tau = \tau_X$. Thus the twist of X by its torsor of lines is the unique quasi-split twist of X. Since the twisting does not affect S_X we see from Lemma 2.2 that the twist of X by τ_X is isomorphic to X_0. This proves (a). The uniqueness in (b) is immediate from Lemma 2.2. QED

If X is given by (3), then, by the remark in the end of the previous section, X_0 is given by

$$\mathrm{Tr}_{L/k}(P'(\theta)^{-1}u^2) = \mathrm{Tr}_{L/k}(P'(\theta)^{-1}\theta u^2) = 0,$$

or, equivalently, by

$$\sum_{i=1}^{5} P'(\theta_i)^{-1}z_i^2 = \sum_{i=1}^{5} P'(\theta_i)^{-1}\theta_i z_i^2 = 0. \tag{6}$$

When all the roots θ_i of $P(x)$ are in k, the last set of equations describes a split del Pezzo surface of degree 4.

We obtain the following classification of del Pezzo surfaces of degree 4: their isomorphism classes are in a natural bijection with pairs $(S, [\lambda])$, where S is a reduced closed 5-element subscheme of $\mathbb{P}^1_k$, considered up to projective equivalence, and $[\lambda] \in \mathrm{H}^1(k, G_S)$. If S is given by $P(x) = 0$ and $\lambda \in L^*$, then the twisted surface X_λ is given by

$$\mathrm{Tr}_{L/k}(\lambda P'(\theta)^{-1}u^2) = \mathrm{Tr}_{L/k}(\lambda P'(\theta)^{-1}\theta u^2) = 0. \quad (7)$$

Quasi-split surfaces are those for which $[\lambda]$ is trivial, and split surfaces are those for which $[\lambda]$ is trivial and S is the disjoint union of five copies of $\mathrm{Spec}(k)$.

3 Kummer surfaces attached to curves of genus 2

3.1 Multiplication by 2 on the Kummer surface

Let C be a curve of genus 2, and let $W \subset C$ be the closed subscheme of Weierstrass points of C. We denote by $M = k[W]$ the corresponding 6-dimensional étale k-algebra. The canonical map represents C as the double covering $\kappa : C \to \mathbb{P}^1_k$ ramified at $\kappa(W)$. Let ι be the hyperelliptic involution on C (the deck transformation of κ). Let J be the Jacobian of C, and let $S^2(C)$ be the symmetric square of C, i.e. the smooth projective surface defined as the quotient of $C \times C$ by the involution that swaps the two factors. Consider the curve $L \subset S^2(C)$ whose points are the unordered pairs $\{x, \iota(x)\}$, for all $x \in C(\bar{k})$. It is clear that $L \simeq \mathbb{P}^1_k$. The Abel map $\mathrm{Ab} : S^2(C) \to J$ sending $\{A, B\}$ to the class of the divisor $A + B - \kappa^{-1}(\infty)$, where ∞ is some fixed k-point of $\mathbb{P}^1_k$, is the contraction of L to the identity in J. It is well known that $J[2] = \mathrm{Ab}(S^2 W)$. It is also well known that $J[2]$ is naturally isomorphic to the k-group scheme $R^1_{M/k}(\mu_2)/\mu_2$, defined as the kernel of the norm map $\mathrm{R}_{M/k}(\mu_2)/\mu_2 \to \mu_2$.

The quotient of J by the antipodal involution $x \mapsto -x$ is the singular Kummer surface K_{sing}. Let $\tilde{J}$ be the blow-up of J in the 16 points of $J[2]$. The antipodal involution extends to $\tilde{J}$, and the quotient of $\tilde{J}$ is the desingularized Kummer surface K. We also define a partial desingularization K_0 as the blowing up of K_{sing} at the image of $0 \in J(k)$. Alternatively, K_0 is the quotient of $S^2(C)$ by the involution that maps $\{A, B\}$ to $\{\iota(A), \iota(B)\}$. Finally, K_0 also has the involution σ coming from the involution on C^2 that sends the ordered pair (A, B) to $(\iota(A), B)$. The quotient K_0/σ is the same as the quotient of C^2 by the action of the dihedral group of order 8 generated by ι acting on each factor, and the involution swapping the factors. Therefore, $K_0/\sigma = S^2(\mathbb{P}^1_k) = \mathbb{P}^2_k$. We obtain a commutative diagram where the horizontal arrows are contractions, and the vertical arrows are finite morphisms of degree 2:

$$\begin{array}{ccccc} & & C^2 & & \\ & & \downarrow & & \\ \tilde{J} & \to & S^2(C) & \to & J \\ \downarrow & & \downarrow & & \downarrow \\ K & \to & K_0 & \to & K_{\rm sing} \\ & & \downarrow & & \\ & & \mathbb{P}^2_k & & \end{array}$$

It is clear that $\phi : K_0 \to \mathbb{P}^2_k$ is a double covering ramified in the six $\overline{k}$-lines, which are the images of the six curves $C_P \subset S^2(C)$ whose points are $\{P, x\}$, where P is a fixed Weierstrass point from $W(\overline{k})$, and $x \in C(\overline{k})$. Note by the way that these lines are tangent to a common conic, namely $\phi(L)$, where $L \simeq \mathbb{P}^1_k$ is the set of points $\{x, \iota(x)\}$, $x \in C(\overline{k})$. The six lines are in general position in the sense that no three of them have a common point. The fifteen singular points of K_0 go to the intersection points of pairs of these six lines.

The multiplication by 2 on J gives rise to a morphism $\tilde{J} \to S^2(C) = \tilde{J}/J[2]$ which is a torsor under $J[2]$. It descends to a morphism $f : K \to K_0 = K/J[2]$, whose restriction to a certain open subset is a torsor under $J[2]$. Indeed, $J[2]$ acts on K, and the set of points with non-trivial stabilizers is $(J[4] \setminus J[2])/\iota$. This is a $J[2]$-invariant set of 120 $\overline{k}$-points of K. Let K' be its complement in K, and let $K_{0,\rm sm}$ be the smooth locus of K_0. It is clear by construction that f sends K' to $K_{0,\rm sm}$, and that $f : K' \to K_{0,\rm sm}$ is a torsor under $J[2]$. We point out that f sends each of the 16 lines on K to L. We get a commutative diagram, where the right arrows are contractions, the left arrows are finite morphisms of degree 2, and the vertical arrows are finite morphisms of degree 16:

$$\begin{array}{ccccccc} K' & \subset & K & \leftarrow & \tilde{J} & \to & J \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_{0,\rm sm} & \subset & K_0 & \leftarrow & S^2(C) & \to & J \end{array}$$

The description of the desingularized Kummer surface as an intersection of three quadrics in $\mathbb{P}^5_k$ is known since J. Plücker and F. Klein. See [7], [6], [5, Ch. 6] for the case $k = \mathbb{C}$, and [3, Ch. 16], [8] for the case of the arbitrary field of characteristic different form 2. We give a new proof of this classical statement using some basic facts from the theory of torsors due to Colliot-Thélène and Sansuc. Our proof works over any field of characteristic not equal to 2 that contains more than five elements. If k is such a field we can choose a coordinate on $\mathbb{P}^1_k$ so that $\kappa(W) \subset \mathbb{A}^1_k$. Let $Q(x)$ be the monic polynomial defining W, and let θ be the image of x in $M = k[x]/(Q(x))$.

Theorem 3.1 *The desingularized Kummer surface K is isomorphic to the closed subvariety of $\mathbb{P}^5_k = \mathbb{P}(\mathrm{R}_{M/k}\mathbb{A}^1_M)$ given by three quadratic equations*

$$\mathrm{Tr}_{M/k}(Q'(\theta)^{-1}u^2) = \mathrm{Tr}_{M/k}(Q'(\theta)^{-1}\theta u^2) = \mathrm{Tr}_{M/k}(Q'(\theta)^{-1}\theta^2 u^2) = 0, \qquad (8)$$

where u is a variable in $\mathbb{A}^1_M$.

Proof We refer to [11, Def. 2.3.2] for the definition of the type of a torsor. Recall that $J[2]$ is self-dual because of the Weil pairing $J[2]\times J[2]\to\mu_2$, so that the k-groups $\widehat{J[2]}$ and $J[2]$ are canonically isomorphic.

We claim that there is a natural isomorphism $J[2](\overline{k})\tilde{\longrightarrow}\mathrm{Pic}(\overline{K}_{0,\mathrm{sm}})_{\mathrm{tors}}$, and that this isomorphism is the type of the torsor $f: K'\to K_{0,\mathrm{sm}}$. To prove this we note that K' is the complement of a finite subset in the smooth, projective and geometrically integral surface K, and hence $\overline{k}[K']^*=\overline{k}^*$ and $\mathrm{Pic}(\overline{K}')=\mathrm{Pic}(\overline{K})$. The latter abelian group is torsion free since K is a K3 surface. Now the exact sequence [11, (2.5)] takes the form

$$0\to J[2](\overline{k})\to \mathrm{Pic}(\overline{K}_{0,\mathrm{sm}})\to\mathrm{Pic}(\overline{K}').$$

This gives an isomorphism of Γ-modules $J[2](\overline{k})\tilde{\longrightarrow}\mathrm{Pic}(\overline{K}_{0,\mathrm{sm}})_{\mathrm{tors}}$. This map is the type of the torsor $f: K'\to K_{0,\mathrm{sm}}$ by Lemma 2.3.1 and the remark after Def. 2.3.2 of [11].

Recall that $\phi: K_0\to\mathbb{P}^2_k$ is a double covering ramified exactly in the images of the six lines $\kappa(P)\times\mathbb{P}^1_{\overline{k}}$, $P\in W(\overline{k})$, under the morphism $(\mathbb{P}^1_{\overline{k}})^2\to S^2(\mathbb{P}^1_{\overline{k}})=\mathbb{P}^2_{\overline{k}}$. We choose coordinates in $\mathbb{P}^2_k$ in such a way that this morphism sends $\{(a:b),(c:d)\}$ to $(ac:-ad-bc:bd)$. Then the lines have the form $(x\theta_i:-x-y\theta_i:y)$, and so their equations are

$$t_0+t_1\theta_i+t_2\theta_i^2=0.$$

Thus K_0 is given by

$$y^2=a\mathrm{N}_{M/k}(t_0+t_1\theta+t_2\theta^2),$$

for some $a\in k^*$. (More precisely, K_0 is obtained by gluing together three affine surfaces obtained by putting $t_i=1$ in this equation, which is possible since $\dim_k M$ is even.) The curve $\phi(L)\subset\mathbb{P}^2_k$ is the image of the diagonal $\mathbb{P}^1_k\subset(\mathbb{P}^1_k)^2$, and so is the set of points $(r^2:-2rs:s^2)$; in fact, $\phi(L)$ is the conic tangent to the six ramification lines. We see that $\phi^{-1}(\phi(L))$ is given by $y^2=a\mathrm{N}_{M/k}(r-s\theta)^2$, which shows that $a\in k^{*2}$, so we can take $a=1$. Thus K_0 has the equation

$$y^2=\mathrm{N}_{M/k}(t_0+t_1\theta+t_2\theta^2).$$

Let $Z\subset\mathbb{P}^5_k$ be the closed subvariety defined by (8), or, equivalently, by

$$\sum_{i=1}^6 Q'(\theta_i)^{-1}z_i^2=\sum_{i=1}^6 Q'(\theta_i)^{-1}\theta_i z_i^2=\sum_{i=1}^6 Q'(\theta_i)^{-1}\theta_i^2 z_i^2=0. \tag{9}$$

An easy calculation shows that Z is smooth, and hence is a K3 surface. The k-group $\mathrm{R}_{M/k}(\mu_2)/\mu_2$ acts on $\mathbb{P}^5_k=\mathbb{P}(\mathrm{R}_{M/k}\mathbb{A}^1_M)$ by changing the signs of the coordinates z_i. The natural morphism $Z\to Z/(\mathrm{R}_{M/k}(\mu_2)/\mu_2)$ sends u to u^2, so that

$Z/(\mathrm{R}_{M/k}(\mu_2)/\mu_2)$ is the subset of $\mathbb{P}^5_k = \mathbb{P}(\mathrm{R}_{M/k}\mathbb{A}^1_L)$ with M-coordinate $w = u^2$, given by

$$\mathrm{Tr}_{M/k}(Q'(\theta)^{-1}w) = \mathrm{Tr}_{M/k}(Q'(\theta)^{-1}\theta w) = \mathrm{Tr}_{M/k}(Q'(\theta)^{-1}\theta^2 w) = 0.$$

In particular, $Z/(\mathrm{R}_{M/k}(\mu_2)/\mu_2) \simeq \mathbb{P}^2_k$ is the projectivization of the 3-dimensional subspace of $\mathrm{R}_{L/k}\mathbb{A}^1_L$ defined by these equations. This space is spanned by 1, θ, θ^2, i.e. we can write $w = t_0 + t_1\theta + t_2\theta^2$, where t_0, t_1, t_2 are coordinates over k. The quotient of Z by the action of the subgroup $R^1_{M/k}(\mu_2)/\mu_2$ of elements of norm 1 is identified with K_0 by the morphism $g : Z \to K_0$ given by $y = \mathrm{N}_{M/k}(u)$. It is obvious that $R^1_{M/k}(\mu_2)/\mu_2$ acts freely on the open subset of $\mathbb{P}^5_k$ consisting of the points with at most one zero coordinate. Let Z' be the intersection of this open subset with Z. The image $g(Z')$ is precisely $K_{0,\mathrm{sm}}$, hence $g : Z' \to K_{0,\mathrm{sm}}$ is a torsor under $R^1_{M/k}(\mu_2)/\mu_2 = J[2]$. The set $Z \setminus Z'$ is finite, and the same arguments as in the beginning of the proof show that the types of $g : Z' \to K_{0,\mathrm{sm}}$ and $f : K' \to K_{0,\mathrm{sm}}$ are the same.

By the exact sequence of Colliot-Thélène and Sansuc (see [11], (2.22)) to prove that these two torsors are isomorphic it is enough to find a k-point N on $K_{0,\mathrm{sm}}$ with k-points in $f^{-1}(N)$ and in $g^{-1}(N)$. Note that $f^{-1}(L)$ is the union of the 16 lines on K; moreover, one of them, namely, the line corresponding to the identity in J, is defined over k. On the other hand, $g^{-1}(L)$ is given by the equations $u^2 = (r - s\theta)^2$, $\mathrm{N}_{M/k}(u) = y$. The line $u = r - s\theta$ lies in Z and projects isomorphically onto L. This proves that Z' and K' are isomorphic as torsors over $K_{0,\mathrm{sm}}$. QED

We finish this section with some geometric remarks. Let C'_P be the image of C_P in J. The Riemann–Roch theorem on C implies that $C'_P \cap C'_R = \{0, (P-R)\}$, so that 0 is the only common point of these six curves on J. Let $D_P \subset J$ be the inverse image of C'_P under the multiplication by 2 map. Since each C'_P contains 0, each curve D_P contains $J[2] \subset J$. Since the curves C'_P are translations of one of them by points of order 2, the curves D_P are linearly equivalent. More precisely, $D_P \in |4\Theta|$, where $\Theta \in \mathrm{Pic}(\overline{J})$ is the class of the theta-divisor C'_P for some $P \in W(\overline{k})$. The curves D_P are invariant under the antipodal involution. The linear system $|4\Theta - J[2]|$ defines a morphism from $\tilde{J}$ to $\mathbb{P}^5_k$ whose image is K embedded in $\mathbb{P}^5_k$ as an intersection of three quadrics (see [5], p. 786). The images D'_P of the D_P in K define a basis of $\mathrm{H}^1(K, \mathcal{O}(1))$. These curves can also be viewed as the inverse images of the six lines in K_0, where $\phi : K_0 \to \mathbb{P}^2_k$ is ramified. Thus the D'_P are the coordinate hyperplane sections. As a smooth intersection of three quadrics, each of these curves is a canonical curve of genus 5.

3.2 The case of a rational Weierstrass point: from Kummer to del Pezzo

Now suppose that C has a Weierstrass k-point R. Write $\kappa(W)$ as the disjoint union of $\kappa(R)$ and a reduced 5-element subscheme $S = S_C \subset \mathbb{P}^1_k$. This gives a decomposition of the algebra of functions $M = k[W]$ into the direct sum $M = L \oplus k$, where $L = k[S]$. We continue to assume that $|k| > 5$, so we can choose a coordinate on $\mathbb{P}^1_k$ in such a way that $\kappa(W) \subset \mathbb{A}^1_k$. Let θ_6 be the coordinate of $\kappa(R)$. Then $Q(x) = P(x)(x - \theta_6)$, where $P(x) = \prod_{i=1}^5 (x - \theta_i) = \mathrm{N}_{L/k}(x - \theta)$. Then S is the closed subscheme of $\mathbb{A}^1_k$ defined by $P(x) = 0$, and $L = k[x]/(P(x))$.

The map $(id, \mathrm{N}_{L/k})$ identifies $\mathrm{R}_{L/k}(\mu_2)/\mu_2$ with $R^1_{M/k}(\mu_2)/\mu_2$, thus $J[2]$ is the k-group $G = \mathrm{R}_{L/k}(\mu_2)/\mu_2$ of Section 2. The projective space

$$\mathbb{P}^5_k = \mathbb{P}(\mathrm{R}_{M/k}\mathbb{A}^1_M) = \mathbb{P}(\mathrm{R}_{L/k}\mathbb{A}^1_L \times \mathbb{A}^1_k)$$

contains $\mathbb{P}^4_k = \mathbb{P}(\mathrm{R}_{L/k}\mathbb{A}^1_L)$ as a hyperplane. The projection

$$\pi : \mathbb{P}^5_k \setminus \{(0:0:0:0:0:1)\} \to \mathbb{P}^4_k = \mathbb{P}(\mathrm{R}_{L/k}\mathbb{A}^1_L)$$

is a $J[2]$-equivariant morphism.

Proposition 3.2 *Let X be the quasi-split del Pezzo surface of degree 4 defined by the polynomial $P(x)$. If X is embedded into $\mathbb{P}^4_k$ as the zero set of equations (6), then the restriction of π to K is a $J[2]$-equivariant finite morphism $K \to X$ of degree 2. This double covering is ramified in the hyperplane section $K \cap \mathbb{P}(\mathrm{R}_{L/k}\mathbb{A}^1_L)$ given by $z_6 = 0$, which is a canonical curve of genus 5.*

Proof The elimination of z_6 from (9) gives (6). The ramification divisor of π is the curve D'_R described at the end of the previous section. QED

In particular, any quasi-split del Pezzo surface of degree 4 is the quotient of K by the involution whose fixed point set is the curve D'_R.

The k-group $\mathrm{R}_{M/k}(\mu_2)/\mu_2$ is the direct product of $J[2] = G$ and the subgroup $\mu_2 \subset \mathrm{R}_{M/k}(\mu_2)/\mu_2$ which changes the sign of the coordinate z_6 corresponding to the rational Weierstrass point R. The morphism $\pi : K \to X$ can be viewed as passing to the quotient by the action of this subgroup μ_2. Thus the morphism $K \to K/(\mathrm{R}_{M/k}(\mu_2)/\mu_2) \simeq \mathbb{P}^2_k$ can be written either as the composition of $\pi : K \to X$ and $X \to X/G \simeq \mathbb{P}^2_k$, or as the composition of $K \to K/G = K_0$ and $\phi : K_0 \to \mathbb{P}^2_k$.

The k-group $J[2] = G$ acts on the projective surfaces J, K and X, thus for any $\lambda \in L^*$ representing the cohomology class $[\lambda] \in \mathrm{H}^1(k, G) = L^*/k^*L^{*2}$ we can consider the twisted surfaces J_λ, K_λ and X_λ. Here J_λ is a 2-covering of J, whereas X_λ is the same as in the end of Section 2 and is given by (7). Since $\pi : K \to X$ is $J[2]$-equivariant we obtain a natural morphism $K_\lambda \to X_\lambda$ (cf. [9, §6]). Thus in the case of a rational Weierstrass point for every $\lambda \in L^*$ we obtain the following commutative diagram:

$$\begin{array}{ccccccc} X_\lambda & \leftarrow & K_\lambda & \leftarrow & \tilde{J}_\lambda & \rightarrow & J_\lambda \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^2_k & \leftarrow & K_0 & \leftarrow & S^2(C) & \rightarrow & J \end{array}$$

Here the morphisms in the upper row are $J[2]$-equivariant, and the vertical arrows are the factorization morphisms by the action of $J[2]$. We note that the 16 lines on X_λ are the images of the 16 lines on the Kummer surface K_λ.

Corollary 3.3 *For any del Pezzo surface X of degree 4 there exists a curve C of genus 2, and a 2-covering J_λ of the Jacobian J of C that has a dominant rational map to X.*

The above construction produces such a curve C with equation $y^2 = aP(x)(x - \theta_6)$; this curve is uniquely determined by X up to the quadratic twist by a and the choice of the sixth Weierstrass point $x = \theta_6$ in $\mathbb{P}^1_k \setminus S_X$.

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Department of Mathematics, South Kensington Campus, Imperial College London, SW7 2BZ England, U.K.

Institute for the Information Transmission Problems, Russian Academy of Sciences, 19 Bolshoi Karetnyi, Moscow, 127994 Russia

a.skorobogatov@imperial.ac.uk