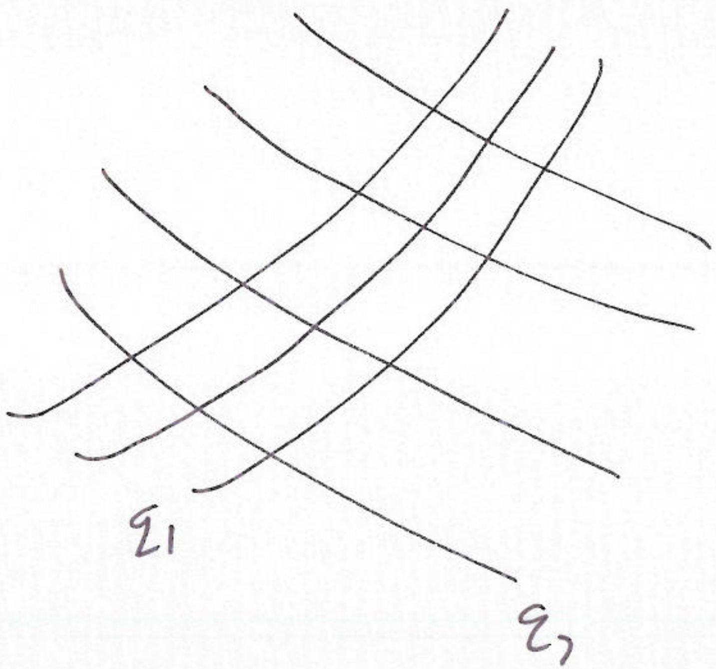
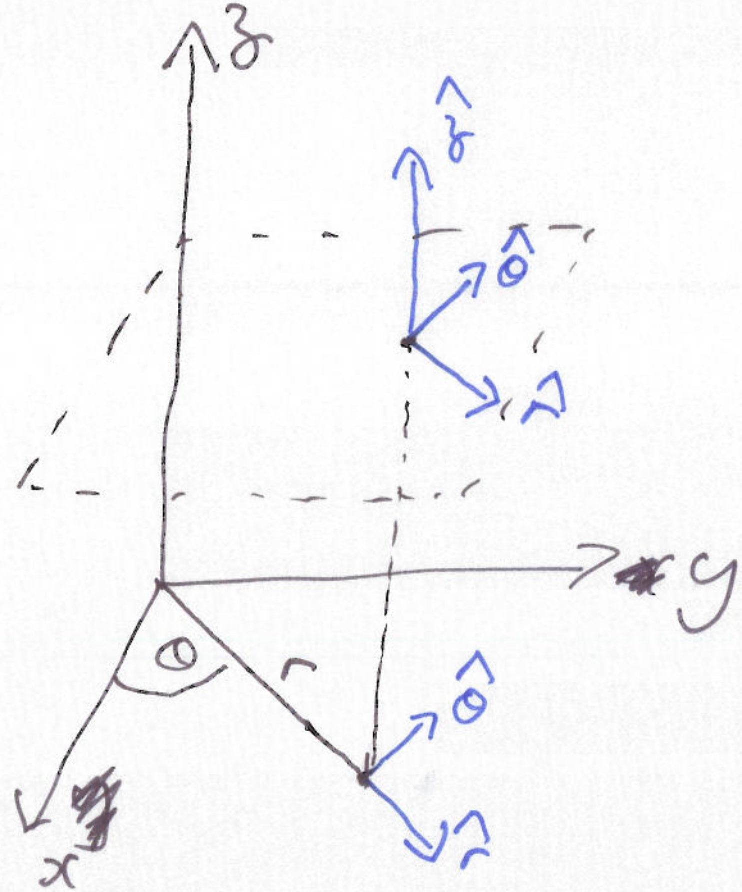
Orthogonal Curvilinear Coordinates



More general coordinates systems

Examples

I Cylindrical Polars



A point (x, y, z) can be represented by (r, θ, z)

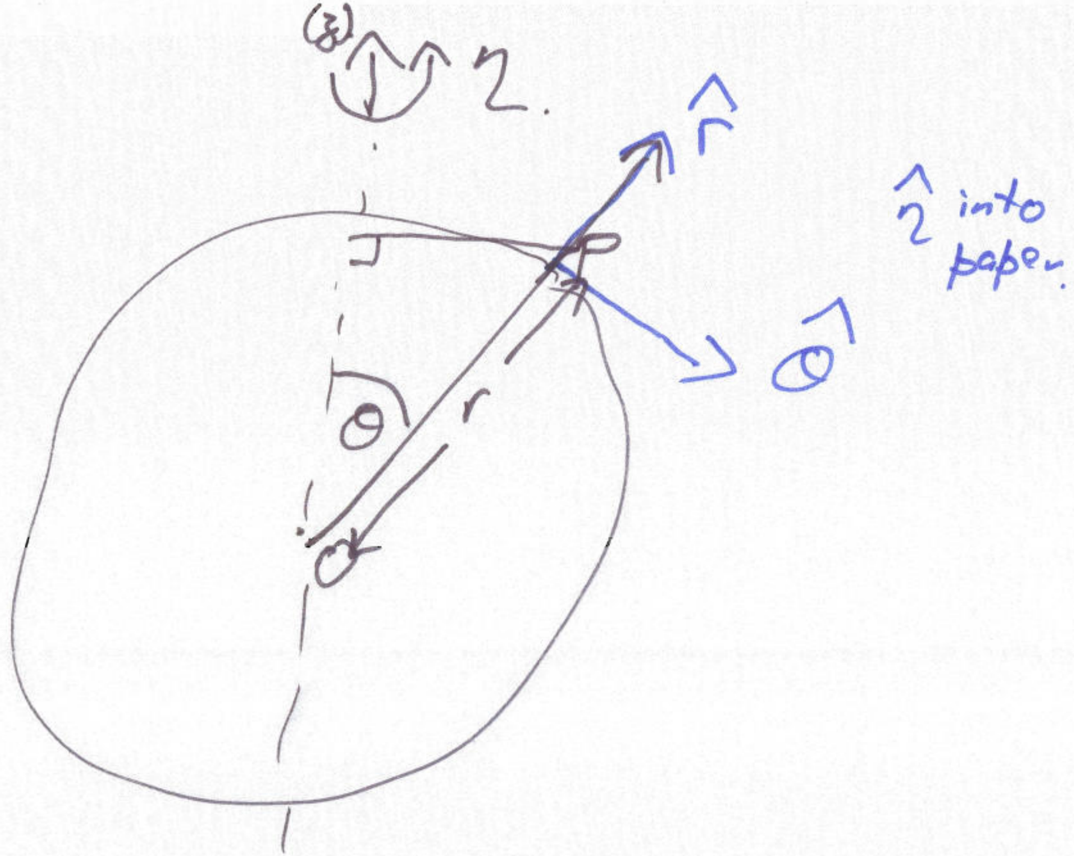
where (r,θ) are polar coords. in the (x,y) plane.

So $\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} r^2 = x^2+y^2$.

The coordinate vectors are as shown. They point in the direction of increasing coordinate

II Spherical Polar Coordinates

Essentially lines of latitude & longtitude.



r is distance from the origin
$r^2 = x^2+y^2+z^2$.

θ is angle between z axis and OP, i.e $r\cos\theta = z$

η measures angle around the z axis.

Notation is horrible!

r in cylindricals $\neq$ r in sphericals

θ " " $\neq$ θ in spher.

θ " " $=$ η in spher.

Insane. But live with it.

$r\sin\theta$ is distance from $\hat{z}$ axis

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\}$$

$r \geq 0$

$0 \leq \theta \leq \pi$

$0 \leq \eta \leq 2\pi$

The General Case

Relabel (x, y, z) as (x_1, x_2, x_3)

Introduce new coordinates

q_1, q_2, q_3

where x_1, x_2, x_3 can be expressed in terms of q_1, q_2, q_3

$$\left.\begin{aligned} &x_1(q_1, q_2, q_3) \\ &x_2(q_1, q_2, q_3) \\ &x_3(q_1, q_2, q_3) \end{aligned}\right\} \text{known functions}$$

$\underline{x} = \underline{x}(\underline{q})$ in a more compact form.

Now make a small change

$\underline{q} \longrightarrow \underline{q} + \delta \underline{q}$

Specifically, keep q_2, q_3 constant

$q_1 \longrightarrow q_1 + \delta q_1$ (δq_1 is small)

Define $\dfrac{\partial \underline{x}}{\partial q_1} = \lim_{\delta q_1 \to 0} \left(\dfrac{\underline{x}(q_1 + \delta q_1) - \underline{x}(q_1)}{\delta q_1} \right)$

$$= h_1 \underset{\sim}{e_1}$$

i.e $\underline{e}_1$ is <u>a unit vector</u> in the direction of increasing q_1, and h_1 is called a <u>scale factor</u>.

Similarly $\dfrac{\partial \underline{x}}{\partial q_2} = h_2 \underline{e}_2$ and $\dfrac{\partial \underline{x}}{\partial q_3} = h_3 \underline{e}_3$.

The <u>coordinate system</u> (q_1, q_2, q_3) is <u>orthogonal</u> iff the vectors $\underline{e}_1, \underline{e}_2$ & $\underline{e}_3$ are orthogonal,

i.e. $\underset{\sim}{e_i} \cdot \underset{\sim}{e_j} = \delta_{ij}$

The system is "<u>right-handed</u>" if

$\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$.

If $\delta \underline{q}$ is general, then

$\underset{\sim}{\delta x} = h_1 \delta q_1 \underset{\sim}{e_1} + h_2 \delta q_2 \underline{e}_2 + h_3 \delta q_3 \underline{e}_3$

$\left[\underline{x}(q_1 + \delta q_1, q_2 + \delta q_2, q_3 + \delta q_3) - \underline{x}(q_1, q_2, q_3) \right.$

$\left. = \delta q_1 \dfrac{\partial \underset{\sim}{x}}{\partial q_1} + \delta q_2 \dfrac{\partial \underline{x}}{\partial q_2} + \delta q_3 \dfrac{\partial \underline{x}}{\partial q_3} + \ldots \right]$

Note:

$$|\delta \underline{x}|^2 = h_1^2 \delta q_1^2 + h_2^2 \delta q_2^2 + h_3^2 \delta q_3^2$$

(using orthogonality).

<u>Notation</u>: Often we would use

h_r, h_θ, h_z in sphericals.

$\updownarrow \quad \updownarrow \quad \updownarrow$ in an obvious way.

$r \quad \theta \quad z$

<u>Example</u>: Cylindrical Polars.

$q_1 = r, \quad q_2 = \theta, \quad q_3 = z$

$$\begin{aligned} x_1 &= r\cos\theta \\ x_2 &= r\sin\theta \\ x_3 &= z. \end{aligned} \quad \begin{pmatrix} = x \\ = y \\ = z \end{pmatrix}.$$

So

$$\underline{x} = (x_1, x_2, x_3)$$

$$\frac{\partial \underline{x}}{\partial q_1} = \frac{\partial}{\partial r}(r\cos\theta, r\sin\theta, z)$$

$$= (\cos\theta, \sin\theta, 0). = h_1 \underline{e_1}$$

$$\left|\frac{\partial \underline{x}}{\partial q_1}\right| = 1 = h_1$$

So $\underset{\sim}{e_r} = (\cos\theta, \sin\theta, 0)$

$\equiv$

$\underset{\sim}{e_1}$

$$\frac{\partial \underset{\sim}{x}}{\partial q_2} = \frac{\partial}{\partial \theta}(r\cos\theta, r\sin\theta, z)$$

$$= (-r\sin\theta, r\cos\theta, 0).$$

$$\underline{h_2} = \left|\frac{\partial \underline{x}}{\partial q_2}\right| = (r^2\sin^2\theta + r^2\cos^2\theta)^{1/2} = \underline{r}.$$

So

$\underset{\sim}{e}_2 \equiv e_\theta = (-\sin\theta, \cos\theta, 0)$.

Finally

$$\frac{\partial \underset{\sim}{x}}{\partial q_3} = \frac{\partial \underset{\sim}{x}}{\partial z} = (0, 0, 1).$$

$$h_3 \equiv h_z = 1$$

$$\underset{\sim}{e}_3 = (0, 0, 1).$$

$$|\delta \underset{\sim}{x}|^2 = \delta r^2 + r^2 \delta\theta^2 + \delta z^2$$

$h_r^2 = 1$ $\quad h_\theta^2 \quad$ $h_z^2 = 1$.

What about Jacobians?

Recall

$$\int f(x,y)\, dx\, dy$$

$$= \int J f \, du\, dv$$

where

$$J = \frac{\partial(x,y)}{\partial(u,v)}$$

Or in 3-D

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)}$$

$$= \det \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} \\ \frac{\partial x_3}{\partial q_1} & \frac{\partial x_2}{\partial q_2} \end{vmatrix}$$