Algebra III M3P8, M4P8

Solutions Sheet 1

1. (a) and (b) are easy.

(c) It is not hard to find zero divisors in R, hence this is not an integral domain, hence not a field. R has no non-zero nilpotents. An idempotent is a continuous function that only takes values 0 and 1. Hence it is either identically 0 or identically 1. R^* is the set of functions with non-zero values.

2. (a) If $x^n = 0$, then $(ax)^n = a^n x^n = 0$, so the only non-obvious property is closedness under addition. If $x^n = y^m = 0$, then the binomial formula shows that $(x+y)^{m+n} = 0$.

(b) If $x^n = 0$, then $(1 + x + x^2 + \dots x^{n-1})(1+x) = 1$.

(c) These are elements of the form $\overline{mp}$, $m \in \mathbb{Z}$. Indeed, $\overline{mp}^n = \overline{m^n p^n} = 0$. All other elements are of the form $\overline{a}$, where $hcf(a, p^m) = 1$. Hence there are integers b, c such that $ab + p^m c = 1$, but then $\overline{b}$ is the multiplicative inverse of $\overline{a}$.

(d) We just did it.

3. (a) is obvious.

(b) No, because $(1,0)(0,1) = (0,0)$ so $A \times B$ has zero divisors.

(c) $nil(A \times B) = nil(A) \times nil(B)$

(d) is easy.

4. (a) is easy.

(b) We have $r = er + (1-e)r$. Conversely, if $r = x + y$, $x \in eR$, $y \in (1-e)R$, then $er = ex + ey = ex = x$ and $(1-e)r = (1-e)x + (1-e)y = (1-e)y = y$.

(c) $e = \overline{3}$ will do. Then $1 - e = \overline{4}$. We obtain $\mathbb{Z}/6 = \overline{3}\mathbb{Z}/6 \times \overline{4}\mathbb{Z}/6$.

(d) There exist integers a and b such that $ap^m + bq^n = 1$. Then $e = ap^m$ is an idempotent. Indeed, $e^2 - e = ap^m(ap^m - 1) = -abp^m q^n$.