M345N10 Second Project – Using Multigrid in Reaction-Diffusion systems.

This project counts for 40% of the entire module. It was released 12th Feb and it is due in by 23:59 on Monday 5th March. It should be submitted electronically on Blackboard.

When I retire, if my pension still permits it, I plan to buy a rectangular orchard $100m \times 200m$. I shall use the apples it produces to make cider, which I shall then offer free to all my ex-students. Unfortunately, there is a parasite which attacks the apple trees and which may ruin the entire plan. The parasite has a population density $u(x, y, t)$ obeying the equations and boundary conditions given below.

(1) Modify the Multigrid solver for the Poisson equation with Dirichlet boundary conditions so that it works on a $1 \times L$ rectangle, where $L > 1$ is an integer. Test the method using a suitable exact solution of your choice.

(2) Now compose a scheme using Multigrid to solve the 2-dimensional diffusion equation for $u(x, y, t)$ with a given source Q,

$$u_t = u_{xx} + u_{yy} + Q(x, y, t, u) \qquad \text{in} \quad 0 < x < 1, \quad 0 < y < 2, \quad t > 0, \tag{1}$$

with the boundary conditions

$$u = 0 \qquad \text{on} \quad x = 0, 1, \quad y = 0, 2 \qquad \text{while} \quad u = u_0(x, y) \qquad \text{at} \quad t = 0. \tag{2}$$

Use a Crank-Nicolson algorithm, finding the approximation at the new time level using a modification of your Multigrid code from part (1). If Q depends on t, evaluate it explicitly at $t + \frac{1}{2}k$. If Q depends on u, you may either evaluate it at the old time-level, OR linearise it, as discussed in lectures. Devise a suitable test to show that your program works to your satisfaction.

(3) Use your code to find the population density u when, for a constant A, the function

$$Q = Au(1-u)(2-u) \qquad \text{with} \qquad -100 < A < 100. \tag{3}$$

Investigate what happens for different values of A and for different starting states u_0. Identify any qualitative changes you discover.

(4) The behaviour of the parasites varies periodically with the time of day. Investigate what happens when

$$Q = f(x, y) \cos \omega t, \qquad \text{for a function } f \text{ of your choice.}$$

After some time, we expect the solution u to settle down to a time-periodic state, so that $u \to u_\infty$ where $u_\infty(x, y, t) = u_\infty(x, y, t + 2\pi/\omega)$. Find u_∞, for at least two values of ω, one small and one large. Comment on the solution structure.

General notes: (a) Most of you will modify the Matlab routines supplied, but it is quite ok to use any sensible language. If in doubt, contact me.
(b) If you are unable to get the Multigrid routine in parts (1) & (2) to work for the diffusion equation you may still attempts parts (3) and (4) using another method, e.g. ADI or even a slow explicit method.
(c) You have a fair amount of free choice in this project. You may choose your timestep k and parameters A, ω and functions f and u_0. Sometimes the results may differ for different initial states and amplitudes. It is to be hoped your choices will not resemble anybody else's too much.