Algebra III M3P8, M4P8

Solutions Sheet 2

1. Let p be a prime number. Consider $\mathbb{Z}[\frac{1}{p}]$ which is defined as the set of rational numbers with only powers of p in denominators. It's easy to check that this is a subring, and actually an integral domain since it contains 1.

2. (a) With all the hints given it is a just a calculation.

(b) The fact that O_K is closed under $+$ and $-$ is clear, so it remains to check that $(\frac{1}{2}+\frac{\sqrt{d}}{2})^2 \in O_K$.

(c) Any $z \in K$ is a root of the polynomial $f(t) = t^2 - \mathrm{Tr}(z)t + \mathrm{N}(z)$ which has integer coefficients if $z \in O_K$. Conversely, if $z \in K$ and $g(t)$ is a polynomial with coefficients in $\mathbb{Z}$ such that $g(z) = 0$, then $g(\overline{z}) = 0$. It follows that if $z \notin \mathbb{Q}$ and $g(t)$ is monic and quadratic, then $g(t)$ is $f(t) = (t-z)(t-\overline{z})$.

(d) $z \in O_K^*$ if and only if $zw = 1$ for some $w \in O_K$, but then $\mathrm{N}(z)\mathrm{N}(w) = 1$, hence $\mathrm{N}(z) = \pm 1$. Conversely, if $\mathrm{N}(z) = \pm 1$, then $z^{-1} = \pm\overline{z} \in O_K$.

We only need to determine O_K^* when $d < 0$ is congruent to 1 modulo 4. Let $z \in O_K^*$. From the description of O_K we see that $2z = a + b\sqrt{d}$, where $a, b \in \mathbb{Z}$. If $\mathrm{N}(z) = 1$, then $a^2 - db^2 = 4$. If $d \leq -7$, then $b = 0$, so $O_K^* = \{\pm 1\}$. But if $d = -3$, we obtain that O_K^* is the group of all complex roots of 1 of order dividing 6, i.e. $O_K = \{\pm 1, \frac{1}{2}(\pm 1 \pm \sqrt{-3})\}$.

(e) No, because the factors are associates.

(f) The first part of the definition of a Euclidean domain is clear from the multiplicativity of the norm. Now let $\alpha, \beta \in O_K$, $\beta \neq 0$. Write $\alpha/\beta = x_1 + x_2\sqrt{-3}$ for some $x_1, x_2 \in \mathbb{Q}$. If x_1 and x_2 are both in $\frac{1}{2} + \mathbb{Z}$, then $\alpha/\beta \in O_K$, so that $\beta|\alpha$. Otherwise let n_1 (resp. n_2) be the nearest integer to x_1 (resp. to x_2). Then the norm of $(x_1 - n_1) + (x_2 - n_2)\sqrt{-3}$ is less than 1. Thus $\alpha = (n_1 + n_2\sqrt{-3})\beta + \delta$, where $\delta \in O_K$ and $\mathrm{N}(\delta) < \mathrm{N}(\beta)$. QED

3. (a) $x^2 + x + 1$

(b) $9-2i = (7-i)+(2-i)$, and $7-i = 2-i+5 = (2-i)+(2-i)(2+i) = (2-i)(3+i)$, so $2-i$ is a gcd (there are four different gcd's here).

(c) Let's use the fact that O_K is a UFD. Up to units, $2+2\sqrt{-3}$ is the same as 2^2, and $3-3\sqrt{-3}$ is the same as 6. Since 2 and 3 are coprime, 2 is a gcd.