## Last Time:

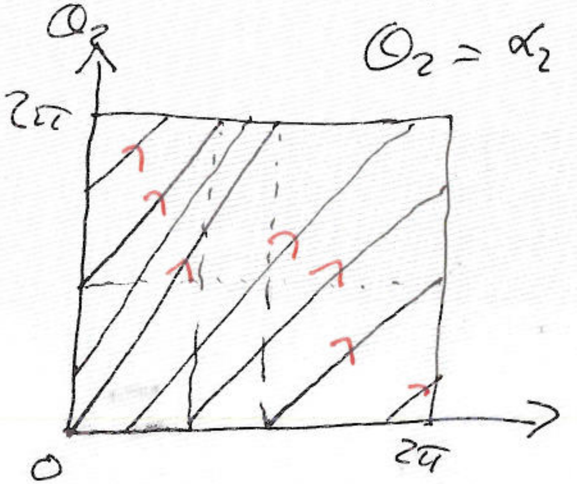
Modelled interaction between Sleep/wake cycles & Circadian rhythm using

$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$$
$$\dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$$

Initially, take $K_1 = 0 = K_2$

$\dot{\theta}_1 = \omega_1$ (constants).
$\dot{\theta}_2 = \omega_2$

$\theta_1 = \alpha_1 + \omega_1 t$
$\theta_2 = \alpha_2 + \omega_2 t$




E.g $\alpha_1 = 0 = \alpha_2$.

$\frac{\theta_1}{\theta_2} = \frac{\omega_1}{\omega_2}$

If $\frac{\omega_1}{\omega_2}$ is rational orbit (trajectory) closes on itself.

If it's irrational, then orbit cannot ever close. However, it

(a) is dense on the torus $T^2$

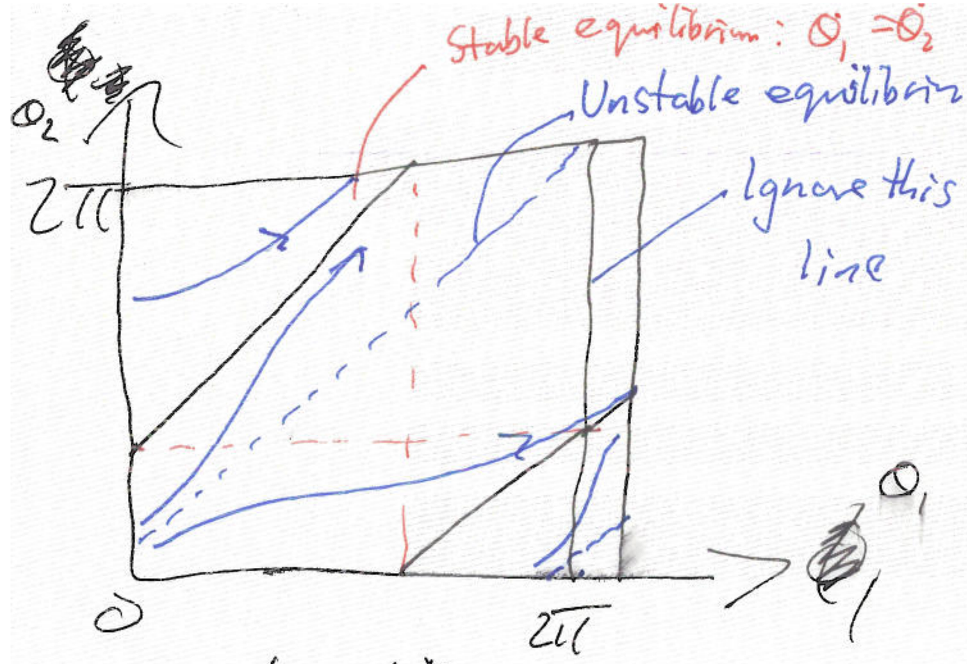
Every trajectory gets arbitrarily close to every point in phase plane.

(b) We say this behaviour is quasi-periodic

(periodic iff $\frac{\omega_1}{\omega_2}$ is rational).

Quasi-periodicity only occurs on the torus!

(NB, Physical behaviour is fairly similar for a large number of circuits necessary to join up as for never joining. But mathematically very different).




As $\phi \to \phi^*$

$$\dot{\theta}_1 \to \omega_1 - K_1 \sin\phi^*$$

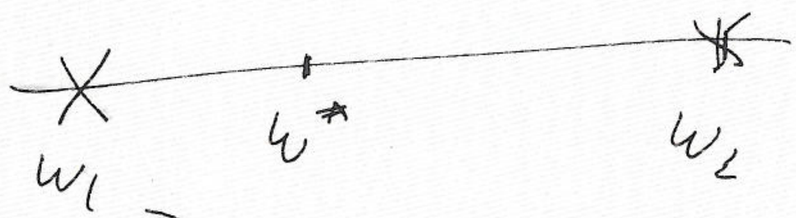
$$\dot{\theta}_2 \to \omega_2 + K_2 \sin\phi^*$$

and $\sin\phi^* = \dfrac{\omega_1 - \omega_2}{K_1 + K_2}$

$$\dot{\theta}_1 = \dot{\theta}_2 = \underline{\omega^*}$$

where $\omega^* = \dfrac{K_1\omega_2 + K_2\omega_1}{K_1 + K_2}$

$\omega^*$ is called a <u>compromise</u> frequency. a weighted average of the natural frequencies $\omega_1$ & $\omega_2$




Any initial trajectory asymptotically approaches the stable trajectory (with the compromise frequency).

Return to the coupled system

$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$$
$$\dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$$
$$(K_1 \geq 0,\ K_2 \geq 0)$$

Obtain a single equation by subtracting:

$$\dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2$$
$$\star - (K_1 + K_2)\sin(\theta_1 - \theta_2)$$

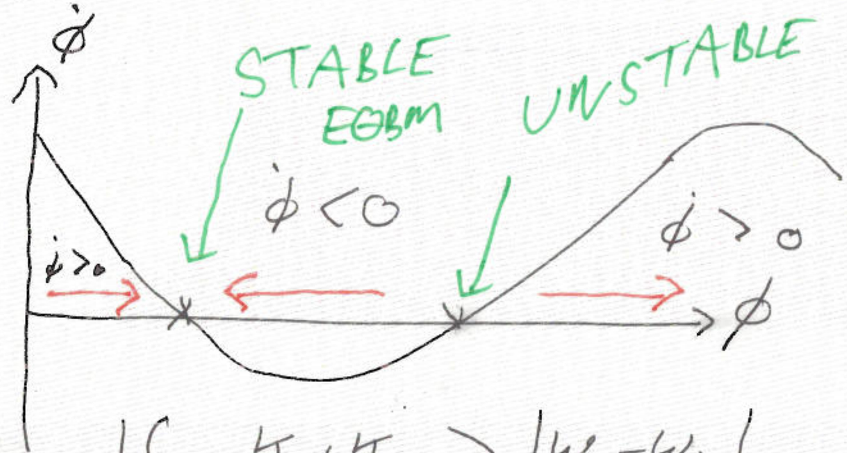
Define the phase difference

$$\phi = \theta_1 - \theta_2$$
$$\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2$$

$$\dot{\phi} = (\omega_1 - \omega_2) - (K_1 + K_2)\sin\phi$$

Could solve this exactly.
More illuminating to think about it.

$\dot{\phi} = f(\phi)$ Draw $f(\phi)$




If $K_1 + K_2 > |\omega_1 - \omega_2|$
there are equilibrium points
where $\dot{\phi} = 0$

Initial value of $\phi$, determines whether $\dot{\phi} > 0$ or $\dot{\phi} < 0$.

Diagram shows one equilibrium is stable, other unstable

So eventually $\phi \to \phi^*$, a unique stable equilibrium.

$$\phi^* = \frac{\omega_1 - \omega_2}{K_1 + K_2}$$

If $K_1 + K_2 < |\omega_1 - \omega_2|$ however, we have no equilibria. The scenario is very like the irrational ratio unlinked case — trajectories meander around, never intersecting (but no longer straight lines)

How can we begin to understand complex dynamical systems?

Introduce the idea of

Poincaré Maps

System undergoes a saddle-node bifurcation as $K_1 + K_2 = |\omega_1 - \omega_2|$.

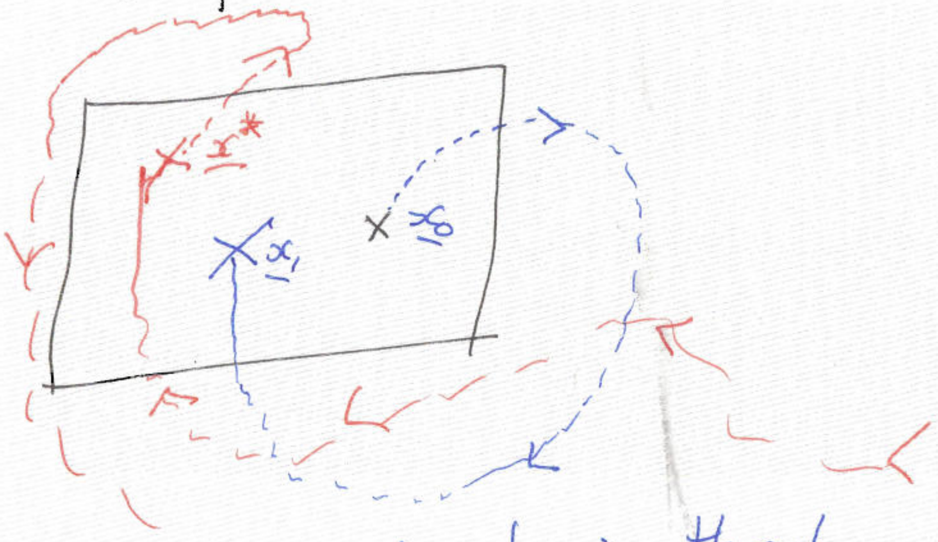
"Same Poincaré"

Suppose we have an $N$-dimensional system

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

$\underline{x}$ is an $N$-vector.

Define an $(N-1)$ dimensional surface $S$.




Suppose a trajectory passing through the point $\underline{x}_0$ on $S$, hits $S$ again at $\underline{x}_1 \in S$.

This defines a map from $S \to S$.

$$\underline{x}_1 = F(\underline{x}_0)$$

(Now let $\underline{x}_0$ be any point on $S$).

($S$ must be transverse to the flow).

This is an easier problem.

Easier to understand properties of the map than of the full ODE system.

Suppose $F$ has a fixed point, $\underline{x}^*$, so that

$$\underline{x}^* = F(\underline{x}^*)$$

Then the system has a closed trajectory which intersects $S$ precisely once.

These notes will appear on www.ma.ic.ac.uk/~ajm8/buck.pdf.