Last time:

$$\nabla\phi = \underbrace{\frac{1}{h_1}\frac{\partial\phi}{\partial q_1}}_{F_1} \underset{\sim}{e}_1 + \dots \dots$$

$\underset{\sim}{F}$ =

$$\nabla\cdot\underset{\sim}{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3F_1) + \dots\right]$$

The Laplacian

∇^2 sometimes written Δ

The most important operator in Mathematics

$$\nabla^2\phi = \nabla\cdot(\nabla\phi)$$

$$= \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2}$$

in Cartesian.

In general (q_1, q_2, q_3)

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial q_3}\right)\right]$$

Not worth remembering, but the special forms in cylindrical & spherical polars are.

Cylindrical (r, θ, z)

$h_1 = 1$, $h_2 = r$, $h_3 = 1$.

$$\boxed{\nabla^2\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial\phi}{\partial z}\right)\right]}$$

$$\boxed{= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}}$$

Spherical Polars (r, θ, η)

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta$$

$$\nabla^2\phi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r}\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial\eta}\left(\frac{r}{r\sin\theta}\frac{\partial\phi}{\partial\eta}\right)\right]$$

or

$$\boxed{\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\eta^2}}$$

What about curl?

$$\nabla\times\left(F_1\underset{\sim}{e_1} + F_2\underset{\sim}{e_2} + F_3\underset{\sim}{e_3}\right)$$

Consider

$$\nabla\times\left(F_1\underset{\sim}{e_1}\right).$$

Recall that $\nabla q_1 = \frac{1}{h_1}\underset{\sim}{e_1}$

$$\nabla\times(F_1\underset{\sim}{e_1}) = \nabla\times(F_1h_1\nabla q_1)$$

$$= \nabla(F_1h_1)\times\nabla q_1 + \underbrace{F_1h_1\nabla\times\nabla q_1}_{=0}$$

$$= \nabla(F_1h_1)\times\frac{\underline{e_1}}{h_1}$$

$$= \left(\frac{1}{h_1}\frac{\partial}{\partial q_1}(F_1h_1)\underset{\sim}{e_1} + \frac{1}{h_2}\frac{\partial}{\partial q_2}(F_1h_1)\underline{e_2} + \dots\right)\times\frac{\underline{e_1}}{h_1}$$

$\underline{e_3}\times\underline{e_1} = \underline{e_2}$ etc.

So

$$\nabla\times(F_1\underset{\sim}{e_1}) = \frac{-1}{h_1h_2}\frac{\partial}{\partial q_2}(F_1h_1)\underset{\sim}{e_3} + \frac{1}{h_1h_3}\frac{\partial}{\partial q_3}(F_1h_1)\underset{\sim}{e_2}$$

Similarly for $\nabla \times (F_2 \underline{e}_2)$ etc.

Finally have

$$\nabla \times \underline{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \underline{e}_1 & h_2 \underline{e}_2 & h_3 \underline{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

E.g.

Cylindrical polars. (r, θ, z).

$h_1 = 1 \quad h_2 = r \quad h_3 = 1$

$$\nabla \times \underline{F} = \frac{1}{r} \begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{e}_z \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ F_r & rF_\theta & F_z \end{vmatrix}$$

$$= \underset{\sim}{e_r} \left[\frac{\partial F_z}{r \partial \theta} - \frac{\partial F_\theta}{\partial z} \right] +$$

$$+ \underline{e}_\theta \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right]$$

$$+ \underline{e}_z \left[\frac{1}{r} \frac{\partial}{\partial r} (rF_\theta) - \frac{1}{r} \frac{\partial}{\partial \theta} F_r \right]$$

Factors of r appear in cylindrical coordinates!

<u>Laplace's Equation &</u>

<u>Poisson's Equation</u>

Most important equation in Physics.

Laplace: $\nabla^2 \phi = 0$

Poisson: $\nabla^2 \phi = f(\underset{\sim}{x})$

General solution of Poisson can be written:

$$\phi = \phi_c + \phi_p$$

where ϕ_p is a <u>particular</u> solution to $\nabla^2 \phi = f$

and ϕ_c is the general solution to $\nabla^2 \phi = 0$.

Applications

(1) Fluid Mechanics

velocity $\underline{u}(\underline{x})$.

If flow is incompressible.

$$\nabla \cdot \underline{u} = 0$$

If No vorticity (irrotational)

$\Rightarrow \nabla \times \underline{u} = 0$

$\Rightarrow \underline{u} = \nabla \phi$ for a <u>potential</u> ϕ.

$$\underline{\underline{\nabla \cdot \underline{u} = 0 \Rightarrow \nabla^2 \phi = 0}}$$

(2) Diffusion

$$\frac{\partial c}{\partial t} = \nabla^2 c$$

$\rightarrow$ equilibrium

$$0 = \nabla^2 c.$$

(3) Electrostatics.

The electric field $\underline{E}$ is related to charge density ρ

ⓔ

by $\nabla \cdot \underline{E} = \rho / \varepsilon_0$ ← constant)

In steady case, Maxwell Equation

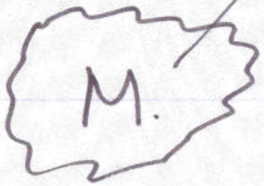
$\nabla \times \underset{\sim}{E} = 0$

$\Rightarrow \underline{E} = -\nabla \phi$.

$\Rightarrow \nabla^2 \phi = -\rho / \varepsilon_0$

Poisson.

(4) Gravitation



← mass density ρ.

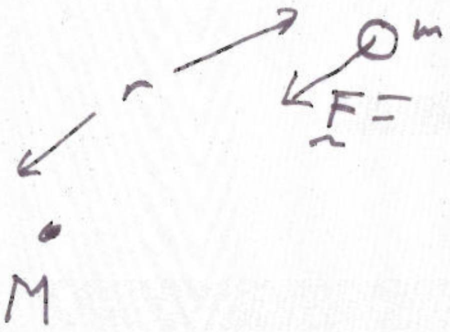
Gravitational force $\underline{F} = \nabla \phi$

↑ potential energy.

$$\boxed{\nabla^2 \phi = 4\pi G \rho}$$

This is not obvious.

Consider a point mass at the origin.



$$\underset{\sim}{F} = -\frac{GMm}{r^2}\hat{r}$$

What is $\nabla \cdot \underset{\sim}{F}$? Use Spherical polars (r, θ, η)

$$F_r = -\frac{GMm}{r^2}$$

So

$$\nabla \cdot \underset{\sim}{F} = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial r}\left(\overset{\sim \frac{1}{r^2}}{F_r}\, r^2 \sin\theta\right) + 0 + 0$$

(Using the formula you worked out yesterday).

So $\nabla \cdot \underline{F} = 0$. / $\underline{\text{Except at } r = 0}$

(as r^2 cancels

$$\nabla \times \underset{\sim}{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h e_r & h e_\theta & h e_\eta \\ \partial_r & 0 & 0 \\ F_r & 0 & 0 \end{vmatrix}$$

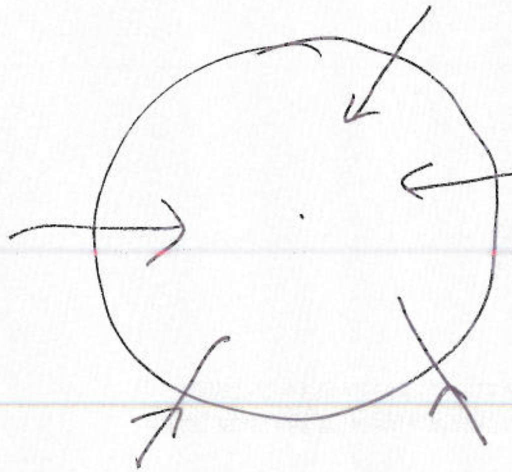
$$= 0$$

So $\nabla^2 \phi = 0$. except at $r = 0$.

$\nabla^2 \phi = 4\pi G\rho$

Use divergence theorem!

$$\int_V \nabla \cdot \underline{F} \, dV = \oint_S \underline{F} \cdot \hat{n} \, dS.$$



$$F_r = -\frac{GMm}{r^2}\hat{r}$$

$$\int \underline{F} \cdot \hat{n} \, dS = 4\pi r^2 \frac{GMm}{r^2}$$

$$\neq 0.$$