Algebra III M3P8, M4P8

Test 2

5 December 2017

1. For each of the prime numbers $p = 2, 3, 5, 7$ determine whether the principal ideal generated by $x^2 + 1$ is a maximal ideal of the polynomial ring $\mathbb{F}_p[x]$, where $\mathbb{F}_p$ is a finite field with p elements.

Solution. (8 marks, 2 for each part)

$p = 2$ In this case $x^2 + 1 = (x + 1)^2$, so the ideal $(x + 1)$ is bigger than $(x^2 + 1)$.

$p = 3$ Since -1 is not a square modulo 3, the polynomial $x^2 + 1$ is irreducible over $\mathbb{F}_3$, hence the ideal is maximal (by a result from lectures).

$p = 5$ Modulo 5 we have that -1 is the square of 2, hence $x^2 + 1 = (x - 2)(x + 2)$, thus $(x - 2)$ is a bigger ideal.

$p = 7$ One checks that -1 is not a square modulo 7, so the polynomial $x^2 + 1$ is irreducible over $\mathbb{F}_7$, hence the ideal is maximal.

2. Let α be the real root of the equation $x^7 - 5 = 0$. Determine $[\mathbb{Q}(\alpha) : \mathbb{Q}]$, with a complete proof.

Solution. (4 marks)

By Eisenstein's criterion $x^7 - 5$ is irreducible over $\mathbb{Q}$ (take $p = 5$). Hence this is the minimal polynomial of α. It follows that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 7$.

3. Determine $[\mathbb{Q}(e^{\frac{\pi i}{4}}) : \mathbb{Q}]$, with a complete proof.

Solution. (8 marks)

We have $e^{\frac{\pi i}{4}} = \frac{1+i}{\sqrt{2}}$. Since $(e^{\frac{\pi i}{4}})^2 = i$, we see that $\mathbb{Q}(\sqrt{-1}) \subset \mathbb{Q}(e^{\frac{\pi i}{4}})$, where $\sqrt{-1} = i$. But then also $\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(e^{\frac{\pi i}{4}})$, hence $\mathbb{Q}(e^{\frac{\pi i}{4}})$ contains $\mathbb{Q}(\sqrt{-1}, \sqrt{2})$. Since $e^{\frac{\pi i}{4}}$ clearly belongs to $\mathbb{Q}(\sqrt{-1}, \sqrt{2})$, we have

$$\mathbb{Q}(e^{\frac{\pi i}{4}}) = \mathbb{Q}(\sqrt{-1}, \sqrt{2}).$$

In the lectures we proved that $\mathbb{Q}(\sqrt{-1}, \sqrt{2})$ has degree 4 over $\mathbb{Q}$. (We represented this field as a tower of quadratic extensions

$$\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{-1}, \sqrt{2}),$$

and used the theorem about the multiplicativity of the degree in a tower of field extensions.) Any other correct proof is fine.