Algebra III M3P8, M4P8

Solutions Sheet 3

1. It is clear that $F_1 \cap F_2$ is a subring of K as it is closed under $+$, $-$ and $\times$. If $x \in F_1 \cap F_2$, $x \neq 0$, then $x^{-1} \in K$ belongs to $F_1 \cap F_2$, so it is a field.

2. (1) x^2+1 has no roots in $\mathbb{Z}/3$ hence is irreducible. Now conclude by a result from lectures.
(2) $|F^*| = 8$. Let α be the image of x in $F = R/I$. Check that α has order 8.

3. (1) Check that 3 and $3+2i$ are irreducibles. Since R is a PID by a result from lectures we conclude that these elements generate maximal ideals.
(2) 9 and 13, respectively.

4. (1) x^3+x+1 and x^3+x^2+1.
(2) x^2+1, x^2+x-1, x^2-x-1.

5. (1) This is an easy check of the axioms.
(2) The ideal $(3+i)+(2)$ contains $3+i-2 = 1+i$ which divides $3+i$ and 2. Hence $(3+i)+(2) = (1+i)$. The ideal $(3+i)+(5-2i)$ contains $10 = (3+i)(3-i)$ and $29 = (5-2i)(5+2i)$, and hence contains 1, so this is a generator.

6. Maximal ideals:
no ($(2, x)$ is a bigger ideal),
no (for the same reason),
no ($(3, x^2+1)$ is a bigger ideal),
no ((2) is a bigger ideal),
no ((x) is a bigger ideal),
yes (the quotient can be identified with $\mathbb{Z}/3$ and this is a field).

Recall that an ideal is prime if the quotient ring has no zero divisors.
Prime ideals:
yes (the quotient is an integral domain $\mathbb{Z}/2[x]$),
yes (the quotient is an integral domain $\mathbb{Z}$),
yes (the quotient is an integral domain $\mathbb{Z}[\sqrt{-1}]$),
no (the quotient has zero divisors which are the images of 2 and x),
no (for a similar reason),
yes (every maximal ideal is prime).