Algebra III M3P8, M4P8

Exercise Sheet 4

1. Suppose that F is a finite field with p^n elements where p is a prime.
(a) Prove that $x^{p^n} = x$ for all $x \in F$.
(b) Prove that the function $f : F \to F$ defined by $f(x) = x^p$ is an automorphism of F, i.e. an isomorphism $F \to F$.

2. Construct fields with 16 and 169 elements.

3. Factorize the following polynomials into irreducibles.
(1) $x^4 + 1 \in \mathbb{R}[x]$.
(2) $x^4 + 1 \in \mathbb{Q}[x]$.
(3) $x^3 - 5 \in \mathbb{Z}/11[x]$.
(4) $x^8 - x \in \mathbb{Z}/2[x]$.
(5) $x^2 + \omega x + \omega^2 \in F[x]$, where $F = \{0, 1, \omega, \omega^2\}$ is the field with 4 elements, and $\omega^2 + \omega + 1 = 0$.

4. Let R be a ring (not necessarily with 1) in which $r^2 = r$ for all $r \in R$. Prove that R is commutative.

5. Let R be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Define addition and multiplication on R in the usual way. That is, for $f, g \in R$ and $x \in \mathbb{R}$ set $(f + g)(x) = f(x) + g(x)$ and $fg(x) = f(x)g(x)$.
(a) Let $a \in \mathbb{R}$. Prove that $\{f \in R : f(a) = 0\}$ is a maximal ideal of R, and describe R/I.
(b) For $n \in \mathbb{N}$, let $I_n = \{f \in R : f(x) = 0 \text{ for } |x| > n\}$. Prove that I_n is an ideal of R. Show that $I_1 \subset I_2 \subset I_3 \dots$ and all these ideals are distinct.

6. Let F be a finite field with p^n elements where p is a prime.
(a) If m is a positive integer dividing n, show that the set of roots of $x^{p^m} - x = 0$ in F is a subfield of F of cardinality p^m.
(b) Least all positive integers N for which there is a subfield of F with N elements.