Lecture 25: How the Leopard got his Spots (and his Stripy Tail.)

The final project of this module investigates Alan Turing's theory of morphogenesis, or how animal coat patterns develop. Observations of animal embryos show that up to a certain age the foetus is monochrome and has no pattern on its skin. When it reaches a certain size, however, a complex pattern suddenly appears. Turing had the idea that this could be explained by a size-dependent instability of two chemicals interacting and diffusing at different rates.

Suppose two chemicals with concentrations $u(x, y, t)$ and $v(x, y, t)$ inhabit a two-dimensional surface Ω. We assume their evolution follows PDEs of the form

$$\left.\begin{aligned} u_t &= f(u, v) + d_1 \nabla^2 u \\ v_t &= g(u, v) + d_2 \nabla^2 v \end{aligned}\right\} \quad \text{in } \Omega. \tag{25.1}$$

Here f and g are known functions describing the chemical reactions. As the chemical molecules may be of different sizes, it is likely that the two diffusion rates differ, $d_1 \neq d_2$. This turns out to be a crucial part of the theory. One of u and v may describe a pigment, so that a coloured pattern may appear if (say) u is not constant in space. The boundary conditions we shall impose are that there is no flux of the chemicals across any boundary,

$$\frac{\partial u}{\partial n} = 0 = \frac{\partial v}{\partial n} \qquad \text{on the boundary } \partial\Omega. \tag{25.2}$$

In project 2, you may have found that for a similar problem a non-zero solution can come into existence as a parameter is varied. That problem had Dirichlet boundary conditions ($u = 0$), so that the only constant solution was $u = 0$ everywhere. The Neumann conditions (25.2) permit a constant solution $(u_0,\, v_0)$ to (25.1) if

$$f(u_0,\, v_0) = 0 = g(u_0,\, v_0). \tag{25.3}$$

Turing's theory predicts that if f, g and the other parameters obey certain constraints, that the equilibrium (25.3) may be stable to perturbations which do not vary in space, but unstable to perturbations with a spatial structure. This means that a complex pattern may suddenly appear from nowhere as the parameters change. The theory is a stability analysis, solving linear perturbation equations and predicts that a pattern will begin to form at $t = 0$. It cannot predict what pattern will eventually develop as $t \to \infty$. That would involve solving nonlinear equations numerically, which is where we come in.

The linear theory suggests that the patterns which develop initially may be either stripes or spots. Stripy patterns only vary with one coordinate (e.g. $u = u_0 + \varepsilon \cos \lambda x$) whereas spotty patterns vary with both x and y, e.g. $u = u_0 + \varepsilon \cos \lambda x \cos \mu y$. The pattern that first appears depends on the growth rates of any instabilities, which are determined by the parameters of the problem and the domain size. The final (equilibrium) pattern may or may not look similar to the initial instability.

M345N10 Third Project: Nonlinear Animal Coat Patterns

This project counts for 40% of the entire module. It was officially released 19th March and it is due in by 23:59 on Monday 16th April. It should be submitted on Blackboard.

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(1) Modify your multigrid code from Project 2 to solve the diffusion equation with Neumann boundary conditions in a $1 \times L$ rectangle,

$$u_t = d(u_{xx} + u_{yy}) + Q \qquad \text{in} \quad 0 < x < 1, \quad 0 < y < L, \quad t > 0, \tag{25.4}$$

with the Neumann boundary conditions

$$u_x = \Phi \quad \text{on} \quad x = 0, 1, \quad u_y = \Phi \quad \text{on} \quad y = 0, L, \quad \text{and} \quad u = u_i(x, y) \quad \text{at} \quad t = 0. \tag{25.5}$$

Here Φ denotes 4 different functions on the 4 boundaries. Often these functions will be zero. You will also need to solve for u on the boundaries for these boundary conditions, as discussed in lectures. Devise a suitable test to check your routine is working.

(2) Use your routine to solve the Turing system

$$\left.\begin{aligned} u_t &= \gamma(a - u + u^2 v) + \nabla^2 u \\ v_t &= \gamma(b - u^2 v) \quad + \quad d\nabla^2 v \end{aligned}\right\} \quad \text{in} \quad 0 < x < 1, \quad 0 < y < 2, \quad t > 0, \tag{25.6}$$

with homogeneous Neumann conditions ($\Phi = 0$ on all walls), where γ, a, b and d are given, positive constants. Turing's linear theory suggests choosing values of a, b and d such that

$$\left.\begin{aligned} 0 < b - a < (b + a)^3 < d(b - a) \\ \left[d(b - a) - (b + a)^3\right]^2 > 4d(b + a)^4, \end{aligned}\right\} \tag{25.7}$$

for example $d = 10$, $b = 0.95$, $a = 0.07$. Investigate any patterns which develop for different values of γ which represents the growing animal size. (You may need $10 < \gamma$ before any pattern occurs – experiment.) For your initial state use a perturbation of the (unstable) uniform state (u_0, v_0) and see how it evolves towards equilibrium. Can you find parameter values for which stripy and spotty patterns appear?

(3) Finally, use domain decomposition, as discussed in lectures, to solve the problem in a region formed by gluing one rectangle on the side of another. This aims to model an appendage such as a tail or a leg on the main animal body. Your main aim is to find a steady solution which is predominantly stripy on the tail blending smoothly onto a spotty body, but report on anything interesting you find. It is recommended you split your domain into two overlapping regions, Ω_1 and Ω_2. Each timestep, use your routines to solve in one region using data from the other to calculate the boundary conditions Φ and then alternate between the two regions, a few times.

It may be that you have difficulty getting the multigrid, or domain decomposition routines to work satisfactorily. If so, you may nevertheless attempt the problems using other methods (which will probably be slower or less accurate, but may be easier to implement).

(4) Mastery component (M4/M5 only). Create an A1 poster describing the problem, the model and its solution. A LaTeX template and advice is available on BB for this purpose. You should aim to include essentials but not details, while making the whole comprehensible and attractive to a mathematical audience. The department will print it out for you in colour, if you wish. M3 may also create a poster, but not for credit.