Algebra III M3P8, M4P8

Exercise Sheet 1

All rings here are commutative rings with 1 (the identity for multiplication). Recall that R^* is the group of units (invertible elements) of R. An element $x \in R$ is called *nilpotent* if $x^n = 0$ for some positive integer n. Call $nil(R)$ the set of nilpotent elements of R. An element $e \in R$ is called an *idempotent* if $e^2 = e$.

1. (a) Let I be a set, and let R be the set of functions $I \to \mathbb{R}$. Prove that R is a ring under addition and multiplication of functions.
(b) Now assume that I be an interval in $\mathbb{R}$. Prove that the set of continuous functions is a subring of R.
(c) Is R an integral domain? Is R a field? Find all nilpotent and idempotent elements in R. What is R^*?

2. (a) Let R be a ring. Prove that $nil(R)$ is closed under addition, subtraction and multiplication. [Hint: us the binomial formula] Deduce that for any $r \in R$ and any $x \in nil(R)$ we have $rx \in nil(R)$.
(b) Prove that $x \in nil(R)$ implies that $1 + x \in R^*$. [Hint: think of the geometric progression $1 + x + x^2 + \ldots$] Deduce that for any $a \in R^*$ and any $x \in nil(R)$ we have $a + x \in R^*$.
(c) Find $nil(\mathbb{Z}/p^n)$, where p is prime.
(d) Prove that every element of $\mathbb{Z}/p^n$ is either nilpotent or invertible.

3. (a) Let A and B be two rings. Define addition and multiplication on the set $A \times B$ coordinate-wise. Prove that this turns $A \times B$ into a ring.
(b) Can $A \times B$ be an integral domain for some rings A and B?
(c) Find $nil(A \times B)$.
(d) Prove that $(A \times B)^* = A^* \times B^*$.

4. (a) Let R be a ring. Prove that if $e \in R$ is an idempotent, $e \neq 0$, $e \neq 1$, then so is $1 - e$. Observe that $e(1 - e) = 0$.
(b) Deduce that every $r \in R$ is uniquely written as $x + y$, where $x \in eR$, $y \in (1 - e)R$. Conclude that R with its addition and multiplication operations can be identified with $A \times B$, where $A = eR$, $B = (1 - e)R$.
(c) Find an idempotent $e \in \mathbb{Z}/6$, $e \neq 0$, 1, hence write this ring as a product of two rings.
(d) Find an idempotent $e \in \mathbb{Z}/p^m q^n$, $e \neq 0$, 1, where p and q are distinct primes.