Jacobians

Recall that

$$\int_V f \, dx_1 \, dx_2 \, dx_3$$

$$= \int f |J| \, dq_1 \, dq_2 \, dq_3$$

where

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)}$$

column $\frac{\partial \underline{x}}{\partial q_3}$

$$= \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \frac{\partial x_1}{\partial q_3} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_2}{\partial q_3} \\ \frac{\partial x_3}{\partial q_1} & \frac{\partial x_3}{\partial q_2} & \frac{\partial x_3}{\partial q_3} \end{vmatrix}$$

$$= \left| \frac{\partial \underline{x}}{\partial q_1} \;\vdots\; \frac{\partial \underline{x}}{\partial q_2} \;\vdots\; \frac{\partial \underline{x}}{\partial q_3} \right|$$

$$= \frac{\partial \underline{x}}{\partial q_1} \cdot \left(\frac{\partial \underline{x}}{\partial q_2} \times \frac{\partial \underline{x}}{\partial q_3} \right)$$

Scalar Triple Product.

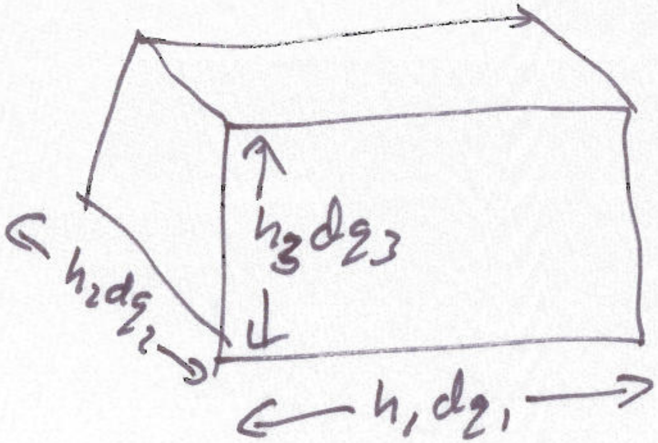
$$= h_1 \underset{\sim}{e_1} \cdot (h_2 \underset{\sim}{e_2} \times h_3 \underline{e}_3)$$

(by definition: see earlier).

$$J = h_1 h_2 h_3 \, (\underline{e}_1 \cdot \underbrace{\underline{e}_2 \times \underline{e}_3}_{= \underset{\sim}{e_1}})$$

$$J = h_1 h_2 h_3.$$

Pictorially



Scale factors give physical lengths corresponding to small coordinate changes

So $\delta V = h_1 h_2 h_3 \, \delta q_1 \delta q_2 \delta q_3$

(as expected) - for <u>orthogonal</u> coordinates

Grad Div and Curl.

∇ ← wonderful symbol.

We want to express grad div and curl in an arbitrary orthogonal coordinate system.

$\nabla\phi$ is a vector: write as

$$\nabla\phi = A_1 \underline{e}_1 + A_2 \underline{e}_2 + A_3 \underline{e}_3$$

where $\underline{e}_1, \underline{e}_2, \underline{e}_3$ are the vectors corresponding to q_1, q_2, q_3.

Make a small change in ϕ,

$\phi \to \phi + d\phi$, as $\underline{x} \to \underline{x} + d\underline{x}$

$q_1 \to q_1 + dq_1$ etc.

By choin rule,

$$\underline{d\phi} = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3$$

$$= (\nabla \phi) \cdot d\underline{x}$$

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3} \right)$$

$$d\underline{x} = (dx_1, dx_2, dx_3)$$

$$= h_1 dq_1 \underline{e}_1 + h_2 dq_2 \underline{e}_2 + h_3 dq_3 \underline{e}_3$$

Also

$$d\phi = \frac{\partial \phi}{\partial q_1} dq_1 + \frac{\partial \phi}{\partial q_2} dq_2 + \frac{\partial \phi}{\partial q_3} dq_3$$

$$\nabla \phi \cdot d\underline{x} = d\phi$$

$$= (A_1 \underline{e}_1 + A_2 \underline{e}_2 + A_3 \underline{e}_3) \cdot (h_1 dq_1 \underline{e}_1 + h_2 dq_2 \underline{e}_2 + h_3 dq_3 \underline{e}_3)$$

$$= A_1 h_1 dq_1 + A_2 h_2 dq_2 + A_3 h_3 dq_3 = d\phi.$$

(by orthogonality $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$)

$$\frac{\partial \phi}{\partial q_1} = A_1 h_1, \quad \frac{\partial \phi}{\partial q_2} = A_2 h_2 \ldots$$

So $A_1 = \frac{1}{h_1} \frac{\partial \phi}{\partial q_1}$ etc.

So

$$\boxed{\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial q_1}\underset{\sim}{e_1} + \frac{1}{h_2}\frac{\partial\phi}{\partial q_2}\underset{\sim}{e_2} + \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}\underset{\sim}{e_3}}$$

Intuitively reasonable:

$h_1 dq_1$ is distance in $\underline{e_1}$ direction just as x_1 is distance in $\hat{x}_1$ direction.

$$\frac{\partial\phi}{\partial x_1} \Longleftrightarrow \frac{\partial\phi}{\underline{h_1}\partial q_1}$$

<u>Check:</u>

Cartesians $h_1 = h_2 = h_3 = 1$

$q_1 = x_1$ etc.

$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\hat{x}_1 + \frac{\partial\phi}{\partial x_2}\hat{x}_2 + \frac{\partial\phi}{\partial x_3}\hat{x}_3$$

as it should.

<u>Cylindrical Polars</u> (r, θ, z)

$h_r = 1$, $h_\theta = r$, $h_z = 1$

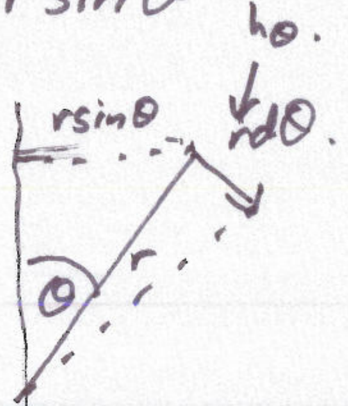
$$\boxed{\nabla\phi(r,\theta,z) = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{\partial\phi}{\partial z}\hat{z}}$$

Spherical Polars (r, θ, η)

Ex: check

$h_r = 1$, $h_\theta = r$, ~~$h_\eta = rs$~~

$h_\eta = r\sin\theta$



$$\boxed{\nabla\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\eta}\hat{\eta}}$$

Divergence: Div $\nabla\cdot$

Suppose

$$\underset{\sim}{F} = F_1\underset{\sim}{e_1} + F_2\underset{\sim}{e_2} + F_3\underset{\sim}{e_3}$$

Then

$$\nabla\cdot\underset{\sim}{F} = \nabla\cdot(F_1\underset{\sim}{e_1}) + \nabla\cdot(F_2\underline{e_2}) + \nabla\cdot(F_3\underline{e_3})$$

$\underset{\sim}{e_1}$ etc is non-constant. How do we work out $\nabla\cdot\underset{\sim}{e_1}$?

We know $\underline{e_1} = \underline{e_2}\times\underline{e_3}$

$$\nabla\cdot(\underline{e_2}\times\underline{e_3}) = (\nabla\times\underline{e_2})\cdot\underline{e_3} - \underline{e_2}\cdot(\nabla\times\underline{e_3})$$

Now we also know:

$$\nabla q_1 = \frac{1}{h_1}\frac{\partial q_1}{\partial q_1}\underline{e_1} = \frac{1}{h_1}\underset{\sim}{e_1} \Rightarrow \underline{\underline{\underline{e_1} = h_1\nabla q_1}}$$

So

$$\nabla \times \underline{e}_2 = \nabla \times (h_2 \nabla q_2)$$
$$= \nabla h_2 \times \nabla q_2 + \cancel{h_2 \nabla \times \nabla q_2}$$
$$= 0.$$

So

$$\nabla \cdot (F_1 \underset{\sim}{e_1}) = \nabla \cdot (F_1 \underline{e}_2 \times \underline{e}_3)$$
$$= \nabla \cdot (\underline{F_1 h_2 h_3}\, \underline{\nabla q_2 \times \nabla q_3})$$
$$= \nabla (F_1 h_2 h_3) \cdot (\nabla q_2 \times \nabla q_3)$$
$$+ \underbrace{\cancel{(F_1 h_2 h_3) \nabla \cdot (\nabla q_2 \times \nabla q_3)}}_{= 0\ (\nabla \times \nabla = 0).}$$
$$= \nabla (F_1 h_2 h_3) \cdot \left(\frac{\underline{e}_2}{h_2} \times \frac{\underline{e}_3}{h_3} \right)$$

(As $\underline{e}_2 = h_2 \nabla q_2$ etc).

$$= \frac{1}{h_2 h_3} \nabla (F_1 h_2 h_3) \cdot \underbrace{(\underline{e}_2 \times \underline{e}_3)}_{= \underset{\sim}{e_1}}.$$

$$= \frac{1}{h_2 h_3} \frac{1}{h_1} \frac{\partial}{\partial q_1} (F_1 h_2 h_3)$$

We conclude, with a leaping heart,

$$\boxed{\nabla \cdot \underset{\sim}{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (F_1 h_2 h_3) + \frac{\partial}{\partial q_2} (F_2 h_3 h_1) + \frac{\partial}{\partial q_3} (F_3 h_1 h_2) \right]}$$

Check:

Cartesians: $h_1 = h_2 = h_3$

$$1\left[\frac{\partial}{\partial x_1} F_1 + \frac{\partial}{\partial x_2} F_2 + \frac{\partial}{\partial x_3} F_3\right] \checkmark$$

Cylindrical Polars (r, θ, z).

$h_r = 1, \; h_\theta = r, \; h_z = 1.$

$$\underset{\sim}{F} = F_r \hat{r} + F_\theta \hat{\theta} + F_z \hat{z}.$$

$$\nabla \cdot \underset{\sim}{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(F_r r) + \frac{\partial}{\partial \theta}(F_\theta \cdot 1) + \frac{\partial}{\partial z}(F_z \cdot 1 \cdot r)\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

Exercise: Derive

$\nabla \cdot \underline{F}$ in Spherical Polars.
