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Sharp spectral inequalities for the Heisenberg Laplacian

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ABSTRACT We obtain sharp inequalities for the spectrum of the Heisenberg Laplacian with Dirichlet boundary conditions in $2N+1$-dimensional domains of finite measure. We also give another proof of an inequality obtained by Strichartz.

1 Introduction

The aim of this paper is to prove uniform inequalities for a class of differential operators with discrete spectrum. In particular, we give a simple proof of a previously known result by Strichartz [24] for hypoelliptic operators.

Let X_1 and X_2 be two vector fields in $\mathbb{R}^3$, expressed in coordinates $x = (x_1, x_2, x_3)$ as

$$X_1 := \partial_{x_1} + \frac{1}{2} x_2 \partial_{x_3}, \qquad X_2 := \partial_{x_2} - \frac{1}{2} x_1 \partial_{x_3}. \tag{1.1}$$

The Lie algebra of left-invariant vector fields on the first Heisenberg group $\mathbf{H}^1$ is spanned by X_1, X_2 and $X_3 := \partial_{x_3}$. $\mathbf{H}^1$ may be described as the set $\mathbb{R}^2 \times \mathbb{R}$ equipped with the group law

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = \left(x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3 + \frac{x_1 y_2 - x_2 y_1}{2} \right).$$

These vector fields satisfy the canonical commutation relation

$$[X_1, X_2] = -X_3.$$

The quadratic form

$$a[u] := \int_{\mathbb{R}^3} \left(|X_1 u(x)|^2 + |X_2 u(x)|^2 \right) dx, \qquad u \in H^1(\mathbb{R}^3), \tag{1.2}$$

defines a self-adjoint operator

$$A := X_1^* X_1 + X_2^* X_2 = -X_1^2 - X_2^2$$

from the Kolmogorov-Hörmander class [8] of second-order hypoelliptic operators. A, usually referred to as the *Heisenberg Laplacian*, is non-elliptic because it is associated to a degenerate metric on $\mathbf{H}^1$, see [26]. The operators

$$Z := \frac{X_1 + iX_2}{\sqrt{2}} \qquad \text{and} \qquad \bar{Z} := \frac{X_1^* - iX_2^*}{\sqrt{2}}$$

are related to creation and annihilation operators; their corresponding energy operator $Z\bar{Z} + \bar{Z}Z$ coincides with A.

Let $\Omega \subset \mathbb{R}^3$ be a domain of finite measure. The closure of the form (1.2), defined on the class of functions $C_0^\infty(\Omega)$, gives us a self-adjoint operator A_Ω which is identified with the operator A with Dirichlet boundary conditions on the boundary $\partial\Omega$. It is well known that the spectrum of this operator is discrete and the corresponding eigenvalues $(\lambda_k)_{k=1}^\infty$ accumulate at ∞.

In a later section we will also study the elliptic operator

$$B := -X_1^2 - X_2^2 - X_3^2 \qquad \text{in } L^2(\mathbb{R}^3),$$

which commutes with A and hence acts on each of its eigenspaces. By the approach used to prove the trace inequality for A we will obtain an estimate which turns out to be identical to the classical Weyl-type asymptotics.

2 Hypoelliptic case: main results

The aim of this article is to obtain a sharp uniform spectral inequality of Berezin-Li-Yau type for the trace $\operatorname{Tr} \varphi_\lambda(A_\Omega)$. Let

$$\varphi_\lambda(t) = (\lambda - t)_+ = \begin{cases} \lambda - t, & \lambda > t, \\ 0, & \lambda \le t. \end{cases}$$

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^3$ be a domain of finite measure and let A_Ω be the operator A with Dirichlet boundary conditions in Ω. Then the spectrum of A_Ω is discrete and*

$$\operatorname{Tr} \varphi_\lambda(A_\Omega) = \sum_k (\lambda - \lambda_k)_+ \le \frac{1}{96} |\Omega| \lambda^3. \tag{2.1}$$

Remark 2.2 *The operator A is unitarily equivalent to a two-dimensional Laplacian with constant magnetic field, see* (6.1). *Note, however, that* (2.1) *cannot be considered as the (semi-classical) phase-volume estimate, which is infinity in the case of the Heisenberg Laplacian with Dirichlet boundary conditions.*

Remark 2.3 *For the Dirichlet Laplacian with constant magnetic field the uniform inequalities for the eigenvalues have been obtained in [6] and [7].*

This result implies several corollaries.

Corollary 2.1 *Let $N(\lambda)$ be the counting function of A_Ω,*

$$N(\lambda) := \{k : \lambda_k < \lambda\}. \tag{2.2}$$

Then under the conditions of Theorem 2.1 we have

$$N(\lambda) \le \frac{9}{2^7}\,|\Omega|\,\lambda^2.$$

Remark 2.4 *Uniform spectral inequalities for the number of eigenvalues below $\lambda > 0$ for Dirichlet Laplacian were obtained by Pólya [21] for tiling domains, in [4] and [5] for bounded domains and in [16], [M] and [23] for domain of finite Lebesgue measure. The sharp constant in this inequality remains unknown.*

Applying the Aizenmann-Lieb principle [1] for $\gamma > 1$ and an argument close in spirit [7] for $\gamma < 1$, we obtain

Corollary 2.2 *Under the conditions of Theorem 2.1 and for any $\gamma > 0$, the eigenvalues of the Dirichlet Heisenberg Laplacian satisfy the Lieb-Thirring inequality*

$$\sum_k (\lambda - \lambda_k)_+^\gamma \le K_\gamma\,|\Omega|\lambda^{\gamma+2}$$

with

$$K_\gamma = \begin{cases} \dfrac{9}{32}\dfrac{\gamma^\gamma}{(\gamma+2)^{\gamma+2}}, & 0 < \gamma \le 1, \\ \dfrac{1}{16}\dfrac{1}{(\gamma+1)(\gamma+2)}, & 1 \le \gamma. \end{cases} \tag{2.3}$$

Remark 2.5 *Note that $\lim_{\gamma\downarrow 0} K_\gamma$ is the constant in Corollary 2.1. However, taking this as the initial value and applying the technique we used to* *estimate* K_γ *'upwards' would have given* $2^{-6}9(\gamma+1)^{-1}(\gamma+2)^{-1}$, $\gamma < 1$, *which decreases at a slower rate.*

In the case where $\partial\Omega$ is smooth, it has been obtained in [19] that the following spectral asymptotic formula holds true:

$$\lim_{\lambda\to\infty} \lambda^{-2} N(\lambda) = \int_\Omega \gamma(x)dx.$$

Here γ is a continuous and strictly positive function, which is given rather implicitly. Theorem 2.1 and Corollaries 2.1, 2.2 allow us to extend this statement, with an explicit right-hand side, to arbitrary domains of finite measure with non-smooth boundaries. Approximating the domain by cylinders and arguing by the variational principle, we obtain the necessary converse inequality, which confirms the asymptotic sharpness of the constant in Theorem 2.1 and Strichartz's Theorem 6.1 [24].

Corollary 2.3 *Under the conditions of Theorem 2.1, for any* $\gamma \geq 1$,

$$\lim_{\lambda\to\infty} \lambda^{-\gamma-2} \sum_k (\lambda - \lambda_k)_+^\gamma = K_\gamma |\Omega|$$

with K_γ *as in* (2.3).

By a standard Tauberian argument we also have

Corollary 2.4 *Under the conditions of Theorem 2.1, for* $0 \leq \gamma < 1$,

$$\lim_{\lambda\to\infty} \lambda^{-\gamma-2} \sum_k (\lambda - \lambda_k)_+^\gamma = \frac{\gamma+3}{\gamma+1} K_{\gamma+1} |\Omega|$$

with K_γ *as in* (2.3).

Another consequence of Theorem 2.1 follows immediately from the duality

$$f(x) \leq g(x),\ x \geq 0 \qquad \Leftrightarrow \qquad g^\star(p) \leq f^\star(p),\ p \geq 0,$$

where the Legendre transform of a convex, non-negative function f is given by $f^\star(p) = \sup_{x\geq 0}(px - f(x))$, $p \geq 0$. Transforming both sides of (2.1) (cf. [14]) we obtain

Corollary 2.5 *Under the conditions of Theorem 2.1 the eigenvalues of the Dirichlet Heisenberg Laplacian satisfy the Li-Yau inequality*

$$\sum_{k=1}^n \lambda_k \geq \frac{8\sqrt{2}}{3} |\Omega|^{-1/2} n^{3/2}.$$

This result constitutes the announced connection with [24]; more precisely, the same inequality follows from combining Theorems 4.2 and 6.1 therein.

3 Elliptic case: main results

Now consider the elliptic operator $B = -X_1^2 - X_2^2 - X_3^2$ in $L^2(\mathbb{R}^3)$. For $\Omega \subset \mathbb{R}^3$ an open set of finite measure we define as before the Dirichlet operator $B_\Omega := P_\Omega B P_\Omega$. In some contrast with Theorem 2.1 we prove

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^3$ be a domain of finite measure and let B_Ω be the operator B with Dirichlet boundary conditions in Ω. Then the spectrum of B_Ω is discrete and*

$$\operatorname{tr}\varphi_\lambda(B_\Omega) = \sum_k (\lambda - \lambda_k)_+ \le \frac{1}{15\pi^2}\,|\Omega|\lambda^{5/2}.$$

Remark 3.2 *The latter expression coincides which the semi-classical constant appearing in the phase-volume Weyl-type asymptotics. Indeed,*

$$\begin{aligned}&\lim_{\lambda\to\infty} \lambda^{-5/2}\operatorname{tr}\varphi_\lambda(A_\Omega)\\ =&\frac{1}{(2\pi)^3}\lim_{\lambda\to\infty}\lambda^{-5/2}\int_\Omega\int_{\mathbb{R}^3}(\lambda - a(x,\xi))_+\,d\xi\,dx = \frac{1}{15\pi^2}\,|\Omega|,\end{aligned}$$

where $a(x,\xi) = \left(\xi_1 + \frac{1}{2}x_2\xi_3\right)^2 + \left(\xi_2 - \frac{1}{2}x_1\xi_3\right)^2 + \xi_3^2$.

4 Hypoelliptic case: results obtained by heat-kernel techniques

So far we have restricted our study of the Dirichlet Heisenberg Laplacian A_Ω to the case where Ω is a domain of finite measure. Put differently, we have studied the Schrödinger-type operator $A - V$ for *potential wells*, i.e., with V being infinite outside and constant inside Ω. However, using heat-kernel methods we obtain the following upper bound on the counting function without this restriction on V.

Theorem 4.1 *Let $N(A - V)$ be the number of negative eigenvalues of $A - V$. Then for any $V \in L^2(\mathbb{R}^3)$.*

$$N(A - V) \le C\int_{\mathbb{R}^3} V(x)_+^2\,dx$$

with

$$C = \min_{a>0} \frac{1}{64a} \frac{1}{e^{-a} + a\,\mathrm{Ei}(-a)} \geq 0.09429.$$

Note that this result is not an improvement in the case of potential wells; then already Corollary 2.1 gives us the constant $9/2^8 \approx 0.03516$. However, the asymptotics on the moments of the eigenvalues can be obtained via heat-kernel estimates in conjuction with Karamata's well-known Tauberian theorem. [10]

Theorem 4.2 *Let* (λ_j) *be a non-decreasing sequence of non-negative numbers such that* $C = \lim_{t\to 0} t^\alpha \sum_{j=0}^\infty e^{-t\lambda_j}$ *exists for some* $\alpha > 0$. *Then for any* $f \in C([0,1])$,

$$\lim_{t\to 0} t^\alpha \sum_{j=0}^\infty f(e^{-t\lambda_j}) e^{-t\lambda_j} = \frac{C}{\Gamma(\alpha)} \int_0^\infty f(e^{-t}) t^{\alpha-1} e^{-t}\, dt.$$

In Section 9, in order to prove Theorem 4.1 we derive the equality

$$e^{-tA}(x,x) = \frac{1}{32t^2},$$

which implies

$$\mathrm{tr} e^{-tA_\Omega} \leq \frac{1}{32t^2} |\Omega|. \tag{4.1}$$

If the converse inequality also holds – which is still to be proved – so ! that

$$\lim_{t\to\infty} t^{-2} \mathrm{tr} e^{-tA_\Omega} = \frac{1}{32} |\Omega|,$$

then this implies, by Theorem 4.2 with $f(s) = s^{-1}\chi_{(1/e,1]}$,

$$\lim_{\lambda\to\infty} \lambda^{-2} N(\lambda) = \frac{1}{32\Gamma(3)} |\Omega|.$$

This gives us the asymptotic counterpart of Corollary 2.1, namely

Corollary 4.1 *Let* $N(\lambda)$ *be the counting function of* A_Ω, *see* (2.2). *Then under the conditions of Theorem 2.1 we have*

$$\lim_{t\to\infty} \lambda^{-2} N(\lambda) = \frac{1}{64} |\Omega|.$$

If we apply Theorem 4.2 with $f(s) = \lambda s^{-1}(1 + \log s)\chi_{(1/e,1]}$ instead (still assuming (4.1) is true), we get

Corollary 4.2 *Under the conditions of Theorem 2.1 we have*

$$\lim_{\lambda\to\infty} \lambda^{-3} \sum_k (\lambda - \lambda_k)_+ = \frac{1}{192}|\Omega|.$$

As we pointed out after Corollary 2.2, this is a better constant than what one would have got by straightforward Aizenmann-Lieb extrapolation.

5 Hypoelliptic case: generalisation to $\mathbb{R}^{2N+1}$

The Lie algebra on the Nth Heisenberg group $\mathbf{H}^N$ is spanned by vector fields $X_1, X_2, \ldots, X_{2N+1}$ in $\mathbb{R}^{2N+1}$. If we let

$$X_{2k-1} := \partial_{x_{2k-1}} + \frac{1}{2}x_{2k}\partial_{x_{2N+1}}, \qquad X_{2k} := \partial_{x_{2k}} - \frac{1}{2}x_{2k-1}\partial_{x_{2N+1}}$$

for $1 \leq k \leq N$ and $X_{2N+1} := \partial_{x_{2N+1}}$, then the only non-zero commutator is $[X_{2k-1}, X_{2k}] = -X_{2N+1}$.

This is the $2N+1$-dimensional version of Theorem 2.1:

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^{2N+1}$ be a domain of finite measure and let $A_{N,\Omega}$, $N \geq 1$, be the operator corresponding to the differential expression*

$$-\sum_{k=1}^{2N} X_k^2$$

defined in a self-adjoint way in $L^2(\mathbb{R}^2)$ with Dirichlet boundary conditions. The spectrum of $A_{N,\Omega}$ is discrete and

$$\operatorname{tr}\varphi_\lambda(A_{N,\Omega}) \leq \frac{2c_N}{(2\pi)^{N+1}(N+1)(N+2)}|\Omega|\lambda^{N+2}$$

with

$$c_N := \sum_{n_1,\ldots,n_N \geq 0} \frac{1}{(2(n_1 + \ldots + n_N) + N)^{N+1}}. \tag{5.1}$$

Remark 5.2 *The multiple sum* (5.1) *can be expressed as a single sum,*

$$c_N = \sum_{k=0}^{\infty} \binom{N+k-1}{k} \frac{1}{(2k+N)^{N+1}}.$$

Unaware of a closed expression for this number we give approximate values for the first constants: $c_2 \approx 2.055 \cdot 10^{-1}$, $c_3 \approx 2.737 \cdot 10^{-2}$, $c_4 \approx 2.929 \cdot 10^{-3}$, $c_5 \approx 2.601 \cdot 10^{-4}$, $c_6 \approx 1.969 \cdot 10^{-5}$.

Without hardly modifying the proofs at all, we can extend Corollaries 2.1 and 2.2 into

Corollary 5.1 *Let $N(\lambda)$ be the counting function of $A_{N,\Omega}$, see* (2.2)*. Then under the conditions of Theorem 5.1 we have*

$$N(\lambda) \le \frac{2c_N(N+2)^{N+1}}{(2\pi)^{N+1}(N+1)^{N+2}}|\Omega|\lambda^{N+1}.$$

Corollary 5.2 *Under the conditions of Theorem 5.1 and for any $\gamma > 0$ the eigenvalues of $A_{N,\Omega}$ satisfy the Lieb-Thirring inequality*

$$\sum_k (\lambda - \lambda_k)_+^\gamma \le K_{N,\gamma}|\Omega|\lambda^{N+\gamma+1}$$

with

$$K_{N,\gamma} = \begin{cases} \dfrac{2c_N(N+2)^{N+1}}{(2\pi)^{N+1}(N+1)} \dfrac{\gamma^\gamma}{(\gamma+N+1)^{\gamma+N+1}}, & 0 < \gamma \le 1, \\ \dfrac{2c_N}{(2\pi)^{N+1}(N+1)(N+2)} \dfrac{6}{(\gamma+1)(\gamma+2)}, & 1 \le \gamma. \end{cases}$$

6 Proof of Theorems 2.1 and 3.1

Denote by $\mathcal{F}_3$ the partial Fourier transform

$$\mathcal{F}_3 u(x', \xi_3) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix_3\xi_3} u(x', x_3)\, dx_3, \qquad x' = (x_1, x_2).$$

Then,

$$\mathcal{F}_3 A \mathcal{F}_3^* = \left(i\partial_{x_1} - \frac{1}{2}x_2\xi_3\right)^2 + \left(i\partial_{x_2} + \frac{1}{2}x_1\xi_3\right)^2 = (i\nabla_{x'} + \xi_3\mathbf{A}(x'))^2, \tag{6.1}$$

where $\mathbf{A}(x') = \frac{1}{2}(-x_2, x_1)$. This is a Laplacian in x' with constant magnetic field ξ_3, on which the Landau levels $\mu_n(\xi_3)$ depend:

$$\mu_n(\xi_3) := |\xi_3|(2n+1), \qquad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

Note that $\mathcal{F}_3 A \mathcal{F}_3^*$ acts as a multiplication operator with respect to the variable ξ_3. By the Spectral theorem,

$$\mathcal{F}_3 A \mathcal{F}_3^* = \sum_{n=0}^{\infty} \mu_n(\xi_3)\Pi_{\xi_3,n}, \qquad \Pi_{\xi_3,n} = \Pi'_{\xi_3,n} \otimes I_{L^2(\mathbb{R})},$$

where the projector $\Pi'_{\xi_3,n}$ has an explicit representation as an integral kernel depending on x' (see, e.g., [9] or [22]), which is constant on the diagonal:

$$\Pi'_{\xi_3,n}(x',x') = \frac{|\xi_3|}{2\pi}. \tag{6.2}$$

Let $P_\Omega : L^2(\mathbb{R}^3) \to L^2(\Omega)$ be the operator of multiplication by χ_Ω, the characteristic function of Ω. The operator A_Ω, which is A with Dirichlet boundary conditions in Ω, can be identified with the operator $P_\Omega A P_\Omega$. By the Berezin-Lieb inequality (see [2], [3], [15] and also [13]), we find

$$\begin{aligned}\mathrm{tr}\varphi_\lambda(A_\Omega) \leq &\mathrm{tr} P_\Omega \varphi_\lambda(A) P_\Omega \\ =&\frac{1}{2\pi}\int_\Omega \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \varphi_\lambda(\mu_n(\xi_3))\Pi_{\xi_3,n}(x',x')\,d\xi_3\,dx \\ =&\frac{1}{2\pi^2}|\Omega| \sum_{n=0}^{\infty} \int_0^\infty (\lambda - \xi_3(2n+1))_+ \xi_3\, d\xi_3.\end{aligned}$$

Substituting $s = \xi_3(2n+1)$ we have

$$\begin{aligned}\mathrm{tr}\varphi_\lambda(A_\Omega) \leq &\frac{1}{2\pi^2}|\Omega| \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \int_0^\infty (\lambda - s)_+ s\, ds \\ =&\frac{1}{2\pi^2}|\Omega| \frac{\pi^2}{8}\frac{\lambda^3}{6} = \frac{1}{96}|\Omega|\,\lambda^3.\end{aligned}$$

This completes the proof of Theorem 2.1.

Similarly to (6.1) we have

$$\begin{aligned}\mathcal{F}_3 B \mathcal{F}_3^* =&\mathcal{F}_3(-(\partial_{x_1} + x_2\partial_{x_3})^2 - (\partial_{x_2} - x_1\partial_{x_3})^2 - \partial_{x_3}^2)\mathcal{F}_3^* \\ =&(i\partial_{x_1} + x_2\xi_3)^2 + (i\partial_{x_2} - x_1\xi_3)^2 + \xi_3^2 \\ =&(i\nabla_{x'} + \xi_3 \mathbf{A}(x'))^2 + \xi_3.\end{aligned}$$

The eigenvalues of this operator are again functions of ξ_3,

$$\nu_n(\xi_3) := |\xi_3|(2n+1) + |\xi_3|^2, \qquad n \in \mathbb{N}_0.$$

Applying the Berezin-Lieb inequality and recycling some of the computations above we find

$$\begin{aligned}
\operatorname{tr}\varphi_\lambda(B_\Omega) \leq & \operatorname{tr} P_\Omega \varphi_\lambda(B) P_\Omega \\
= & \frac{1}{(2\pi)^2}|\Omega| \int_{-\infty}^{\infty} |\xi_3| \sum_{k=0}^{\infty} (\lambda - |\xi_3|(2k+1) - |\xi_3|^2)_+ \, d\xi_3 \\
\leq & \frac{1}{(2\pi)^2}|\Omega| \int_{-\infty}^{\infty} \int_0^{\infty} |\xi_3|(\lambda - 2|\xi_3|t - |\xi_3|^2)_+ \, dt \, d\xi_3 \\
= & \frac{1}{2^3\pi^2}|\Omega| \int_{-\infty}^{\infty} \int_0^{\infty} (\lambda - t - s^2)_+ \, dt \, ds \\
= & \frac{1}{2^4\pi^2}|\Omega| \int_{-\infty}^{\infty} (\lambda - s^2)_+^2 \, ds = \frac{1}{15\pi^2} \, |\Omega| \lambda^{5/2}
\end{aligned}$$

as claimed.

7 Proof of Corollary 2.1

In order to prove Corollary 2.1 we take up an idea from [12]. Obviously, for any $\tau > 0$,

$$N(\lambda) \leq \frac{1}{\tau\lambda} \sum_k ((1+\tau)\lambda - \lambda_k)_+ \leq \frac{1}{96} \, |\Omega| \lambda^2 \, \frac{(1+\tau)^3}{\tau}. \tag{7.1}$$

The minimum value of $(1+\tau)^3\tau^{-1}$ is reached at $\tau = 1/2$. Substituting this value into (7.1) we obtain Corollary 2.1.

8 Proof of Corollary 2.2

The first part of the proof, concerning $0 < \gamma < 1$, is based on the following result obtained in [7]

Lemma 8.1 *Let $0 \leq \gamma < 1$ and $\lambda < \mu$. Then for all $E \geq 0$,*

$$(\lambda - E)_+^\gamma \leq C(\gamma)(\mu - \lambda)^{\gamma - 1}(\mu - E)_+$$

with $C(\gamma) := \gamma^\gamma (1-\gamma)^{1-\gamma}$.

The proof is elementary. Combining this inequality with Theorem 2.1 we have, for $0 < \gamma < 1$ and any $\mu > \lambda$,

$$\begin{aligned}
\sum_k (\lambda - \lambda_k)_+^\gamma \leq & C(\gamma)(\mu - \lambda)^{\gamma-1} \sum_k (\mu - \lambda_k)_+ \\
\leq & C(\gamma)(\mu - \lambda)^{\gamma-1} \frac{1}{96}|\Omega|\mu^3 \leq \frac{9}{32} \frac{\gamma^\gamma}{(\gamma+2)^{\gamma+2}}|\Omega|\lambda^{\gamma+2}
\end{aligned}$$

(choose $\mu = 3\lambda/(\gamma+2)$).

To determine K_γ for $\gamma > 1$ we will use the identity

$$B(\gamma-1,2)(\lambda-t)_+^\gamma = \int_0^\infty (\lambda-t-\mu)_+\mu^{\gamma-2}\,d\mu, \qquad \gamma > 1,$$

B being the beta function. Writing $\sum(\lambda-\lambda_k)_+^\gamma = \int(\lambda-t)_+^\gamma\,dN(t)$ and applying Fubini's theorem,

$$\begin{aligned} B(\gamma-1,2)\lambda^{-\gamma-2}\sum_k(\lambda-\lambda_k)_+^\gamma &= \lambda^{-\gamma-2}\int_0^\infty \mu^{\gamma-2}\sum_k(\lambda-\mu-\lambda_k)_+\,d\mu \\ &\le K_1|\Omega|\int_0^1 s^3(1-s)^{\gamma-2}\,ds \\ &= K_1|\Omega|B(4,\gamma-1) \end{aligned}$$

by Theorem 2.1. Taking the supremum of the left-hand side we obtain, by the definition of K_γ,

$$K_\gamma \le K_1\frac{B(4,\gamma-1)}{B(2,\gamma-1)} = \frac{6}{(\gamma+1)(\gamma+2)}K_1 = \frac{1}{16(\gamma+1)(\gamma+2)}, \quad \gamma > 1.$$

9 Proof of Theorem 4.1

In this section we assume (U,μ) to be a σ-finite measure space. Any selfadjoint non-negative operator H in $L^2(U,\mu)$ generates a contractive semigroup $\{e^{-tH}\}_{t\in\mathbb{R}_+}$. We will assume that the corresponding integral kernels $e^{-tH}(x,y) \ge 0$ a.e. in $\mathbb{R}_+ \times U \times U$. Put

$$M_H(t) := \|e^{-tH/2}\|^2_{L^2\to L^\infty}$$

and suppose this quantity is bounded for all $t > 0$, $M_H(t) = \mathcal{O}(t^\alpha)$, $\alpha > 0$ at zero and integrable at infinity. Moreover, let $G(s)$ be a function on $\mathbb{R}_+$, polynomially growing at infinity and such that $G(s)/s$ is integrable at $s = 0$. We associate to any such G the function

$$g(\sigma) := \int_0^\infty \frac{G(s)}{s}e^{-s/\sigma}\,ds.$$

(Note that $g(1/\sigma)$ is the Laplace transform of $G(s)/s$.) We shall apply the following bound on the counting function, which is occasionally referred to as Lieb's Formula (see [16] and [25]).

Theorem 9.1 *Let H be an operator satisfying the hypotheses above and let $N(H-V)$ be the number of negative eigenvalues of $H-V$. For any*

$V \in L^2(U,\mu)$ and any admissible G,

$$N(H-V) \le \frac{1}{g(1)} \int_0^\infty \frac{dt}{t} \int_U M_H(t) G(tV(x)) \mu(dx). \tag{9.1}$$

When M_H is a homogeneous function, this statement can be simplified in the following way. Writing $M_H(t) = K/t^{\nu/2}$ (and for ease of notation dx instead of $\mu(dx)$) and applying Fubini's theorem to the right-hand side, we find

$$\frac{K}{g(1)} \int_U \left(\int_0^\infty \frac{G(tV(x))}{t^{\nu/2+1}} dt \right) dx = \frac{K}{g(1)} \int_0^\infty \frac{G(s)}{s^{\nu/2+1}} ds \int_U V(x)^{\nu/2} dx.$$

In this case, (9.1) apparently is a (ν-dimensional) Cwikel-Lieb Rozenblyum inequality, whose constant only depends on the power ν.

It is easy to check that the operator A in $L^2(\mathbb{R}^3)$ fits into this framework and so $N(A-V)$ can be estimated using Theorem 9.1. Similarly to the proof of Theorem 2.1 we calculate, for $t > 0$, $x \in \mathbb{R}^3$,

$$\begin{aligned} e^{-tA}(x,x) &= (\mathcal{F}_3^* e^{-t\mathcal{F}_3 A \mathcal{F}_3^*} \mathcal{F}_3)(x,x) \\ &= (2\pi)^{-1} \int_{-\infty}^\infty \sum_{n=0}^\infty e^{-\mu_n(\xi_3)t} \Pi_{\xi_3,n}(x,x) d\xi_3 \\ &= \pi^{-2} \sum_{n=0}^\infty \int_0^\infty \xi_3 e^{-2\xi_3(2n+1)t} d\xi_3 \\ &= \frac{1}{(2\pi t)^2} \sum_{n=0}^\infty \frac{1}{(2n+1)^2} \int_0^\infty s e^{-s} ds = \frac{1}{32t^2}. \end{aligned}$$

This is exactly $M_A(t) = \operatorname{ess\,sup}_{x\in\mathbb{R}^3} e^{-tA}(x,x)$, so $\nu = 4$. One can argue (see [16]) that the optimal G is of the form $G(s) = (s-a)_+$, $a > 0$. Clearly, since

$$\int_0^\infty \frac{G(s)}{s^{\nu/2+1}} ds = \frac{1}{2a} \quad \text{and } g(1) = \int_a^\infty \left(1 - \frac{a}{s}\right) e^{-s} ds = e^{-a} + a\,\mathrm{Ei}(-a),$$

the best constant is the minimum of $[64a(e^{-a} + a\,\mathrm{Ei}(-a))]^{-1}$, as stated in Theorem 4.1.

10 Proof of Theorem 5.1

Extending our previous definitions we let

$$\mathcal{F}_{2N+1} u(x', \xi_{2N+1}) = (2\pi)^{-1/2} \int_{-\infty}^\infty e^{-ix_{2N+1}\xi_{2N+1}} u(x', x_{2N+1}) dx_{2N+1},$$

$x' = (x_1, \ldots, x_{2N})$ and, for future convenience, $x'_k = (x_{2k-1}, x_{2k})$. Then,

$$\begin{aligned}\mathcal{F}_{2N+1} A_N \mathcal{F}^*_{2N+1} &= -\sum_{k=1}^{N} \mathcal{F}_{2N+1}(X^2_{2k-1} + X^2_{2k})\mathcal{F}^*_{2N+1} \\ &= \sum_{k=1}^{N} (i\nabla_{x'_k} + \xi_{2N+1}\mathbf{A}(x'_k))^2,\end{aligned}$$

where $\mathbf{A}$ remains as in Section 6. The eigenvalues of this operator are

$$\mu_{\mathbf{n}}(\xi_{2N+1}) := |\xi_{2N+1}|(2|\mathbf{n}| + N), \qquad \mathbf{n} \in \mathbb{N}_0^N,$$

where $|\mathbf{n}| := n_1 + \ldots + n_N$. By the Spectral theorem,

$$\mathcal{F}_{2N+1} A_N \mathcal{F}^*_{2N+1} = \sum_{\mathbf{n}\in\mathbb{N}_0^N} \mu_{\mathbf{n}}(\xi_{2N+1})\Pi_{\xi_{2N+1},\mathbf{n}},$$

where $\Pi_{\xi_{2N+1},\mathbf{n}} = \Pi'_{\xi_{2N+1},n_1} \otimes \ldots \otimes \Pi'_{\xi_{2N+1},n_N} \otimes I_{L^2(\mathbb{R})}$ with Π' as in Section 6. Clearly,

$$\Pi_{\xi_{2N+1},\mathbf{n}}(x', x') = \frac{|\xi_{2N+1}|^N}{(2\pi)^N},$$

so that

$$\begin{aligned}\mathrm{tr}\varphi_\lambda(A_{N,\Omega}) \leq& \mathrm{tr} P_\Omega \varphi_\lambda(A_N) P_\Omega \\ =& \frac{1}{2\pi}\int_\Omega \int_{-\infty}^{\infty} \sum_{\mathbf{n}\in\mathbb{N}_0^N} \varphi_\lambda(\mu_{\mathbf{n}}(\xi_{2N+1}))\Pi_{\xi_{2N+1},\mathbf{n}}(x', x')d\xi_{2N+1}dx \\ =& \frac{2}{(2\pi)^{N+1}}|\Omega| \sum_{\mathbf{n}\in\mathbb{N}_0^N} \int_0^\infty (\lambda - \xi_{2N+1}(2|\mathbf{n}| + N))_+ \xi^N_{2N+1} d\xi_{2N+1} \\ =& \frac{2}{(2\pi)^{N+1}}|\Omega| \sum_{\mathbf{n}\in\mathbb{N}_0^N} \frac{1}{(2|\mathbf{n}| + N)^{N+1}} \int_0^\infty (\lambda - s)_+ s^N ds \\ =& \frac{2}{(2\pi)^{N+1}}|\Omega| c_N \frac{\lambda^{N+2}}{(N+1)(N+2)}\end{aligned}$$

with c_N as in (5.1).

11 Proof of Corollary 2.3

The asymptotics stated in the corollary will follow from Corollary 2.2 if we can prove the converse bound

$$\sum_k (\lambda - \lambda_k)^\gamma_+ \geq K_\gamma |\Omega| \lambda^{\gamma+2} + o(\lambda^{\gamma+2}), \qquad \lambda \to \infty,\ \gamma \geq 1. \tag{11.1}$$

We first assume the domain to be a cylinder, $\Omega = \Omega' \times (0, L_3)$. By Theorem 2.1 the spectrum of A_Ω consists of eigenvalues λ_k of finite multiplicity. If we denote the corresponding $L^2(\Omega)$-normalised eigenfunctions by ω_k then by Plancherel's theorem $\|\mathcal{F}_3\omega_k\|^2_{L^2(\Omega'\times\mathbb{R})} = 2\pi$. As explained in the proof of Theorem 2.1, the eigenstate of $\mathcal{F}_3 A \mathcal{F}_3^*$ corresponding to $\mu_n(\xi_3)$ has infinite dimensionality and is spanned by the eigenfunctions $(\tau_{\xi_3,m,n})_{m=-n}^\infty$, which we assume to be normalised in $L^2(\Omega)$. Evidently, the projection kernel has the form

$$\Pi'_{\xi_3,n}(x', y') = \sum_{m=-n}^{\infty} \tau_{\xi_3,m,n}(x')\overline{\tau_{\xi_3,m,n}(y')}.$$

We finally introduce a cut-off function $\chi_\varepsilon \in C_0^\infty(\Omega)$ such that $\chi_\varepsilon(x) = 1$ if $\operatorname{dist}(x, \partial\Omega) \geq \varepsilon$ for $\varepsilon > 0$, and will use $\chi_\varepsilon(x)e^{ix_3\xi_3}\tau_{\xi_3,m,n}(x')$ as an approximate eigenfunction.

Setting $\varphi(t) := (\lambda - t)_+^\gamma$ we have

$$\begin{aligned}
\sum_k \varphi(\lambda_k) &= \sum_k \varphi(\lambda_k)\|\omega_k\|_2^2 \\
&= \frac{1}{2\pi}\sum_k \varphi_\lambda(\lambda_k)\sum_{m,n}\int_{-\infty}^{\infty} |(\omega_k, e^{ix_3\xi_3}\tau_{\xi_3,m,n})|^2 d\xi_3 \\
&\geq \frac{1}{2\pi}\sum_k \varphi(\lambda_k)\sum_{m,n}\int_{-\infty}^{\infty} |(\omega_k, \chi_\varepsilon e^{ix_3\xi_3}\tau_{\xi_3,m,n})|^2 d\xi_3 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{m,n}\int_0^\infty \varphi(\nu)(dE_\nu\chi_\varepsilon\tau_{\xi_3,m,n}, \chi_\varepsilon\tau_{\xi_3,m,n})d\xi_3, \quad (11.2)
\end{aligned}$$

where E_ν is the spectral measure of $\mathcal{F}_3 A_\Omega \mathcal{F}_3^*$. However,

$$\begin{aligned}
&\sum_m \int_0^\infty (dE_\nu\chi_\varepsilon\tau_{\xi_3,m,n}, \chi_\varepsilon\tau_{\xi_3,m,n}) \\
&= \sum_m \|\chi_\varepsilon\tau_{\xi_3,m,n}\|_2^2 = \int_{\mathbb{R}^3}\chi_\varepsilon^2(x)\sum_m |\tau_{\xi_3,m,n}(x')|^2 dx \\
&= \int_{\mathbb{R}^3}\chi_\varepsilon^2(x)\Pi'_{\xi_3,n}(x', x')dx = \frac{|\xi_3|}{2\pi}\int_{\mathbb{R}^3}\chi_\varepsilon^2(x)dx,
\end{aligned}$$

which implies

$$1 - C_1\varepsilon \leq \frac{2\pi}{|\xi_3||\Omega|}\sum_m \int_0^\infty (dE_\nu\chi_\varepsilon\tau_{\xi_3,m,n}, \chi_\varepsilon\tau_{\xi_3,m,n}) \leq 1 - c_1\varepsilon,$$

where $C_1 \geq c_1 > 0$ depend on Ω. The convexity of φ gives us, by

Jensen's inequality, the following lower bound to (11.2):

$$\begin{aligned}
&\frac{1-C_1\varepsilon}{(2\pi)^2}|\Omega|\int_{-\infty}^{\infty}\sum_n \varphi\left(\frac{1}{1-c_1\varepsilon}\sum_m\int_0^{\infty}\nu(dE_\nu\chi_\varepsilon\tau_{\xi_3,m,n},\chi_\varepsilon\tau_{\xi_3,m,n})\right)|\xi_3|d\xi_3\\
&\qquad\geq \frac{1-C_1\varepsilon}{(2\pi)^2}|\Omega|\int_{-\infty}^{\infty}\sum_n\varphi\left(\mu_n(\xi_3)+\frac{C_2\varepsilon^{-2}}{1-c_1\varepsilon}\right)|\xi_3|d\xi_3\\
&\qquad\qquad= \frac{1-C_1\varepsilon}{16}|\Omega|\int_0^{\infty}\left(\lambda-\frac{C_2\varepsilon^{-2}}{1-c_1\varepsilon}-s\right)_+^{\gamma} s\,ds\\
&\qquad\qquad\qquad\qquad= (1-C_1\varepsilon)K_\gamma|\Omega|\left(\lambda-\frac{C_2\varepsilon^{-2}}{1-c_1\varepsilon}\right)^{\gamma+2}
\end{aligned}$$

similarly to the proof of Theorem 2.1. Hence (11.1) holds for cylinders. By the variational principle, the lower bound obtained when we approximate an arbitrary domain by decoupled cylinders subject to Dirichlet boundary conditions will not be less than the correct one. Since the constant actually coincides with that of the upper bound, we have proved the statement.

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