Algebra III M3P8, M4P8

Exercise Sheet 3

1. Let F_1 and F_2 be subfields of a field K. Prove that $F_1 \cap F_2$ is also a subfield.

2. Let $R = \mathbb{Z}/3[x]$ and let I be the principal ideal $(x^2 + 1)$.
(1) Show that R/I is a field of 9 elements. (You may quote any results from lectures which you need.)
(2) Let F^* be the multiplicative group consisting of the non-zero elements of R/I. Prove that F^* is cyclic.

3. Let $R = \mathbb{Z}[i]$.
(1) Show that (3) and $(3 + 2i)$ are maximal ideals of R.
(2) How many elements have the fields $R/(3)$ and $R/(3 + 2i)$?

4. (1) List the irreducible cubic polynomials in $\mathbb{Z}/2[x]$.
(2) List the irreducible monic quadratic polynomials in $\mathbb{Z}/3[x]$.
Explain briefly how you obtained your answers.

5. (1) Let I and J be ideals in a commutative ring R. Show that $I + J$, defined as $\{x + y | x \in I, y \in J\}$, is also an ideal of R.
(2) We know that $R = \mathbb{Z}[i]$ is a PID. Find generators of the ideals $(3 + i) + (2)$ and $(3 + i) + (5 - 2i)$.

6. Are these ideals maximal in $\mathbb{Z}[x]$: (2), (x), $(x^2 + 1)$, $(2x)$, (x^2), $(3, x + 1)$? Which of these ideals are prime ideals?