Algebra III M3P8, M4P8

Exercise Sheet 5

1) Let $f : R_1 \to R_2$ be a homomorphism of commutative rings, and let $I \subset R_2$ be an ideal. Let $J = f^{-1}(I) = \{x \in R_1 | f(x) \in I\}$.

(a) Prove that J is an ideal of R_1.
(b) Check that $x + J \mapsto f(x) + I$ defines an injective homomorphism of rings

$$R_1/J \hookrightarrow R_2/I.$$

(c) Deduce that if I is a prime ideal, then so is J.
(d) Assume that I is maximal. Is J always maximal?

2) In each of the following cases determine whether the ideal I of the ring R is principal, whether I maximal, and whether I is prime:

(a) $R = \mathbb{Z}[\sqrt{-10}]$, $I = (\sqrt{-10})$,
(b) $R = \mathbb{Z}[\sqrt{-10}]$, $I = (2, \sqrt{-10})$,
(c) $R = \mathbb{R}[x]$, $I = (x^2, x^3 - x)$,
(d) $R = \mathbb{Z}[x]$, $I = (x^2 + 2, x^3 + 2x)$,
(e) $R = \mathbb{R}[x, y]$, $I = (x, y)$. (You can use without proof that $\mathbb{R}[x, y]$ is a UFD.)

3) Let R be the set of rational numbers with odd denominators.

(a) Prove that R is a subring of $\mathbb{Q}$.
(b) Determine the units and the irreducible elements of R.
(c) Let $I \subset R$ be an ideal. Prove that $I \cap \mathbb{Z}$ is an ideal, and hence determine all ideals in R.
(d) Determine whether R is a PID or a UFD.

4) Let $\mathbb{F}_p$ be a finite field with p elements. Since $\mathbb{F}_p[x]$ is a UFD, the polynomial $x^{p^n} - x$ is uniquely written as a product of monic irreducible polynomials. What are the degrees of these irreducible polynomials?