**Lecture 24: Journey to the Centre of the Earth and the distant Solar System**

The final project of this module consists of the solution of the Convection Equations in an annulus. This is intended as a simplified (two-dimensional) model of thermally-driven convection between two spheres. This problem is important in the centre of the earth, whose outer core consists of a spherical shell of liquid iron. It also occurs in the recently discovered underground oceans in the moons of the outer planets. We hope our 2-D annular model will capture many of the important features of the problem.

Convection occurs because fluids expand when heated. Their density decreases and they become buoyant. Gravity can then drive fluid motion, which then advects heat around. Suitably non-dimensionalised, the governing equations for the temperature $T$, pressure $p$ and velocity $\mathbf{u}$ are

$$\left.\begin{aligned} T_t + \mathbf{u}\cdot\nabla T &= \nabla^2 T \\ P_r^{-1}(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) &= -\nabla p - R_a T\,\widehat{g} + \nabla^2\mathbf{u} \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}\right\} \tag{24.1}$$

There are two parameters in the problem: The Prandtl number, $P_r$ is the ratio of the fluid kinematic viscosity to the thermal diffusivity. The Rayleigh number $R_a$ is a measure of the thermal driving force, as defined last lecture. In deriving (24.1), it is assumed that the density decreases linearly with temperature, and that the fluid speed is much less than the speed of sound, permitting the Boussinesq approximation. Gravity acts in the $\widehat{g}$-direction. In two-dimensions, we may once again eliminate the pressure by taking the curl. The vorticity $\nabla\wedge\mathbf{u}$ then has a single component $\omega$, related to a streamfunction $\psi$ by

$$-\omega = \nabla^2\psi. \tag{24.2}$$

In our annular model, we will consider flow in the domain $1 < r < b$ in terms of polar coordinates $(r,\theta)$. The temperature boundary conditions are $T = 1$ on $r = 1$ and $T = 0$ on $r = b$. Gravity acts in the negative $\widehat{r}$-direction. The $r$- and $\theta$-components of the velocity are respectively $\psi_\theta/r$ and $-\psi_r$. The equations for $\psi(r,\theta,t)$ and $T(r,\theta,t)$ reduce to

$$\left.\begin{aligned} rT_t + \psi_\theta\,T_r - \psi_r\,T_\theta &= (rT_r)_r + \frac{1}{r}T_{\theta\theta} \\ P_r^{-1}\left(r\omega_t + \psi_\theta\,\omega_r - \psi_r\,\omega_\theta\right) &= -R_a T_\theta + (r\omega_r)_r + \frac{1}{r}\omega_{\theta\theta}. \end{aligned}\right\} \tag{24.3}$$

The problem obviously is very similar to the Rayleigh-Bénard convection considered in lectures and for which programs are available on Blackboard. Here too, there is a purely conductive solution, with $\mathbf{u} = 0$ and $T = T(r)$. What we expect to happen is that for $R_a \leqslant R_c$ the only solution is the conductive solution, but for $R_a > R_c$ this solution is unstable, and a flow (*convection*) develops. We want to investigate the critical value of $R_a$, and the convection patterns formed for different values of $P_r$, $R_a$ and maybe $b$.

**M345N10 Third Project – An Annular Model of Planetary Convection**

*This project counts for 50% of the entire module. It was released 8th March and it is due in by 23:59 on Monday 10th April. It should be submitted electronically on Blackboard.*

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Once more, you may choose the value of $b$, your grid size and your testing procedures as you think appropriate, but describe your choices clearly.

**(1)** Write a routine to advect a quantity $Q$ by $Q_t + \mathbf{u} \cdot \nabla Q = 0$ in the annulus $1 < r < b$ by a given velocity field expressed in its $r$- and $\theta$-coordinates. You should aim for the routine to be 2nd order in space and time and to conserve $Q$. If you wish, you may assume $Q = 0$ on $r = 1$ and $r = b$. You may assume that $\nabla \cdot \mathbf{u} = 0$ and that $\mathbf{u} = 0$ on the boundary. Test how well $Q$ is conserved in your routine by running it for an interval with suitable velocity fields, and then running them backwards, as we did in lectures for 2-D.

**(2)** Now modify your routine from project 2 to solve the diffusion equation $Q_t = \nabla^2 Q$ by multigrid. When you are happy things are working, combine your two routines to solve the advection diffusion equation for the temperature $T(r, \theta, t)$

$$T_t + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T \qquad \text{in } 1 < r < b \quad \text{with} \quad T\big|_{r=1} = 1, \qquad T\big|_{r=b} = 0. \tag{24.4}$$

As in lectures, use an operator splitting approach, wherein you first advect $T$ and then diffuse it, ignoring the other process in each part of the algorithm.

For a suitable velocity field, run your code until it settles down to an equilibrium for various values of $\kappa$, and comment on the differing features of the solution.

**(3)** Now modify your code in (2) to apply to the vorticity equation

$$\omega_t + \mathbf{u} \cdot \nabla \omega = \nabla^2 \omega + F, \qquad -\omega = \nabla^2 \psi, \tag{24.5}$$

with $\psi = \psi_r = 0$ on the boundaries. Use the Dirichlet condition to find $\psi$ from $\omega$, and then use the Neumann condition to approximate a Dirichlet condition for $\omega$, as in the lecture notes. Choose some forcing $F$ and test your routine to your satisfaction.

**(4)** Finally, combine everything to solve the full problem as specified in (24.2) & (24.3) overleaf. It might speed things up to seed the instability with a small, $\theta$-dependent perturbation at $t = 0$ – this perturbation should decay for $R_a < R_c$, but grow for $R_a > R_c$. Identify an approximate value of $R_c$, and then investigate the solution for various values of $P_r$ and $R_a$. Describe any interesting behaviour you find. Does the number of rolls vary with the parameters at all? Does the solution always settle down to an equilibrium, or does it sometimes produce a time-periodic solution? What effect does varying the Prandtl number have? What about varying $b$? It is not expected that you will be able to investigate the parameter-space fully.

It may be that you have difficulty getting either the advection or the multigrid routines to work satisfactorily. If so, you may nevertheless attempt part (4) using other methods (which will probably be slower or less accurate, but may be easier to implement) For example, you could use upwinding or SOR techinques.

**(5)** Mastery component (M4/M5 only). Create an A1 poster describing the problem, the model and its solution. A LaTeX template and advice is available on BB for this purpose. You should aim to include essentials but not details, while making the whole comprehensible and attractive to a mathematical audience. The department will print it out for you in colour, if you wish. M3 may also create a poster, but not for credit.